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Professor Ľubomír Snoha, 60 years old

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On 18 March 2015, we celebrated the 60th birthday of our colleague, Prof. RNDr. Ľubomír Snoha, DSc., DrSc. His arrival to our Department of Mathematics in 1978 has significantly influenced its further development. With his admirable diligence, commitment and general knowledge (which he acquired during his studies at the Faculty of Natural Sciences, Comenius University in Bratislava), he proved that even small departments can achieve excellent results at the international level.

Ľubomír Snoha attended the secondary school in Lučenec between 1970 and 1973. After graduation in 1973, he enrolled at the Faculty of Natural Sciences, Comenius University in Bratislava, where he studied Mathematics and Physics from 1973 to 1978. Due to his excellent study results (Honours diploma) and admirable achievements within his master's thesis (on a theorem of Sophie Piccard and on points of connectivity and Darboux continuity of real functions), that were later published, he managed to acquire the post-graduate academic degree RNDr (Rerum Naturalium Doctor), only several months after finishing his studies in 1978. He was recommended by the Professors Štefan Znám and Jaroslav Smítal from the Faculty of Natural Sciences, Comenius University, to work as an assistant at the then Pedagogical Faculty in Banská Bystrica.

In 1978, he was appointed as an assistant to the Department of Mathematics of the above-mentioned faculty, and works there to this day. Today, the department is a part of the Faculty of Natural Sciences of Matej Bel University. In 1979, immediately after finishing his duty in the army, he went onto to hold a position of teaching, and in 1980, he was appointed to the Faculty of Mathematics and Physics at the Comenius University in Bratislava, where he studied within an external research assistantship (today known as a postgraduate doctoral study) in the field of Mathematical Analysis. During his studies there, he was tutored by Prof. Jaroslav Smítal, under whose guidance he elaborated two works within ŠVOČ (Student Scientific and Professional Activity), the master's thesis and the RNDr dissertation. Being an enthusiast, he regularly attended the scientific seminars that were led by Prof. Tibor Šalát (seminar on real functions) and Prof. Jaroslav Smítal (seminar on dynamical systems) at the Faculty of Mathematics and Physics, Comenius University in Bratislava. In 1986, he became a Candidate of Sciences of the Faculty of Mathematics and Physics, after he had defended the doctoral dissertation on dynamical systems, in which he completed the characterization of minimal periodic orbits of continuous maps of an interval (the same results were independently proven by Ll. Alseda, J. Llibre and R. Serra). At the age of 34, he managed to obtain the scientific-pedagogical degree "docent" (equivalent to associate professor) at the Faculty of Mathematics and Physics, Comenius University (in 1989). In 2005, he was awarded an academic degree DSc (the Doctor of Science) by the Academy of Sciences of the Czech Republic and in 2007, an academic degree DrSc (the Doctor of Science) by the Comenius University in Bratislava. In 2008, the Silesian University in Opava awarded him the scientific-pedagogical degree of a Professor.

Initially, he dealt with the theory of real functions, however, he soon moved to the theory of discrete dynamical systems, in which he is a highly respected figure today. He deals mainly with topological dynamics, chaos theory and low-dimensional dynamics. He began with interval dynamics, explored minimal periodic orbits, generic and dense chaos and mappings of type 2^∞. For instance, he found a full characterization of generically chaotic maps and densely chaotic maps. He also published several papers with V. Jiménez López; one of their outstanding findings is that there are no continuous piecewise linear maps (with finitely many pieces and no constant piece) of type 2^∞. Along with interval dynamics, he engaged in dynamics of triangular maps (skew products), dynamics on graphs, dendrites and spaces with a free interval, as well as in dynamics on metric spaces. He explored, for example, ω-limit sets, topological transitivity, topological entropy, stroboscopical property, scrambled sets etc. In collaboration with Ll. Alseda, S. Kolyada and J. Llibre, they investigated the connection between qualitative dynamical properties and possible values of topological entropy. In an extensive paper with F. Blanchard and W. Huang they investigated in great depth the question of how large (from the topological point of view) scrambled sets may be. In the theory of non-autonomous dynamical systems, his seminal paper with S. Kolyada on topological entropy is best known. Jointly with J. Auslander and S. Kolyada they introduced the notion of the functional envelope of a dynamical system. With M. Misiurewicz and S. Kolyada they explored the topology of the space of transitive interval maps; they propose the name Dynamical Topology for the investigation of topological properties of spaces of maps that can be described in dynamical terms.

His favourite subjects within his field of scientific research are minimal dynamical systems, i.e. the systems with all orbits dense. He is a highly reputable expert in the field of topological structure of minimal sets and topological properties of minimal maps. Jointly with S. Kolyada and S. Trofimchuk they showed that minimal maps in compact metric spaces are almost one-to-one. They also proved that proper minimal sets on compact connected 2-manifolds are nowhere dense. Further, they described topological structure of minimal sets of fibre-preserving maps in graph bundles. Another deep result on minimality, now obtained jointly with F. Balibrea, T. Downarowicz, R. Hric and V. Špitalský, is that an almost totally disconnected compact metric space admits a minimal map if and only if either it is a finite set or it has no isolated point. Let us also mention that very recently, with T. Downarowicz and D. Tywoniuk, they have constructed a continuum that is uniquely minimal in the sense that its group of self-homeomorphisms is isomorphic to $\mathbb{Z}$ and all the self-homeomorphisms, except of the identity, are minimal.

He has been an author or co-author of approximately 50 scientific works that have been followed by many respected foreign mathematicians. He has given lectures on many foreign scientific conferences, very often as an invited speaker. He is an editor of Journal of Difference Equations and Applications (Taylor and Francis), Non-autonomous Dynamical Systems (de Gruyter) and Acta Universitatis Matthiae Belii, series Mathematics.

Prof. Snoha spent a year as a visiting professor at University of Murcia. He has visited Max Planck Institute for Mathematics in Bonn several times, and spent a semester in Stefan Banach International Mathematical Center in Warsaw. He has also visited many other research institutes and universities (Barcelona, Murcia, Lisbon, Marseille, Paris, Oberwolfach, Vienna, Warsaw, Kiev, Santiago, Talca, Vadodara, Hefei, ...).

Since 1995, he has been running a scientific seminar on Dynamical Systems at Matej Bel University. He is the team leader of the dynamical systems group whose members are also his former PhD students, Roman Hric, Vladimír Špitalský and Matúš Dirbák.

Ľ. Snoha loves teaching. He has been teaching mainly subjects from the field of mathematical analysis and dynamical systems. Apart from scientific and pedagogical activities at the university, he has dealt with gifted pupils in elementary and secondary schools. In the 1980s, he was an organizer of ten summer camps for young 'mathematicians' (in fact schoolchildren) from central Slovakia and he led mathematics correspondence courses in problem solving for secondary school students from central Slovakia for five years. Moreover, he has lectured dozens of seminars for gifted secondary school students, mainly from grammar schools in the regions of central Slovakia.

He has always endeavoured to build the department, faculty, university, mainly for the purpose of maintaining a high level of science. In 1992, after merging the former faculties in Banská Bystrica, the present Matej Bel University came into existence, in which he was appointed the Vice-Rector for Science and as such, he urged to enhance the role of science at the university. In other words, his main idea was to substitute the regional research for international standards.

Our colleague, Ľ. Snoha, has left a deep mark in our department during his 37 years of work and has succeeded in building an outstanding school of dynamical systems. His opinions have had a positive influence on the direction of the department in recent years. Thanks also to him, the Department of Mathematics has been valued as one of the best mathematics departments in Slovakia today.

Dear Ľubo, on the occasion of your jubilee, we would like to thank you for your on-going and outstanding work you have done for our department and mathematics in Slovakia. We wish you a good health, a success not only in scientific but also in pedagogical activities, and a lot of professional and personal satisfaction.

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Maximal designs and configurations - a survey

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Abstract
It is the aim of this article to provide a unified view for, and a survey of, a class of problems that occur often in combinatorics, graph theory and related areas but also in "real life".

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Introduction

It is the aim of this article to provide a unified view for, and a survey of, a class of problems that occur often in combinatorics, graph theory and related areas but also in "real life".

We want to discuss a situation which typically is as follows. Given is a finite set $\mathcal{F}$ of objects called *figures*, and a symmetric irreflexive relation R on $\mathcal{F}$ (the *compatibility rule*) which specifies when two figures are compatible. An $(\mathcal{F}, R)$-configuration or simply a *configuration* is a set of pairwise compatible figures. A configuration C is *maximal* if there is no $f \in \mathcal{F}$, $f \notin C$ such that $f \cup C$ is also a configuration. In other words, maximality is here with respect to inclusion.

More generally, the compatibility rule R is a function from a subset of the power set of $\mathcal{F}$ into $\{0, 1\}$ but we will restrict ourselves to examples which are all of the simpler type above.

The *size* of a configuration is the number of its figures. An $(\mathcal{F}, R)$-configuration is *maximum* if it is maximal and contains the largest possible number of figures. Maximum configurations are sometimes called *maximum packings* or just *packings*.

Our interest will be mainly in the possible sizes of maximal $(\mathcal{F}, R)$-configurations, i.e. in the *spectrum* $Sp(\mathcal{F}, R)$ defined by

$$Sp(\mathcal{F}, R) = \{m \colon \text{there exists a maximal } (\mathcal{F}, R)\text{-configuration of size } m\}.$$

To determine the spectrum $Sp(\mathcal{F}, R)$, one usually needs to determine first the size of the smallest maximal, and maximum configurations, that is, the size of the smallest and the largest element of $Sp(\mathcal{F}, R)$.

We may envisage a procedure under which one tries to build maximal (maximum) configurations of given kind in a naive way: given any configuration, try to enlarge it by adjoining another figure subject to the compatibility rule, then another one, and so on, until this is no longer possible, i.e. you get “stuck”. The elements of the spectrum represent sizes of all possible outcomes of such a process.

What follows is a (non-exhaustive) survey of problems falling under this framework, both completely solved (unfortunately, very few), partially solved (a few more) and those that remain largely open. This article may be viewed as an expanded and updated version of [61].

Our first example is a problem that has been solved completely.

1 Maximal sets of 1-factors

The figures here are 1-factors of the complete graph K_{2n} on a given set of $2n$ vertices; two such 1-factors are compatible if they are edge-disjoint. Let $M(2n)$ be the spectrum for maximal sets of 1-factors, i.e.

$$M(2n) = \{m\colon \text{there exists a maximal set of } m \text{ edge-disjoint 1-factors in } K_{2n}\}.$$

As a corollary to Dirac's Theorem (see, e.g., [70]) one obtains immediately

$$M(2n) \subseteq \{n, n+1, \ldots, 2n-1\}.$$

Trivially, $2n-2 \notin M(2n)$ since the complement of the union of $2n-2$ one-factors is itself a 1-factor. Furthermore, $n \notin M(2n)$ if n is even [16]. On the other hand, when k is odd and $k \in \{n, n+1, \ldots, 2n-1\}$ then $k \in M(2n)$, as shown by the following simple construction.

Let $Z_k \cup \{a_i : i = 1, 2, \ldots, 2n-k\}$ be the set of vertices of K_{2n} and let the 1-factor F be defined by

$$\begin{aligned} F = \{&\{a_1, 0\}, \{a_2, 1\}, \{a_3, k-1\}, \ldots, \{a_{2n-k-1}, \frac{1}{2}(2n-k-1)\}, \\ &\{a_{2n-k}, k - \frac{1}{2}(2n-k-1)\}, \{\frac{1}{2}(2n-k-1)+1, k-\frac{1}{2}(2n-k-1)-1\}, \\ &\{\frac{1}{2}(2n-k-1)+2, k-\frac{1}{2}(2n-k-1)-2\}, \ldots, \{\frac{1}{2}(k-1), \frac{1}{2}(k+1)\}\} \end{aligned}$$

(the edges in the last two lines are used only when $k \neq n$).

Developing F modulo k yields a maximal set of 1-factors, since the complement of the union of these 1-factors contains an odd component K_{2n-k}.

The case of even k turned out to be much more difficult. It was shown in [60] that for k even, $k \in M(2n)$ if and only if $\frac{1}{3}(4n+4) \leq k \leq 2n-4$.

Explicitly, we have for small values of n;

$$\begin{aligned} &M(4) = \{3\}, M(6) = \{3, 5\}, M(8) = \{5, 7\}, M(10) = \{5, 7, 9\}, \\ &M(12) = \{7, 9, 11\}, M(14) = \{7, 9, 11, 13\}, M(16) = \{9, 11, 12, 13, 15\}, \ldots, \\ &M(30) = \{15, 17, 19, 21, 22, 23, 24, 25, 27\} \text{ and so on.} \end{aligned}$$

Although the spectrum for maximal sets of 1-factors has thus been completely determined, several further problems arise when one puts additional conditions on the 1-factors comprising the set in question. Among several possible variations of the above problem that have been treated, at least to some degree, in the literature, is the one concerning maximal *perfect* sets of 1-factors. In this variation of the problem, two 1-factors are compatible if they are edge-disjoint and their union is a hamiltonian cycle. Let $M_{perf}(2n)$ be the spectrum of maximal perfect sets of 1-factors. It is currently not known whether the maximum possible value $2n-1$ is a member of $M_{perf}(2n)$ for all n since to determine this is equivalent to determining whether there exists a perfect 1-factorization of K_{2n} for all n. The latter remains a difficult unsolved problem (cf., e.g., [56]).

It is easily verified that $M_{perf}(4)=\{3\}, M_{perf}(6)=\{3,5\}, M_{perf}(8)=\{5,7\}$ but the determination of $M_{perf}(2n)$ becomes much more difficult for larger orders. Petrenjuk [58], [59] determined the sets $M_{perf}(2n)$ for $2n=10,12$: $M_{perf}(10)=\{5,6,7,9\}$, $M_{perf}(12)=\{6,7,8,9,10,11\}$. It is established in [56] that $M_{perf}(14)=\{7,8,9,10,11,12,13\}\cup I$ where either $I=\emptyset$ or $I=\{6\}$. Matan Ziv-Av (private communication) has now shown that $6\notin M_{perf}(14)$.

When n is an odd prime then $n\in M_{perf}(2n)$ but not much else seems to be known about $M_{perf}(2n)$.

One may, of course, consider also the situation where the union of any two 1-factors in a set $\mathcal{F}$ of disjoint 1-factors is isomorphic to a fixed 2-regular factor Q, not necessarily a hamiltonian cycle. Such a set has been called *Q-uniform* or simply *uniform*. Let $M_Q(2n)=\{s\colon$ there exists a maximal Q-uniform set of s 1-factors of $K_{2n}\}$. The following results concerning small maximal uniform sets have been established in [56] with the aid of computer (below Q is represented just as a partition of $2n$).

$$\begin{aligned}
&M_{4+4}(8)=\{7\}, M_{6+4}(10)=\{3,9\},\\
&M_{4+4+4}(12)=\{3\}, M_{6+6}(12)=\{3,5,11\}, M_{8+4}(12)=\{6,9\},\\
&M_{6+4+4}(14)=\{3,5,7\}, M_{8+6}(14)=\{5,6,7\}, M_{10+4}(14)=\{5,6,7,8\},\\
&M_{4+4+4+4}(16)=\{7,15\}, M_{6+6+4}(16)=M_{8+4+4}(16)=\{3,4,5,7\},\\
&\{5,6,7,8,9,10\}\subseteq M_{8+8}(16).
\end{aligned}$$

2 Maximal sets of 2-factors

The figures here are 2-factors in the complete graph on a given set of n vertices; two 2-factors are compatible if they are edge-disjoint.

Let $M^{(2)}(n)=\{m\colon$ there exists a maximal set of m edge-disjoint 2-factors of $K_n\}$. Petersen's theorem about the existence of a 2-factor in any regular graph of even degree (cf. [70]) implies that for odd n,

$$M^{(2)}(n)=\{\frac{1}{2}(n-1)\}.$$

The situation is somewhat more involved for n even. This is due to the fact that for odd d, there exist regular graphs of degree d without proper regular factors. König [53] calls such graphs *primitive*. An obvious extension of König's example for $d=3$ yields a primitive graph of odd degree d $(d>1)$ with $(d+1)^2$ vertices. It is shown in [42] that this is the minimum number of vertices a primitive graph of odd degree d can have. This implies that the spectrum $M^{(2)}(n)$ for n even is the following interval:

$$M^{(2)}(n)=\{\lfloor\frac{1}{2}(n-\sqrt{n})\rfloor, \lfloor\frac{1}{2}(n-\sqrt{n})\rfloor+1,\ldots,\frac{1}{2}(n-2)\}.$$

In the next two examples, the figures are still 2-factors but of a restricted type.

3 Maximal sets of hamiltonian cycles

The figures here are *connected* 2-factors of K_n, that is, hamiltonian cycles; two hamiltonian cycles are compatible if they are edge-disjoint.

Let

$$MH(n) = \{m\colon \text{there exists a maximal set of } m \text{ hamiltonian cycles in } K_n\}.$$

Put

$$Dir(n) = \{\lfloor \frac{1}{4}(n+3)\rfloor, \lfloor \frac{1}{4}(n+3)\rfloor + 1, \ldots, \lfloor \frac{1}{2}(n-1)\rfloor\}.$$

It follows directly from Dirac's theorem and a result of Nash-Williams (cf. [70]) that $MH(n) \subseteq Dir(n)$. One would like to show that, in fact, equality takes place here. To achieve this, consider the following.

Let n be even, $n = 2k$, and let m be a positive integer, $2m \leq k$. Let G be a regular graph of degree $2k - 4m$ with $2k - 2m$ vertices, and let $H = \bar{K}_{2m} \nabla\, G$. Similarly, let n be odd, $n = 2k + 1$, m be a positive integer, $2m + 1 \leq k$, and let G be a regular graph of degree $2k - 4m - 1$ with $2k - 2m$ vertices, and let $H = \bar{K}_{2m+1} \nabla\, G$ (here ∇ denotes the join, cf. [70]).

In order to show that $MH(n) = Dir(n)$, it clearly suffices to show that the graph H, with G suitably chosen, has a hamiltonian decomposition. Indeed, the complement $\bar{H}$ of H is disconnected, and so the set of hamiltonian cycles in any hamiltonian decomposition is maximal. The corresponding proof that H has a hamiltonian decomposition for G suitably chosen is given in [42].

The above provides another example of a problem with completely determined spectrum.

4 Maximal sets of Δ-factors

The figures are Δ-factors of K_n, that is, 2-factors whose each component is a triangle (sometimes also called triangle-factors); two Δ-factors are compatible when they are edge-disjoint. Clearly, here we must have $n \equiv 0 \pmod 3$.

Let $\Delta(n) = \{m\colon$ there exists a maximal set of Δ-factors of $K_n\}$.

A classical result of Corrádi and Hajnal [12] states that a graph with $n = 3k$ vertices and minimum degree at least $2k$ has a Δ-factor. Thus a maximal set of Δ-factors on $3k$ vertices must contain at least $\frac{k}{2}$ triangle-factors. This implies

$$\Delta(n) \subseteq \{\lceil \frac{n}{6}\rceil, \lceil \frac{n}{6}\rceil + 1, \ldots, \lfloor \frac{n-1}{2}\rfloor\}.$$

It is easily seen that $\Delta(3) = \Delta(6) = \{1\}$, $\Delta(9) = \{4\}$.

For every odd k, there is a maximal (in fact, maximum) set of $\frac{3k-1}{2}$ Δ-factors in K_{3k}. For every even $k \geq 6$, there is maximal set of $\frac{3k-2}{2}$ Δ-factors in K_{3k}. This just restates the fact that for every $n \equiv 3 \pmod 6$ there exists a Kirkman triple system $KTS(n)$ of order n, and for every $n \equiv 0 \pmod 6$, $n \geq 18$, there exists a nearly Kirkman system $NKTS(n)$ of order n [13].

Furthermore, it is not difficult to establish that $2 \notin \Delta(12)$, while $5 \notin \Delta(12)$ folows from the nonexistence of a nearly Kirkman triple system of order 12. Thus $\Delta(12) = \{3, 4\}$.

On the other hand, it is not easy to establish that $\Delta(15) = \{4, 5, 6, 7\}$ (see [30]). More precisely, it is difficult to show $3 \notin \Delta(15)$; no computer-free proof of this fact is known to us. (In [30], *all* maximal sets of Δ-factors in K_{15} are enumerated.)

It is proved in [60] that $\Delta(18) = \{4, 5, 6, 7, 8\}$ (this involved showing $3 \notin \Delta(18)$), $\Delta(21) = \{4, 5, 6, 7, 8, 9, 10\}$, and $\Delta(24) = \{4, 5, 6, 7, 8, 9, 10, 11\}$. It is also shown that $\{6, 7, 8, 9, 10, 11, 12, 13\} \subseteq \Delta(27)$ but whether or not $5 \in \Delta(27)$ remains undecided. Similarly, $[6, 14] \subseteq \Delta(30)$ but whether $5 \in \Delta(30)$ is undecided.

It was conjectured in [60] that the spectrum $\Delta(n)$ contains the interval $[\lceil \frac{n}{6} \rceil, \frac{n-1}{2}]$, and proved that $[\lceil \frac{n}{6} \rceil, \lceil \frac{n}{4} \rceil] \in \Delta(n)$. Several further constructions for maximal sets of Δ-factors are given in [60] but especially for k in the interval $[\frac{n}{4}, \frac{n}{3}]$, new ideas appear to be needed. Also, for $k = \lceil \frac{n}{6} \rceil$ when $n \equiv 0, 9$ or $12 \pmod{18}$, not a single maximal set of k Δ-factors is known to exist (and $\lceil \frac{n}{6} \rceil \notin \Delta(n)$ for $n = 9, 12, 18$). So, e.g., whether or not $6 \in \Delta(33)$ remains an open problem.

5 Maximal partial latin squares and latin cubes

The figures are elements of $N \times N \times N$, i.e. ordered triples from a set N of n elements; two such triples are compatible if they agree in at most one coordinate. We can take $N = \{1, 2, \dots, n\}$.

It is somewhat more convenient to think of a partial latin square as an $n \times n$ array whose cells are either empty or contain an element of N such that no element occurs in a cell of any row or column more than once. A partial latin square is then maximal if no further nonempty cell can be filled without violating this condition.

Let $ML(n)$ be the spectrum of maximal partial latin squares of order n, that is,

$$ML(n) = \{m\colon \text{there exists a maximal partial latin square of order } n \text{ with exactly } m \text{ nonempty cells}\}.$$

The set $ML(n)$ was investigated in [44]. Clearly, if $t < \frac{n^2}{2}$ or if $t = n^2 - 1$ then $t \notin ML(n)$. It is shown in [44] that when either

(i) $t = \frac{n^2}{2} + k, 1 \le k \le \frac{n}{2}$ where k is odd and n is even, or

(ii) $t = \lceil \frac{n^2}{2} \rceil + k, 1 \le k \le \frac{n-1}{2}$ where k is odd and n is odd,

we also have $t \notin ML(n)$.

It was also shown in [44] that the spectrum $ML(n)$ contains all integers t in the interval $[\frac{n^2}{2}, n^2 - 2]$ except possibly when (1) $t = \frac{n^2}{2} + k$, n even, k odd, $\frac{n}{2} < k \le n - 1$, or when (2) $t = \frac{n^2+1}{2} + k$, n odd, k odd, $\frac{n-1}{2} \le k \le n - 1$. It is conjectured in [44] that these possible exceptions are in fact true exceptions.

Recently, in [5] an analogous question was studied for partial latin cubes. Here the figures are elements of $N \times N \times N \times N$, i.e. ordered quadruples from a set N of n elements. Two such quadruples are compatible if they agree in at most two coordinates. One can picture a partial latin cube as a set of layers where each layer is a partial latin square, and no element occurs in the same row or column of distinct layers. A partial latin cube is then maximal if no further cell can be filled without violating this.

Let $ML^{(3)}(n) = \{m\colon$ there exists a maximal partial latin cube of order n with exactly m nonempty cells$\}$.

Neither $n^3 - 1$ nor $n^3 - 2$ can belong to $ML^{(3)}(n)$. In [5] it is shown that, unlike for maximal partial latin squares, there exist maximal partial latin cubes with substantially less than half of its n^3 cells filled. In fact, while any maximal partial latin cube must contain at least $t > (1 - \frac{1}{\sqrt{2}} n^3 > 0.29289 n^3$ nonempty cells, there exist maximal partial latin cubes with $\frac{n^3}{3} + O(n^2)$ nonempty cells. For instance, when $n \equiv 1 \pmod 3$, there exists a maximal partial latin cube with at most $\frac{n^3+9n^2-6n-4}{3}$ nonempty cells.

A large portion of spectrum is determined in [5]: when n is even, $n \ge 10$ then $[\frac{n^3}{2}, n^3 - 3] \subseteq ML^{(3)}(n)$, and when n is odd, $n \ge 21$ then $[\frac{n^3+n}{2}, n^3 - 3] \subseteq ML^{(3)}(n)$. But for less than "half-full" maximal partial latin cubes, gaps remain (cf. also [55]).

In the same paper [5], the spectra $ML^{(3)}(n)$ for $n = 2, 3, 4$ are determined almost completely, with only three values in the case of $n = 4$ remaining in doubt. In particular, it is shown that $ML^{(3)}(2) = \{4, 5, 8\}$,
$ML^{(3)}(3) = \{9, 12, 14, 15, \ldots, 24, 27\}, ML^{(3)}(4) = \{31, 32, \ldots, 61, 64\} \cup S$ where
$S \subseteq \{28, 29, 30\}$.

6 Row-maximal latin rectangles and maximal latin parallelepipeds

Here the figures are permutations of n symbols, say $1, 2, \ldots, n$; two such permutations are compatible if they are discordant, i.e. do not agree in any position.

In 1945, M. Hall Jr. proved [41] that if $r < n$ then any $r \times n$ latin rectangle can be extended to a $(r + 1) \times n$ latin rectangle. His proof is a nice application of Philip Hall's Theorem on systems of distinct representatives.

It follows that the spectrum of row-maximal $r \times n$ latin rectangles

$$MLR(n) = \{r\colon \text{there exists a row-maximal } r \times n \text{ latin rectangle}\}$$

consists of a single element, namely n.

The situation changes dramatically as one tries to extend M. Hall's result to three dimensions. Now the figures are $(n \times n)$ latin squares; they are compatible if they are disjoint. A latin $(n \times n \times r)$-parallelepiped is maximal if it cannot be extended to a latin $(n \times n \times (r + 1))$-latin parallelepiped. Let $MLC(n) = \{r$: there exists a maximal $n \times n \times r$-latin parallelepiped$\}$.

Horák [43] was the first to show that for all $n = 2^k$, there exist infinitely many Latin $(n \times n \times (n - 2))$-parallelepipeds that cannot be completed to a Latin cube of order n and are therefore maximal. In [31], [50] further results on maximal $(n \times n\ (n - 2))$-latin parallelepipeds were obtained. (Clearly, any latin $(n \times n\ (n - 1))$-latin parallelepiped can be extended to a latin cube of order n.)

Subsequently Kochol [51], [49], [52] proved that for any r, n such that $\frac{n}{2} < r \leq n - 2$ there exists a *noncompletable* $n \times n\ r$ latin parallelepiped. In [8] both noncompletable and nonextendible (that is, maximal) latin parallelepipeds are investigated. A maximal $5 \times 5 \times 2$ and a $6 \times 6 \times 3$ latin parallelepiped is produced, and a construction is given showing that for all even $m > 2$, there exists a maximal $(2m - 1) \times (2m - 1) \times (m - 1)$-latin parallelepiped. In particular, that shows the existence of a maximal $7 \times 7 \times 3$-latin parallelepiped.

The above are first examples of maximal latin parallelepipeds that are less than "half-full". But clearly, lots of work remains towards determining the spectrum $MLC(n)$.

7 Row-maximal orthogonal latin rectangles

The figures are pairs of permutations of degree n. Two pairs (P_1, P_1') and (P_2, P_2') are compatible if (P_1, P_2) and (P_1', P_2') are both discordant, and the two $2 \times n$ latin rectangles $\binom{P_1}{P_2}$ and $\binom{P_1'}{P_2'}$ are orthogonal.

Let $MOR(r, n)$ be a pair of row-maximal orthogonal latin $r \times n$ rectangles. Let the spectrum for row-maximal orthogonal latin $(r \times n$ rectangles be $MOR(n) = \{r$: there exists a $MOR(r, n)\}$.

For small values of n, we have

$$\begin{aligned} &MOR(1) = MOR(2) = \{1\}, MOR(3) = \{3\}, MOR(4) = \{3, 4\},\\ &MOR(5) = \{3, 5\}, MOR(6) = \{3, 4, 5\}, MOR(7) = \{3, 4, 5, 6, 7\},\\ &MOR(8) = \{3, 4, 5, 6, 7, 8\}. \end{aligned}$$

Several partial results are obtained in [45] towards settling the following conjecture.

Conjecture. For $n \geq 7, MOR(n) = \{r : \frac{n}{3} < r \leq n\}$.

In particular, it is shown in [45] that $MOR(r, n)$ exists if $n \geq 7$ and

(i) $\frac{n}{3} < r \leq \frac{n}{2}$, except possibly when $(r, n) = (6, 12)$

(ii) $\frac{7}{9}n \leq r \leq n$

(iii) $\frac{2n-1}{3} \leq r \leq n-1, r$ odd

(iv) $\frac{3}{7}n < r < \frac{3}{4}n,\ r \equiv 3 \pmod 6,\ n \equiv 1 \pmod 2$

(v) $\frac{3}{5}n < r \leq \frac{3}{4}n,\ r \equiv 3 \pmod 6,\ n \equiv 0 \pmod 2$

(vi) $\frac{n}{2} \leq r \leq \frac{3n+2}{4},\ r \equiv 0 \pmod 2,\ n \equiv 0 \pmod 2$.

On the other hand, there exist no $MOR(r, n)$ for $r \leq \frac{n}{4}$.

Several recursive constructions for row-maximal orthogonal latin rectangles are given in [45]. These, together with the above results, suffice to show, for instance, that $\{r : 11 \leq r \leq 30\} \subseteq MOR(30)$ where the set on the left conincides with the conjectured spectrum. But in general, quite a few undecided cases remain.

8 Maximal sets of mutually orthogonal Latin squares

This is an extremely important topic, because of its connections to the existence of finite projective planes.

The figures here are latin squares of order n on N; two latin squares are compatible if they are orthogonal. Recall that two latin squares $A = (a_{ij}), B = (b_{ij})$ are orthogonal if $|\{(a_{ij}, b_{ij}) : i, j = 1, \ldots, n\}| = n^2$, that is, when A and B are superimposed, each ordered pairs (a, b) with $a \in A$, $b \in B$ will appear exactly once.

Let $L(n)$ be the spectrum of sizes of maximal sets of mutually (pairwise) orthogonal latin squares (MOLS) of order n, i.e.

$$L(n) = \{r\colon \text{there exists a maximal set of } r \text{ MOLS of order} n\}.$$

The maximum number of MOLS of order n cannot exceed $n - 1$, and equals $n - 1$ whenever n is a prime power. Thus $L(n) \subseteq \{1, 2, \ldots, n-1\}$. To determine $L(n)$ in its entirety would involve, among other things, to settle the existence question for finite projective planes of order n. Worse yet, even *max* $L(n)$ remains undetermined for all values of n other than prime powers or $n = 6$. Nevertheless, any progress towards determining $L(n)$ is very desirable.

A latin square without an orthogonal mate is called a *bachelor square*. It has been now determined that bachelor squares exist for all $n > 3$ [29], [69]. Thus $1 \in L(n)$ for all $n > 3$. A latin square which has an orthogonal mate but is not contained in any set of three mutually orthogonal squares is called *monogamous* (cf. [17]). A monogamous latin square is known to exist for all orders $n > 6$ except possibly when $n = 2p$ for some prime $p \geq 7$. Thus $2 \in L(n)$ for all $n > 6$ except possibly when $n = 2p$ for some prime $p \geq 7$.

To determine the set $L(n)$ even for relatively small values of n is not an easy task. For example, whether or not $4 \in L(8)$ had been an open question for good forty years before it was recently settled [24].

The set $L(n)$ has now been determined for all $n \leq 9$. For $n \leq 7$ this has been done by Drake [22]; the last two outstanding values for $n = 8$ and $n = 9$ have been settled in [24]. In particular, we have

$$L(3) = \{2\}, L(4) = \{1,3\}, L(5) = \{1,4\}, L(6) = \{1\}, L(7) = \{1,2,6\},$$
$$L(8) = \{1,2,3,7\}, L(9) = \{1,2,3,4,5,8\}.$$

We have further $\{1,2\} \subseteq L(10)$, $\{1,2,3,4,10\} \subseteq L(11)$, $\{1,2,3,5\} \subseteq L(12)$, $\{1,2,3,4,6,12\} \subseteq L(13)$,$\{1,2,3,4,7,8,11,15\} \subseteq L(16)$ (cf. [1]), [3]).

As for general results, a theorem of Bruck [7] implies that for $n > 4$, $n-2 \notin L(n)$, $n-3 \notin L(n)$. If $n = q_1.q_2. \ldots . q_r$ is the prime power factorization of n then $min\ (q_i - 1) \in L(n)$.

If $p \geq 7$ is a prime, $p \equiv 3 \pmod 4$, then $\frac{p-3}{2} \in L(p)$. If $p \geq 13$ is a prime, $p \equiv 1 \pmod 4$, then $\frac{p-1}{2} \in L(p)$. If q is a prime power then $q^2 - q - 1 \in L(q^2)$, and if $q = p^r$, $p \geq 5$, then $q^2 - q - 2,\ q^2 - q \in L(q^2)$ (cf. [1]). For many additional results on maximal sets of MOLS, see [1], [3], [25], [27], [28], [38], [47], [48] and references therein.

9 Maximal partial Steiner triple systems

The figures are 3-subsets (triples) of a given v-set; two triples are compatible if they intersect in at most one element.

Alternatively, the figures are triangles in the complete graph K_v; two triangles are compatible if they are edge-disjoint.

Let $S^{(3)}(v)$ be the spectrum for maximal partial Steiner triple systems (STS), i.e.

$$S^{(3)}(v) = \{m\colon \text{there exists a maximal partial STS of order } v \text{ with exactly } m \text{ triples}\}.$$

The largest element $M^{(3)}(v)$ of $S^{(3)}(v)$ was determined already in the 1840's by Kirkman (and since then repeatedly by many others) :

$$\begin{aligned} M^{(3)}(v) &= v(v-1)/6 & v &\equiv 1,3 \pmod 6 \\ &= [v(v-1)-8]/6 & v &\equiv 5 \pmod 6 \\ &= v(v-2)/6 & v &\equiv 0,2 \pmod 6 \\ &= [v(v-2)-2]/6 & v &\equiv 4 \pmod 6). \end{aligned}$$

The smallest element $m^{(3)}(v)$ of $S^{(3)}(v)$ was determined in 1974 by Novák [57]: $m^{(3)}(v) = (v^2 + \delta_v)/12$ where

$$\begin{array}{rcrl} \delta_v & = & -2v+36 & v \equiv 0,8 \pmod{12} \\ & = & -1 & v \equiv 1,5 \pmod{12} \\ & = & -2v & v \equiv 2,6 \pmod{12} \\ & = & 3 & v \equiv 3 \pmod{12} \\ & = & -2v+4 & v \equiv 4 \pmod{12} \\ & = & 11 & v \equiv 7,11 \pmod{12} \\ & = & 15 & v \equiv 9 \pmod{12} \\ & = & -2v+16 & v \equiv 10 \pmod{12} \end{array}$$

The spectrum $S^{(3)}(v)$ for odd v was determined completely by Severn [65].

Let $R(v)$ be the interval $\{m^{(3)}(v), M^{(3)}v)\}$. It was shown in [65] that

$$\begin{array}{rcll} S^{(3)}(v) & = & R(v) \setminus \{M^{(3)} - 1\} & \text{if } v \equiv 1,3 \pmod 6 \\ & = & R(v) & \text{if } v \equiv 5 \pmod 6). \end{array}$$

For even v, the situation is slightly more complicated. The spectrum $S^{(3)}(v)$ in this case has been determined "almost completely" by [65] who left only a few open cases.

These have been settled in [14] so that the spectrum $S^{(3)}(v)$ has now been completely determined:

For v even, $S^{(3)}(v) = R(v) \setminus Q(v)$, where

$$Q(v) = \{r\colon m^{(3)}(v) < s < Y(v) \text{ and } s - \frac{v-1}{2} \equiv 1 \pmod 2),$$

and

$$\begin{aligned} Y(v) &= 12k^2 + 2k && \text{if } v = 12k \\ &= 12k^2 + 6k + 1 && \text{if } v = 12k + 2 \\ &= 12k^2 + 10k + 2 && \text{if } v = 12k + 4 \\ &= 12k^2 + 14k + 5 && \text{if } v = 12k + 6 \\ &= 12k^2 + 18k + 6 && \text{if } v = 12k + 8 \\ &= 12k^2 + 22k + 11 && \text{if } v = 12k + 10 \end{aligned}$$

(cf. [14], [13]).

10 Maximal partial 4-cycle systems

The figures here are quadrangles (cycles with 4 edges, 4-cycles); two quadrangles are compatible when they are edge-disjoint.

Let $S^{(4)}(n) = \{r\colon$ there exists a maximal set of quadrangles with exactly r quadrangles$\}$. Let the smallest and largest element of $S^{(4)}(n)$ be $m^{(4)}(n)$ and $M^{(4)}(n)$, respectively.

The numbers $M^{(4)}(n)$ have been determined completely in [64] (cf. also [32]). For odd n, when $n \equiv 1 \pmod 8$, there exists a 4-cycle system of order n, thus $M^{(4)}(n) = \frac{n(n-1}{8}$, the number of 4-cycles in such a system. For $n \equiv 3, 5, 7 \pmod 8$, there exists a maximum packing of 4-cycles whose leave is a triangle, a 2-regular graph with 6 edges (and thus either a 6-cycle, or two vertex-disjoint triangles, or a "bowtie"), or a pentagon, respectively [64]. Thus we have

$$\begin{aligned} M^{(4)}(n) &= \lfloor \tfrac{n(n-1)}{8} \rfloor && \text{if } n \equiv 1 \text{ or } 3 \pmod 8 \\ &= \lfloor \tfrac{n(n-1)}{8} \rfloor - 1 && \text{if } n \equiv 5 \text{ or } 7 \pmod 8. \end{aligned}$$

In order to determine the spectrum $S^{(4)}(n)$, it is necessary to know the values of $m^{(4)}(n)$ but therein lies the difficulty: to determine the maximum number of edges in an n-vertex graph without 4-cycles is a difficult unsolved problem [33], [34], [37], [10], [71].

Nevertheless, the values $ex(n; C_4)$, the largest number of edges in a graph with n vertices without a 4-cycle, has been determined exactly for all $n \leq 31$ [37], [71] which makes it possible to determine $min^{(4)}(n)$, and also the whole spectrum $S^{(4)}(n)$ for certain small values of n. No exact formula for $ex(n; C_4)$ appears to be known although it is known that $ex(n; C_4) < \frac{1}{4}n(1 + \sqrt{4n-3})$ when $n > 3$, and asymptotically $ex(n; C_4) \simeq \frac{1}{2}n^{\frac{3}{2}}$.

[The value of $ex(n; C_4)$ has also been determined exactly for $n = q^2 + q + 1$ when q is either a power of 2 [33] or when q is a prime power greater than 13 [34]].

While knowing the maximum number of edges in an n-vertex graph is, in turn, a necessary step in determining $m^{(4)}(n)$, what is actually needed is the maximum number of edges in an n-vertex eulerian and antieulerian graph without 4-cycles, according as n is odd and even, respectively. This numbers are usually somewhat smaller than the former; for example, the maximum number of edges in a 9-vertex graph without 4-cycles is 13 [10], that in a 9-vertex *eulerian* graph is 12. Similarly, for example, the maximum number of edges in a 10-vertex graph without 4-cycles is 16 [10], that in an *antieulerian* graph is 13.

Clearly, $\frac{n(n-1}{8} - 1 \notin S^{(4)}(n)$ when $n \equiv 1 \pmod 8$.

The bounds given in [10] plus ad hoc considerations enable one to determine the spectrum $S^{(4)}(n)$ for small values of n. In particular, we have
$S^{(4)}(4) = S^{(4)}(5) = \{1\}, S^{(4)}(6) = \{3\}, S^{4)}(7) = \{3, 4\}, S(4)(8) = \{5, 6\}$
$S^{(4)}(9) = \{6, 7, 9\}, S^{(4)}(10) = \{8, 9, 10\}, S^{(4)}(11) = \{10, 11, 12, 13\}$,
$S^{(4)}(12) = \{12, 13, 14, 15\}$, $S^{(4)}(13) = \{15, 16, 17, 18\} \cup I$ where $I = \emptyset$ or $I = \{14\}$. At this time, I am unable to decide whether $14 \in S^{(4)}(13)$ or not.

11 Maximal partial 5-cycle systems

The figures here are pentagons (cycles with 5 edges, 5-cycles); two pentagons are compatible if they are edge-disjoint.

Let

$$S^{(5)}(n) = \{r\colon \text{there exists a maximal set of pentagons with exactly } r \text{ pentagons}\}.$$

Let the smallest and largest element of $S^{(5)}(n)$ be $m^{(5)}(n)$ and $M^{(5)}(n)$, respectively.

The numbers $M^{(5)}(n)$ have been determined completely in [63]:

$$\begin{aligned} M^{(5)}(n) &= \lfloor \tfrac{e_n}{5} \rfloor && \text{if } n \not\equiv 7, 9 \pmod{10} \\ &= \lfloor \tfrac{e_n}{5} \rfloor - 1 && \text{if } n \equiv 7, 9 \pmod{10} \end{aligned}$$

where $e_n = \frac{n(n-1)}{2}$ or $\frac{n(n-2)}{2}$ according as n is odd or even.

To determine $m^{(5)}(n)$ turned out to be much more difficult. The first step in this was to obtain bounds on $m^{(5)}(n)$ by determining extremal graphs not containing a pentagon. While for $n \geq 7$ the maximal size of a graph with n vertices without a pentagon is $\lfloor \frac{n^2}{4} \rfloor$, for a nonbipartite graph the maximal size is $f(n) = \lfloor \frac{n^2}{4} \rfloor - n + 4$, a slight improvement [63]. Furthermore, a nonbipartite eulerian graph (all degrees even) without a pentagon with an odd number of vertices $n \geq 11$ has at most $\lfloor \frac{n^2}{4} \rfloor - n + 3$ edges. It follows that for a maximal size $E(n)$ of an eulerian graph without a pentagon we have

$$\begin{aligned} E(n) &= \tfrac{n^2}{4} && \text{if } n \equiv 0 \pmod 4 \\ &= \tfrac{(n-1)^2}{4} && \text{if } n \equiv 1 \pmod 4 \\ &= \tfrac{n^2-4}{4} && \text{if } n \equiv 2 \pmod 4 \\ &= \tfrac{n^2-2n-3}{4} && \text{if } n \equiv 3 \pmod 4. \end{aligned}$$

When $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, the extremal graph is $K_{\frac{n}{2},\frac{n}{2}}$ and $K_{\frac{n+1}{2},\frac{n-3}{2}}$, respectively. When $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, one of the extremal graphs is $K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $K_{\frac{n+1}{2},\frac{n-3}{2}}$, respectively.

Similarly, let $A(n)$ be the maximal size of an antieulerian (all degrees odd) graph without a pentagon. Then

$$\begin{aligned} A(n) &= \tfrac{n^2-4}{4} && \text{if } n \equiv 0 \pmod 4 \\ &= \tfrac{n^2}{4} && \text{if } n \equiv 2 \pmod 4, \end{aligned}$$

and the extremal graphs are $K_{\frac{n+2}{2},\frac{n-2}{2}}$ and $K_{\frac{n}{2},\frac{n}{2}}$, respectively.

Let

$$\begin{aligned} \Delta_n &= \lceil \tfrac{n(n-1)}{2} - \tfrac{E(n)}{5} \rceil && \text{if } n \text{ is odd,} \\ &= \lceil \tfrac{n(n-1)}{2} - \tfrac{A(n)}{5} \rceil && \text{if } n \text{ is even.} \end{aligned}$$

It is shown in [63] that

$$\begin{array}{rcll} m^{(5)}(n) & \geq & \Delta_n & \text{if } n \geq 11 \\ & \geq & \Delta_n + 1 & \text{if } n \equiv 13, 14, 15, 16, 17, 18 \pmod{20} \\ & \geq & \Delta_n + 2 & \text{if } n \equiv 4, 8 \pmod{20},\ n \geq 24. \end{array}$$

Equality in the above for all $n \geq 11$ is then established by three special constructions (see [63]).
Clearly, the spectrum $S^{(5)}(n)$ is a subset of the interval $[m^{(5)}(n), M^{(5)}(n)]$. This spectrum has not been determined completely yet, except when $n \equiv 3 \pmod{40}$. In [15], the following conjecture on the shape of the spectrum $S^{(5)}(n)$ was formulated.

Conjecture. For any $n \geq 6$, there is a number z_n (for $n \geq 45$, $z_n - m^{(5)}(n) \geq \frac{n}{5} - 5$) such that

(i) if $t \in [m^{(5)}(n), z_n]$ then $t \in S^{(5)}(n)$ if and only if t has the same parity as $m^{(5)}(n)$;

(ii) if $t \in [z_n, M^{(5)}(n)]$ then $t \in S^{(5)}(n)$.

It is shown in [15] that (i) holds for all $n \geq 45$. This has required determining the maximum number of edges in a pentagon-free nonbipartite eulerian (antieulerian) graph.

The conjecture has been proved in full only for $n \equiv 3 \pmod{40}$ (see [15]). If $n = 40k + 3$, $k \geq 2$, then $m^{(5)}(40k+3) = 80k^2 + 12k + 1$, $M^{(5)}(40k+3) = 160k^2 + 20k$, $z_{40k+3} = m^{(5)}(40k+3) + 8k - 1$, and
$S^{(5)}(40k+3) = \{80k^2+12k+1, 80k^2+12k+3, \ldots, 80k^2+20k-1, 80k^2+12k+1, 80k^2+20k+2, \ldots, 160k^2+20k\}$.

For $n \not\equiv 3 \pmod{40}$, part (ii) of the Conjecture remains open.

To determine the spectra in the following two sections appears quite difficult.

12 Maximal sets of disjoint Steiner triple systems

The figures are Steiner triple systems on a given v-set; they are compatible if they are disjoint, i.e. they have no triple in common. Here, of course, we must have $v \equiv 1$ or $3 \pmod 6$.

Let $DS(v) = \{m$: there exists a maximal set of m pairwise disjoint STS(v)s$\}$. It is well known that $DS(7) = \{2\}$, a result by Cayley that goes back to the middle of the 19th century. For $v > 7$, the largest element of $DS(v)$ was determined in [54], [68]: $max\ DS(v) = v - 2$.

The only other general results are:

(1) for $v \geq 7$, every Steiner triple system of order v has a disjoint mate, thus $1 \notin DS(v)$ [67],

(2) $v - 4 \in DS(v)$ for $v = 5.2^i - 1, i \geq 1$,

(3) $v - 5 \in DS(v)$ for $v = 2^{i+2} - 1, 5.2^i - 1, i \geq 1$ [12].

Cooper [9] determined $DS(9)$ (follows also from [12]): $DS(9) = \{4, 5, 7\}$. He also determined the isomorphism classes of all maximal sets of disjoint STS(9)s. At this point, for no other values of v has $DS(v)$ been determined.

13 Maximal sets of orthogonal Steiner triple systems

A property stronger than disjointness is the orthogonality property. Two Steiner triple systems $(V, \mathcal{B}_1), (V, \mathcal{B}_2)$ are *orthogonal* if they are disjoint, and, moreover, whenever $\{x, y, a\}, \{w, z, a\} \in \mathcal{B}_1$, $\{x, y, b\}, \{w, z, c\} \in \mathcal{B}_2$ then $b \neq c$. In other words, whenever two pairs of elements occur with the same third element in triples in one of the systems, they must occur with different third elements in the triples of the other system. Orthogonal STSs were originally introduced for the purpose of constructing Room squares.

For all $v \equiv 1, 3 \pmod 6$, $v \neq 9$, there exists a pair of orthogonal STS(v) [11]; there exists no such pair for $v = 9$. Moreover, it is shown in [18] that a set of three pairwise orthogonal STS(v) exists for all $v \equiv 1, 3 \pmod 6$, except when $v \leq 15$, and except possibly for 24 values of v, all of which are $\equiv 3 \pmod 6$, and smallest of which is $v = 21$. Many multiple sets of pairwise orthogonal STS(v) were constructed in [39] where it is shown, among other things, that for any positive integer t there exists a set of t pairwise orthogonal STS(v) provided $v \equiv 1 \pmod 6$ and v is sufficiently large. No maximality of these sets seems to have been investigated though, and while some of the sets constructed may indeed be maximal, it appears hard to either verify or disprove maximality.

Concerning *maximal* sets of orthogonal STS(v), let $OM(v) = \{r\colon$ there exists a maximal set of r orthogonal STS(v)$\}$. It is known that $OM(7) = \{2\}, OM(9) = \{1\}, OM(13) = \{1, 2\}, OM(15) = \{1, 2\}$ [36] but hardly anything else. It is believed that $max\ OM(v) \leq \frac{v-1}{2}$ but no nontrivial upper bound on $max\ OM(v)$ has been proved.

14 Row-maximal Room rectangles

Let N be a $2n$-set; a *Room* $(r, 2n)$*-rectangle* on N is an $r \times 2n - 1$ array $(r \leq 2n - 1)$ whose cells are either empty or contain a 2-subset of N. Each element of N occurs in exactly one cell of each row and in at most one cell of each column, and no 2-subset appears more than once in the array. A Room $(2n - 1, 2n)$-rectangle is called a *Room square* (of order $2n$, or of side $2n - 1$). A Room $(r, 2n)$ rectangle is row-maximal if no further row can be added to it to produce a Room $(r + 1, 2n)$-rectangle.

The figures are pairs (f, α) where f is a 1-factor of K_{2n} on a given $2n$-set N, and α is an injection from f into $\{1, 2, \ldots, 2n - 1\}$. Two figures $(f_1, \alpha_1), (f_2, \alpha_2)$ are compatible if $\alpha_1^{-1}(i) \cap \alpha_2^{-1}(i) = \emptyset$ whenever $\alpha_1^{-1}(i) \neq \emptyset$ and $\alpha_2^{-1}(i) \neq \emptyset$. Less formally, the figures are rows with $2n - 1$ cells of which $n - 1$ are empty such that the n nonempty cells contain a partition of N into 2-subsets; two such rows are compatible if no element occurs in any of the $2n - 1$ columns more than once.

Here we have the following result.

Let

$$MRR(2n) = \{r\colon \text{there exist a row-maximal Room } (r, 2n)\text{-rectangle}\}.$$

A row-maximal Room $(r, 2n)$-rectangle exists if

(i) $(r, 2n) = (1, 4)$

(ii) $n \leq r \leq 2n - 1$ except when $(r, 2n) \in \{(2, 4), (3, 4), (5, 6)\}$.

Indeed, (i) is trivial while (ii) follows from the fact that there exists no Room square of order 4 or 6 (i.e. there exists no Room $(3, 4)$-rectangle or Room $(5, 6)$-rectangle, and there exists no Howell design $H(2, 4)$ [19]. It remains to be observed that while a Howell design $H(5, 8)$ does not exist, either, a row-maximal Room $(5, 8)$-rectangle does:

12	34	56	78	–	–	–
–	–	–	13	24	57	68
67	–	14	–	58	23	–
–	15	–	26	–	48	37
38	–	27	–	16	–	45

(For a general reference on Room squares and Howell designs, see, e.g., [19]).

15 Packings of dominoes

A different kind of a problem on packing dominoes onto a square $n \times n$ board was considered in [40]. It is trivial to see that the maximum number $M^{(d)}(n)$ of dominoes (1×2 tiles) that may be packed onto an $n \times n$ board (without overlap) is $\frac{n^2}{2}$ if n is even, and $\frac{n^2}{2} - 1$ when n is odd. The authors of [40] were interested in the *minimum* number $m^{(d)}(n)$ of dominoes that can be placed on an $n \times n$ board in such a way that no further domino can be placed on it without an overlap. It is shown in [40] that $m^{(d)}(n) = \frac{n^2}{3}$ if $n \equiv 0 \pmod 3$, and $m^{(d)}(n) > \frac{n^2}{3} + \frac{n}{111}$ otherwise, provided n is large.

While $m^{(d)}(n) = \lfloor \frac{n^2+2}{3} \rfloor$ for $2 \leq n \leq 12$, the exact value of $m^{(d)}(n)$ for $n \not\equiv 0 \pmod 3$ is not known. The best upper bound known is $m^{(d)}(n) < \frac{n^2}{3} + \frac{n}{12} + 1$. In any case, in any maximal packing of dominoes roughly at least two thirds of the cells must be covered.

Let $S^{(d)}(n) = \{r$: there exists a maximal packing of the $n \times n$ board with exactly m dominoes$\}$. The constructions given in [40] allow one to deduce that

$$S^{(d)}(n) = \left\{ \frac{n^2}{3}, \frac{n^2}{3} + 1, \ldots, \lfloor \frac{n^2}{2} \rfloor \right\} \text{ when } n \equiv 0 \pmod 3, \text{ and}$$

$$\left\{ \frac{n^2}{3} + \frac{n}{12} + 1, \frac{n^2}{3} + \frac{n}{12} + 2, \ldots, \lfloor \frac{n^2}{2} \rfloor \right\} \subseteq S^{(d)}(n) \text{ for } n \not\equiv 0 \pmod 3.$$

The case of maximal packing of "dominoes" on triangular and hexa(gonal) boards is also considered in [40]. For example, a "domino" for a hexaboard is a pair of two neighbouring hexagonal cells. Hexa board itself has a triangular shape and consists of n rows containing a total of $\binom{n}{2}$ hexagonal cells.

While clearly one can cover the entire n-hexaboard (a board with n rows) by $\lfloor \frac{n(n+1)}{4} \rfloor$ dominoes (except for one hexagonal cell when $\binom{n}{2}$ is odd), in this case one is also able to determine the minimum number of dominoes in a maximal packing; this minimum equals $\lfloor \frac{n(n+1)}{6} \rfloor$. Thus this case turns out to be easier than that of the regular $n \times n$ board (cf. [40]).

Conclusion and some open problems

This survey cannot, and does not attempt to, encompass all situations where the spectrum problem for maximal designs and configurations arises - this would anyway be virtually impossible. There are many further examples of problems of the kind similar to those explored above. To name just a few further examples of problems that have been studied to various degrees of depth in the literature, maximal sets of orthogonal hamiltonian cycles [46], maximal sets of orthogonal hamiltonian decompositions [46], maximal sets of disjoint 1-factorizations [2], [9] maximal sets of orthogonal 1-factorizations (or, equivalently, dimension-maximal Room cubes) [2], maximal k-cliques [21], [23], [26], [35], maximal partial projective planes [20], and several others come to mind.

A concept closely related to maximal configurations is that of *premature* configurations (see, e.g., [4], [62]). While maximal configurations are not extendible, premature configurations are not completable (to maximum configurations) but may themselves not be maximal. Although in the context of some of the problems discussed above, premature configurations have been explored in the literature, for example, premature sets of 1-factors (cf. Section 1), premature sets of latin parallelepipeds (cf. Section 6), or premature sets of MOLS (cf. Section 8), we refrained in this article from discussing premature configurations in more detail.

The spectrum problems treated in Sections 1, 2, 3, 9 and 14 have been solved completely. In the remaining sections, many open problems remain. Some open problems that I would like to see solved, or at least seriously attacked, are:

(i) Maximal partial Steiner systems $S(2,4,v)$.
Here the figures are 4-subsets of a given v-set; two 4-subsets are compatible if they intersect in at most one element. Let $S^4(v)$ be the spectrum for maximal partial Steiner systems $S(2,4,v)$, and let $m^4(v)$ and $M^4(v)$ be the smallest and largest element of $S^4(v)$, respectively. The numbers $M^4(v)$ have been determined by Brouwer [6] (cf. also [66]. The numbers $m^4(v)$ and the spectrum $S^4(v)$ have not been determined yet.

(ii) Maximal partial Room squares.
Here I am not aware of any results in this direction.

It is hoped that by bringing together the most up-to-date results on these and potentially many other similar or related problems a renewed interest will be generated.

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Initial Coefficient Bounds for a Comprehensive Subclass of Sakaguchi Type Functions

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Abstract
In this paper, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions defined in the open unit disk. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

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1 Introduction and Definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [8] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \ , \ \ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \ , \ \ \left(|w| < r_0(f) \ , \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

*corresponding author.

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \qquad (z \in U)$$

if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0) = 0 \text{ and } \quad |w(z)| < 1 \qquad (z \in U)$$

such that

$$f(z) = g(w(z)) \qquad (z \in U).$$

Lewin [15] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Netanyahu [17] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and obtained estimates on the initial coefficients. Bounds for the initial coefficients of several classes of functions were also investigated in ([1], [3], [7], [9], [13], [14], [16], [19], [21], [22], [23]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [6], [10], [11], [12]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, ...\}$) is still an open problem.

Motivated by the earlier work of Sakaguchi [20] on the class of starlike functions with respect to symmetric points denoted by S_S consisting of functions $f \in A$ satisfy the condition $\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0$, $(z \in U)$, we introduce a new subclass of the function class Σ of bi-univalent functions, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

Definition 1. Let $h : U \longrightarrow \mathbb{C}$, be a convex univalent function such that $h(0) = 1$ and $h(\bar{z}) = \overline{h(z)}$, for $z \in U$ and $\mathrm{Re}(h(z)) > 0$. A function $f \in \Sigma$ is said to be in the class $S_\Sigma^\lambda(\beta, s, t, h)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad e^{i\beta} \frac{[(s-t)z]^{1-\lambda} f'(z)}{[f(sz) - f(tz)]^{1-\lambda}} \prec h(z) \cos\beta + i \sin\beta, \quad z \in U \tag{1.2}$$

and

$$e^{i\beta} \frac{[(s-t)w]^{1-\lambda} g'(w)}{[g(sw) - g(tw)]^{1-\lambda}} \prec h(w) \cos\beta + i \sin\beta, \quad w \in U \tag{1.3}$$

where $g(w) = f^{-1}(w)$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\lambda \geq 0$.

Remark 2. If we set $h(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in the class $S_\Sigma^\lambda(\beta, s, t, h)$, we have $S_\Sigma^\lambda\left(\beta, s, t, \frac{1+Az}{1+Bz}\right)$ and defined as

$$f \in \Sigma, \quad e^{i\beta} \frac{[(s-t)z]^{1-\lambda} f'(z)}{[f(sz) - f(tz)]^{1-\lambda}} \prec \frac{1+Az}{1+Bz} \cos\beta + i \sin\beta, \quad z \in U$$

and

$$e^{i\beta}\frac{[(s-t)w]^{1-\lambda}\,g'(w)}{[g(sw)-g(tw)]^{1-\lambda}}\prec\frac{1+Aw}{1+Bw}\cos\beta+i\sin\beta,\quad w\in U$$

where $g(w)=f^{-1}(w)\,,\ s,t\in\mathbb{C}$ with $s\neq t,\ |t|\leq 1,\ \beta\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\lambda\geq 0$.

Remark 3. If we set $h(z)=\dfrac{1+(1-2\alpha)z}{1-z},\quad 0\leq\alpha<1$, in the class $S_{\Sigma}^{\lambda}(\beta,s,t,h)$, we have $S_{\Sigma}^{\lambda}(\beta,s,t,\alpha)$ and defined as

$$f\in\Sigma,\quad \mathrm{Re}\left\{e^{i\beta}\frac{[(s-t)z]^{1-\lambda}\,f'(z)}{[f(sz)-f(tz)]^{1-\lambda}}\right\}>\alpha\cos\beta,\quad z\in U$$

and

$$\mathrm{Re}\left\{e^{i\beta}\frac{[(s-t)w]^{1-\lambda}\,g'(w)}{[g(sw)-g(tw)]^{1-\lambda}}\right\}>\alpha\cos\beta,\quad w\in U$$

where $g(w)=f^{-1}(w)\,,\ s,t\in\mathbb{C}$ with $s\neq t,\ |t|\leq 1,\ \beta\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\lambda\geq 0$.

Lemma 4. *(see [18]) Let the function $\phi(z)$ given by*

$$\phi(z)=\sum_{n=1}^{\infty}B_nz^n$$

be convex in U. Suppose also that the function $h(z)$ given by

$$h(z)=\sum_{n=1}^{\infty}h_nz^n$$

is holomorphic in U. If $h(z)\prec\phi(z),\ z\in U$, then $|h_n|\leq|B_1|\,,\ n\in\mathbb{N}=\{1,2,3,...\}$.

2 Coefficient Estimates

Theorem 5. *Let f given by (1.1) be in the class $\mathrm{S}_{\Sigma}^{\lambda}(\beta,s,t,h)$. Then*

$$|a_2|\leq\sqrt{\frac{2|B_1|\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}},\tag{2.1}$$

and

$$|a_3|\leq\frac{2|B_1|\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}.\tag{2.2}$$

Proof. Let $f\in S_{\Sigma}^{\lambda}(\beta,s,t,h)$, g be the analytic extension of f^{-1} to U and $s,t\in\mathbb{C}$ with $s\neq t,\ |t|\leq 1$ and $\lambda\geq 0$. It follows from (1.2) and (1.3) that there exists $p,q\in P$ such that

$$e^{i\beta}\frac{[(s-t)z]^{1-\lambda}\,f'(z)}{[f(sz)-f(tz)]^{1-\lambda}}=p(z)\cos\beta+i\sin\beta,\qquad(z\in U)\tag{2.3}$$

and

$$e^{i\beta}\frac{[(s-t)w]^{1-\lambda}\,g'(w)}{[g(sw)-g(tw)]^{1-\lambda}}=q(w)\cos\beta+i\sin\beta,\qquad(w\in U)\tag{2.4}$$

where $p(z)\prec h(z)$ and $q(w)\prec h(w)$ have the forms

$$p(z)=1+p_1z+p_2z^2+\cdots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots,$$

respectively. It follows from (2.3) and (2.4), we deduce

$$e^{i\beta}\left[(\lambda-1)(s+t)+2\right]a_2 = p_1\cos\beta, \tag{2.5}$$

$$e^{i\beta}\left\{\left[(\lambda-1)\left(s^2+t^2+st\right)+3\right]a_3 - \tfrac{\lambda(\lambda-1)}{2}(s+t)^2 a_2^2 + (\lambda-1)(s+t)\left[2+(\lambda-1)(s+t)\right]a_2^2\right\}$$
$$= p_2\cos\beta, \tag{2.6}$$

and

$$-e^{i\beta}\left[(\lambda-1)(s+t)+2\right]a_2 = q_1\cos\beta, \tag{2.7}$$

$$e^{i\beta}\left\{2\left[(\lambda-1)\left(s^2+t^2+st\right)+3\right] - \tfrac{\lambda(\lambda-1)}{2}(s+t)^2 + (\lambda-1)(s+t)\left[2+(\lambda-1)(s+t)\right]\right\}a_2^2$$
$$-e^{i\beta}\left[(\lambda-1)\left(s^2+t^2+st\right)+3\right]a_3 = q_2\cos\beta. \tag{2.8}$$

From (2.5) and (2.7) we obtain

$$p_1 = -q_1,$$

By adding (2.6) to (2.8), we get

$$e^{i\beta}\left\{2(\lambda-1)(s+t)\left[2+(\lambda-1)(s+t)\right] + 2\left[(\lambda-1)(s^2+st+t^2)+3\right] - \lambda(\lambda-1)(s+t)^2\right\}a_2^2$$
$$= (p_2+q_2)\cos\beta. \tag{2.9}$$

Since $p, q \in h(U)$, applying Lemma 4, we have

$$|p_m| = \left|\frac{p^{(m)}(0)}{m!}\right| \leq |B_1|, \quad m \in \mathbb{N} \tag{2.10}$$

and

$$|q_m| = \left|\frac{q^{(m)}(0)}{m!}\right| \leq |B_1|, \quad m \in \mathbb{N}. \tag{2.11}$$

Applying (2.10), (2.11) and Lemma 4 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_2| \leq \sqrt{\frac{2|B_1|\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}}.$$

Subtracting (2.8) from (2.6) we have

$$e^{i\beta}\left\{2\left[(\lambda-1)(s^2+t^2+st)+3\right]a_3 - 2\left[(\lambda-1)(s^2+t^2+st)+3\right]a_2^2\right\} = (p_2-q_2)\cos\beta. \tag{2.12}$$

or, equivalently

$$a_3 = \frac{e^{-i\beta}(p_2+q_2)\cos\beta}{2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2} + \frac{e^{-i\beta}(p_2-q_2)\cos\beta}{2[(\lambda-1)(s^2+t^2+st)+3]}.$$

Applying (2.10), (2.11) and Lemma 4 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{2|B_1|\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}.$$

This completes the proof of Theorem 5. $\square$

3 Corollaries and Consequences

Corollary 6. *Let f given by (1.1) be in the class* $\mathrm{S}_\Sigma^\lambda\left(\beta, s, t, \frac{1+Az}{1+Bz}\right)$. *Then*

$$|a_2| \le \sqrt{\frac{2(A-B)\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}}$$

and

$$|a_3| \le \frac{2(A-B)\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}.$$

where $-1 \le B < A \le 1$, $s, t \in \mathbb{C}$ *with* $s \ne t$, $|t| \le 1$ *and* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\lambda \ge 0$.

Corollary 7. *Let f given by (1.1) be in the class* $\mathrm{S}_\Sigma^\lambda(\beta, s, t, \alpha)$. *Then*

$$|a_2| \le \sqrt{\frac{4(1-\alpha)\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}},$$

and

$$|a_3| \le \frac{4(1-\alpha)\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}.$$

where $0 \le \alpha < 1$, $s, t \in \mathbb{C}$ *with* $s \ne t$, $|t| \le 1$ *and* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

If we get $\lambda = 0$ in Theorem 5,

Corollary 8. *Let f given by (1.1) be in the class* $\mathrm{S}_\Sigma^0(\beta, s, t, h)$. *Then*

$$|a_2| \le \sqrt{\frac{|B_1|\cos\beta}{|3-2s-2t+st|}},$$

and

$$|a_3| \le \frac{|B_1|\cos\beta}{|3-2s-2t+st|}.$$

If we get $\lambda = 0$ in Corollary 6,

Corollary 9. *Let f given by (1.1) be in the class* $\mathrm{S}_\Sigma^0\left(\beta, s, t, \frac{1+Az}{1+Bz}\right)$. *Then*

$$|a_2| \le \sqrt{\frac{(A-B)\cos\beta}{|3-2s-2t+st|}}$$

and

$$|a_3| \le \frac{(A-B)\cos\beta}{|3-2s-2t+st|}$$

where $-1 \le B < A \le 1$, $s, t \in \mathbb{C}$ *with* $s \ne t$, $|t| \le 1$ *and* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

If we get $\lambda = 0$ in Corollary 7,

Corollary 10. *Let f given by (1.1) be in the class* $\mathrm{S}_\Sigma^0(\beta, s, t, \alpha)$. *Then*

$$|a_2| \le \sqrt{\frac{2(1-\alpha)\cos\beta}{|3-2s-2t+st|}},$$

and

$$|a_3| \le \frac{2(1-\alpha)\cos\beta}{|3-2s-2t+st|}.$$

where $0 \le \alpha < 1$, $s, t \in \mathbb{C}$ *with* $s \ne t$, $|t| \le 1$ *and* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

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Numerical solution of a class of nonlinear Volterra integral equations using Bernoulli operational matrix of integration

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Abstract

In this paper, a simple efficient method for the numerical solution of a class of nonlinear Volterra integral equations (VIEs) is presented. The approach starts by expanding the existing functions in terms of Bernoulli polynomials. Subsequently, using the new introduced Bernoulli operational matrices of integration and the product along with the so-called collocation method, the considered problem is reduced into a set of nonlinear algebraic equations with unknown Bernoulli coefficients. The error analysis and rate of convergence are also given. Finally, some tests of other authors are included and a comparison has been done between the results.

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1 Introduction

As is noted in [32], Volterra integral equations arise in many physical problems, e.g., heat conduction problem [5], concrete problem of mechanics or physics [44], on the unsteady Poiseuille flow in a pipe [16], diffusion problems [4], electroelastic [14], contact problems [23], etc. Due to this fact that analytical solutions of integral equations either do not exist or are hard to find, many different methods have been proposed to approximate solutions of these equations [1, 7, 8, 13, 21, 25, 27].

Recently, in [2] Aziz and Islam used Haar wavelets and in [34] Maleknejad and Rahimi used ϵ modified block pulse functions (ϵMBPFs) to solve these kinds of equations. A method based on Bernstein polynomials is also presented by Maleknejad, Basirat and Hashemizadeh in [31].

In the present paper, we consider the nonlinear Volterra integral equations of the form

$$u(x) = f(x) + \int_0^x k(x,t)N\big(u(t)\big)dt, \quad x \in \Omega := [0,1], \tag{1.1}$$

*corresponding author

where $u(x)$ is an unknown real valued function and $f(x)$ and $k(x,t)$ are given continuous functions defined, respectively on Ω and $\Omega \times \Omega$, and $N\big(u(x)\big)$ is a polynomial of $u(x)$ with constant coefficients. It follows from the classical theory of Volterra equations (see, for example, [8], [9]) that (1.1) has a unique continuous solution $u^*(x)$ on Ω. Moreover, if functions f and k are r times continuously differentiable on Ω and $S := \{(x,t) : 0 \le t \le x \le 1\}$, respectively, then u^* is r times continuously differentiable on Ω.

The method of this paper consists of reducing (1.1) into a set of nonlinear algebraic equations. The underlying idea employed is the following integral property

$$\int_0^x \Psi(t)dt \simeq P\Psi(x), \tag{1.2}$$

where $\Psi(x) = [\psi_0(x), \psi_1(x), \ldots, \psi_{n-1}(x)]^T$ is the basis vector and P is a square constant matrix called the *operational matrix of integration*. Up to now, the operational matrix of integration P has been derived for several types of basis functions such as Walsh [12], block-pulse [41], Legendre wavelets [40], Haar wavelets [18], Laguerre [20], Chebyshev [19, 37], Legendre [11], Bernstein [45], Bessel [38], Fourier [39] and Jacobi [28]. We are interested here in the use of the Bernoulli polynomials. Some interesting properties of the Bernoulli polynomials are:

- Comparing the structure of the Bernoulli operational matrix of integration P given in (2.17) with the corresponding matrices of other basis functions, we may observe that the setting up of P is simpler.
- The Bernoulli operational matrix of integration P appears to be computationally very attractive because, compared with other types of basis functions, it has more zero elements. Indeed, the nonzero entries of the Bernoulli operational matrix of integration are located only on the superdiagonal and its first column, while the corresponding matrices of the Bessel and the Bernstein polynomials are full and it is an upper triangular matrix for the block-pulse functions and a tridiagonal matrix for the Legendre wavelet basis. The nonzero elements of the shifted Chebyshev and shifted Jacobi operational matrices of integration are located on the subdiagonals, diagonals, superdiagonals and their first columns which are more than the case of Bernoulli polynomials. Also, the shifted Legendre, Laguerre and Hermite operational matrices of integration have the same number of nonzero elements with the Bernoulli operational matrix of integration. A same argument can be made for the operational matrix of derivatives.
- The Bernoulli polynomials have less terms than the shifted Chebyshev, shifted Legendre and shifted Jacobi polynomials which makes them attractive from the computational point of view. For example $B_6(x)$ (the 6th Bernoulli polynomial), has five terms while $T_6(x)$ (the 6th shifted Chebyshev polynomial) and $L_6(t)$ (the 6th shifted legendre polynomial), have seven terms, and this difference will increase by increasing the degree. Hence for approximating an arbitrary function we use less CPU time by applying Bernoulli polynomials as compared to any classical orthogonal polynomials; this issue is claimed in [35] for shifted Legendre polynomials.
- The coefficient of individual terms in Bernoulli polynomials $B_k(t)$, are smaller than the coefficient of individual terms in the shifted Legendre and shifted Chebyshev polynomials $L_k(t)$ and $T_k(t)$, respectively (it can be easily checked by the *Mathematica* software). Since the computational errors in the product are related to the coefficients of individual terms, the computational errors are less by using Bernoulli polynomials [35].

For convenience, we assume that

$$N(u(x)) = u^m(x), \tag{1.3}$$

where m is a positive integer, but the method can be easily extended and applied to any nonlinear VIE of the form (1.1), where $N\big(u(x)\big)$ is a polynomial of $u(x)$ with constant coefficients.

The reminder of the paper is organized as follows. We give a brief review of Bernoulli polynomials and their properties in Sections 2.1 and 2.2. New Bernoulli operational matrices of integration and the product are derived in Section 2.3. In Section 3, how the new introduced Bernoulli operational matrices can be used to reduce the problem (1.1)-(1.3) into a set of nonlinear algebraic equations is explained. The error analysis and rate of convergence are also given in this section. In Section 4, we show that the Bernoulli polynomial coefficients vector of $u^m(x)$ can be computed in terms of the Bernoulli polynomial coefficients vector of $u(x)$. Some numerical examples are presented in Section 5, which show the efficiency and accuracy of the proposed method. Conclusions of the work are given in Section 6.

2 Some properties of Bernoulli polynomials

To facilitate the presentation of the material that follows, we present in this section some background on the Bernoulli polynomials.

2.1 Definition

The generalized Bernoulli polynomials $B_k^{(a)}(x)$ of degree k can be defined by the generating formula [29, Section 2.8]

$$\frac{t^a e^{xt}}{(e^t-1)^a} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k^{(a)}(x), \quad |t| \le 2\pi.$$

If $a=1$, we have the Bernoulli polynomials $B_k^{(1)}(x) \equiv B_k(x)$, and if, further, $x=0$, we have the Bernoulli numbers $B_k(0) = B_k$.
The Bernoulli polynomials satisfy the familiar expansion [15, Section 1.13]

$$\sum_{r=0}^{k-1} \binom{k}{r} B_r(x) = kx^{k-1}, \quad k = 1, 2, \ldots. \tag{2.1}$$

The first five Bernoulli polynomials are as follows

$$\begin{aligned}
B_0(x) &= 1,\\
B_1(x) &= x - \tfrac{1}{2},\\
B_2(x) &= x^2 - x + \tfrac{1}{6},\\
B_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,\\
B_4(x) &= x^4 - 2x^3 + x^2 - \tfrac{1}{30}.
\end{aligned}$$

Also, the Bernoulli polynomials satisfy the following relations ([15], Section 1.13)

$$\begin{aligned} &B'_k(x) = kB_{k-1}(x), \quad k \geq 1, \\ &\textstyle\int_0^1 B_k(x)dx = 0, \quad k \geq 1, \\ &B_k(x+1) - B_k(x) = kx^{k-1}, \quad k \geq 1, \\ &B_k(x) = \sum_{r=0}^{k} \binom{k}{r} B_r x^{k-r}, \quad k \geq 1. \end{aligned} \tag{2.2}$$

With the aid of equation (2.1), the Bernoulli polynomial vector

$$B(x) = [B_0(x), B_1(x), \ldots, B_N(x)]^T, \tag{2.3}$$

can be written of the form

$$B(x) = D^{-1}T_N(x), \tag{2.4}$$

where

$$T_N(x) = [1, x, x^2, \ldots, x^N]^T, \tag{2.5}$$

and D is a lower triangular matrix defined by

$$D = [d_{ij}]_{i,j=0}^N, \quad d_{ij} = \begin{cases} \frac{1}{i+1}\binom{i+1}{j}, & 0 \leq j \leq i, \\ 0, & i < j \leq N. \end{cases}$$

On the other hand, if in the third part of equation (2.2), k varies from 0 to N we have

$$B(x) = \widehat{D}T_N(x), \tag{2.6}$$

where $\widehat{D}$ is a lower triangular matrix as

$$\widehat{D} = [\hat{d}_{ij}]_{i,j=0}^N, \quad \hat{d}_{ij} = \begin{cases} \binom{i}{i-j} B_{i-j}, & 0 \leq j \leq i, \\ 0, & i < j \leq N, \end{cases} \tag{2.7}$$

and $T_N(x)$ is the vector defined by equation (2.5). So, from equations (2.4) and (2.6) we obtain $\widehat{D} = D^{-1}$. The dual matrix of $B(x)$ is defined by

$$\begin{aligned} Q = \int_0^1 B(x)B^T(x)dx &= \int_0^1 \left(\widehat{D}T_N(x)\right)\left(\widehat{D}T_N(x)\right)^T dx \\ &= \widehat{D}\left(\int_0^1 T_N(x)T_N^T(x)dx\right)\widehat{D}^T = \widehat{D}H\widehat{D}^T, \end{aligned} \tag{2.8}$$

where $\widehat{D}$ is the matrix defined in (2.7) and H is the Hilbert matrix

$$H = \int_0^1 T_N(x)T_N^T(x)dx = \left[\frac{1}{i+j+1}\right]_{i,j=0}^N.$$

2.2 Function approximation and error analysis

Let $\mathcal{H} = \mathfrak{L}^2([0,1])$ be the space of square integrable functions with respect to Lebesgue measure on the closed interval $[0,1]$. The inner product in this space is defined by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx, \tag{2.9}$$

and the norm is as follows

$$\|f\|_2 = \langle f, f\rangle^{\frac{1}{2}} = \Big(\int_0^1 f^2(x)dx\Big)^{\frac{1}{2}}. \tag{2.10}$$

Let

$$\mathcal{H}_N = \operatorname{span}\{B_0(x), B_1(x), \ldots, B_N(x)\}. \tag{2.11}$$

Since $\mathcal{H}_N$ is a finite dimensional subspace of $\mathcal{H}$, then it is closed [24, Theorem 2.4-3] and for every given $g \in \mathcal{H}$ there exists a unique best approximation $\bar{g} \in \mathcal{H}_N$ [24, Theorem 6.2-5] such that

$$\|g - \bar{g}\|_2 \leq \|g - f\|_2, \quad \forall f \in \mathcal{H}_N. \tag{2.12}$$

Since $\bar{g} \in \mathcal{H}_N$, there exist unique coefficients g_0, g_1, …, g_N such that

$$g(x) \simeq \bar{g}(x) = \sum_{k=0}^{N} g_k B_k(x) = B^T(x)G, \tag{2.13}$$

where $B(x)$ is the Bernoulli polynomial vector defined in equation (2.3) and G is the Bernoulli polynomial coefficients vector of $g(x)$ defined as

$$G = [g_0, g_1, \ldots, g_N]^T. \tag{2.14}$$

Also, for a positive integer m, $g^m(x)$ may be approximated as

$$g^m(x) \simeq B^T(x)G^{(m)},$$

where $G^{(m)}$ is a column vector whose elements are nonlinear functions of the elements of G. The form of these functions will be explained later in Section 4.

Let us denote by $\mathcal{C}^m(\Omega)$ the space of functions $f : \Omega \to \mathbb{R}$ with continuous derivatives

$$f^{(i)}(x) = \frac{d^i}{dx^i} f(x), \quad x \in \Omega,$$

for all i such that $0 \leq i \leq m$ and by $\mathcal{C}^{m,n}(\Omega \times \Omega)$ the space of functions $f : \Omega \times \Omega \to \mathbb{R}$ with continuous partial derivatives

$$f^{(i,j)}(x,t) = \frac{\partial^{i+j}}{\partial x^m \partial t^n} f(x,t), \quad (x,t) \in \Omega \times \Omega,$$

for all (i, j) such that $0 \leq i \leq m, 0 \leq j \leq n$. The following results are satisfied.

Corollary 1. *[42] Suppose that $g(x) \in \mathcal{C}^N(\Omega)$ is approximated by the truncated Bernoulli series $P_N[g](x) = \sum_{k=0}^{N} g_k B_k(x)$. Then the coefficients g_n can be calculated from the following relation*

$$g_n = \frac{1}{n!}\int_0^1 g^{(n)}(x)dx, \quad n = 0, 1, \ldots, N.$$

It follows from the next corollary that Bernoulli coefficients will decay rapidly with increasing n.

Corollary 2. *[42] Assume that the function $g(x) \in \mathcal{C}^N(\Omega)$ is approximated by Bernoulli polynomials as argued in Corollary 1. Then the coefficients g_n decay as follows*

$$g_n \leq \frac{\bar{G}_n}{n!}, \quad n = 0, 1, \dots, N,$$

where $\bar{G}_n$ denotes the maximum of $g^{(n)}(x)$ in the interval Ω.

The following theorem provides an error term for the approximation presented in Corollary 1.

Theorem 3. *[42] Suppose that $g(x) \in \mathcal{C}^N(\Omega)$ and $P_N[g](x)$ is its approximation in terms of Bernoulli polynomials and $R_N[g](x)$ is the remainder term. Then, the associated formulas are stated as follows*

$$g(x) = P_N[g](x) + R_N[g](x), \quad x \in \Omega,$$

$$P_N[g](x) = \int_0^1 g(x)dx + \sum_{j=1}^{N} \frac{B_j(x)}{j!}(g^{(j-1)}(1) - g^{(j-1)}(0)),$$

$$R_N[g](x) = -\frac{1}{N!}\int_0^1 B_N^*(x-t)g^{(N)}(t)dt,$$

where $B_N^*(x) = B_N(x - [x])$ and $[x]$ denotes the largest integer not greater than x.

Theorem 4. *[43] Suppose $g(x) \in \mathcal{C}^N(\Omega)$ and $P_N[g](x)$ is its approximation in terms of Bernoulli polynomials. Then the error bound would be obtained as follows*

$$E(g) = \|g(x) - P_N[g](x)\|_\infty \leq C\hat{G}(2\pi)^{-N}, \quad x \in \Omega,$$

where C is a positive constant independent of N and $\hat{G}$ is such that

$$\|g^{(i)}(x)\|_\infty \leq \hat{G}, \quad i = 0, 1, \dots, N.$$

The above results can be extended to the case of functions of two (or more) variables. Let $k(x,t) \in H \times H$, then it can be approximated in terms of truncated Bernoulli series as

$$k(x,t) \simeq \sum_{i=0}^{N}\sum_{j=0}^{N} k_{ij}B_i(x)B_j(t) = B^T(x)KB(t), \tag{2.15}$$

where $K = [k_{ij}]_{i,j=0}^N$ is an $(N+1) \times (N+1)$ matrix.

Corollary 5. *[6] Assume that the function $k(x,t) \in \mathcal{C}^{N,N}(\Omega \times \Omega)$ is approximated by the two variable truncated Bernoulli series $P_N[k](x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} k_{ij}B_i(x)B_j(t)$, then the coefficients k_{ij} can be calculated from the following relation*

$$k_{ij} = \frac{1}{i!j!}\int_0^1\int_0^1 \frac{\partial^{i+j}}{\partial x^i \partial t^j}k(x,t)dxdt, \quad i,j = 0, 1, \dots, N.$$

Corollary 6. *Assume that the function $k(x,t) \in \mathcal{C}^{N,N}(\Omega \times \Omega)$ is approximated by Bernoulli polynomials as argued in Corollary 5. Then the coefficients k_{ij} decay as follows*

$$k_{mn} \leq \frac{\bar{K}_{i,j}}{i!j!}, \quad i,j = 0,1,\ldots,N,$$

where $\bar{K}_{i,j}$ denotes the maximum of $\frac{\partial^{i+j}}{\partial x^i \partial t^j} k(x,t)$ in the unit square $\Omega \times \Omega$.

Proof. Since it is trivial we omit the proof. □

Theorem 7. *[43] Suppose $k(x,t) \in \mathcal{C}^{N,N}(\Omega \times \Omega)$ and $P_N[k](x,t)$ be its approximation in terms of Bernoulli polynomials. Then the error bound would be obtained as follows*

$$E(k) = \|k(x,t) - P_N[k](x,t)\|_\infty \leq C\hat{K}N(2\pi)^{-N},$$

where C is a positive constant independent of N and $\hat{K}$ is such that

$$\left\| \frac{\partial^{i+j}}{\partial x^i \partial t^j} k(x,t) \right\|_\infty \leq \hat{K}, \quad i,j = 0,1,\ldots,N.$$

2.3 Operational matrices of integration

In this section, the Bernoulli operational matrices of integration and the product will be derived.

Theorem 8. *Let $B(x)$ be the Bernoulli vector defined in* (2.3)*. Then*

$$\int_0^x B(t)dt \simeq PB(x), \tag{2.16}$$

where P is the $(N+1) \times (N+1)$ operational matrix of integration defined by

$$P = \begin{bmatrix} -B_1 & 1 & 0 & \ldots & 0 \\ \frac{-B_2}{2} & 0 & \frac{1}{2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-B_N}{N} & 0 & 0 & \ldots & \frac{1}{N} \\ \frac{-B_{N+1}}{N+1} & 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{2.17}$$

Proof. It follows from the first part of (2.2) that

$$\int_0^x B_k(t)dt = \frac{1}{k+1}(B_{k+1}(x) - B_{k+1}), \quad k \geq 0.$$

So, the integration of the vector $B(x)$ is given by

$$\int_0^x B(t)dt = P^* B^*(x), \tag{2.18}$$

where P^* is an $(N+1)\times(N+2)$ matrix having the form

$$P^* = \left[\begin{array}{c|c} P & p \end{array}\right] = \left[\begin{array}{ccccc|c} -B_1 & 1 & 0 & \dots & 0 & 0 \\ \frac{-B_2}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-B_N}{N} & 0 & 0 & \dots & \frac{1}{N} & 0 \\ \frac{-B_{N+1}}{N+1} & 0 & 0 & \dots & 0 & \frac{1}{N+1} \end{array}\right],$$

and $B^*(x)$ is an $(N+2)\times 1$ vector of the form

$$B^*(x) = \left[\begin{array}{c} B(x) \\ \hline B_{N+1}(x) \end{array}\right].$$

If we trunctate the term $B_{N+1}(x)$ in the vector $B^*(x)$, i.e., if we drop the vector p in the matrix P^*, relation (2.18) becomes the approximate relation (2.16). □

Note that, the structure of P is simple, since all its elements are zero, except for its first column and its superdiagonal, and hence the Bernoulli basis may be computationally more attractive than other sets of basis functions.

Comparing the structure of the Bernoulli integral operational matrix P (denoted for the moment as P_B) with the corresponding matrices of Walsh P_W, block-pulse P_b, and Laguerre P_L, we may observe that P_B has the following characteristics:

- Using P_B, instead of P_B^* is a rather insignificant approximation, particularly if one considers the fact that $\alpha_N = \frac{1}{N+1}$ diminishes with N. The same approach is applied in the Laguerre case [20], but the approximation there is more significant since the corresponding term α_N in the P matrix is independent of N and is always equal to -1. For the Walsh case, the approximation of the form (2.16) is definitely significant since, for any given N, many non-zero terms in determining P are truncated. Finally, the case of block-pulse functions appears not to involve this type of approximation. This fact may be of great importance, since it could considerably reduce the overall approximation error.
- The accuracy in relation (1.2) depends on two factors, namely, the dimension $(N+1)$ of the basis vector $\Psi(x)$ and the particular $\Psi(x)$ used. From the remarks of the previous paragraph it appears that relation (1.2) could be more accurate if Bernoulli functions were used rather than Walsh or Laguerre functions.

It is to be noted that, using equations (2.13) and (2.16), the integral of any function $g(x)$ can be approximated as

$$\int_0^x g(t)dt \simeq \int_0^x G^T B(t)dt \simeq G^T P B(x).$$

We also need to evaluate the product of $B(x)$ and $B^T(x)$, which is called the product matrix of Bernoulli polynomials. For this purpose, we first approximate the functions $x^k B_i(x)$, for $i,k = 0,1,\dots,N$, in terms of $B(x)$. By using (2.13), we can write

$$x^k B_i(x) \simeq B^T(x) e_{k,i}, \tag{2.19}$$

where $e_{k,i}$ is the Bernoulli polynomial coefficients vector defined as

$$e_{k,i} = [e_0^{k,i}, e_1^{k,i}, \ldots, e_N^{k,i}]^T. \tag{2.20}$$

Using Eqs. (2.8) and (2.19), we obtain

$$e_{k,i} \simeq Q^{-1} \int_0^1 x^k B(x) B_i(x) dx = Q^{-1} \begin{bmatrix} \int_0^1 x^k B_i(x) B_0(x) dx \\ \int_0^1 x^k B_i(x) B_1(x) dx \\ \vdots \\ \int_0^1 x^k B_i(x) B_N(x) dx \end{bmatrix}.$$

Now, for any arbitrary vector $C = [c_0, c_1, \ldots, c_N]^T$ in $\mathbb{R}^{N+1}$ we define the notations

$$\tilde{E}_k = E_k C, \quad k = 0, 1, \ldots, N,$$
$$\tilde{C} = [\tilde{E}_0, \tilde{E}_1, \ldots, \tilde{E}_N],$$

where $\hat{D}$ is the matrix defined by (2.7) and E_k is an $(N+1) \times (N+1)$ matrix with $e_{k,i}$, $i = 0, 1, \ldots, N$, as its columns.

Theorem 9. *Let $C = [c_0, c_1, \ldots, c_N]^T$ be an arbitrary vector in $\mathbb{R}^{N+1}$. Then*

$$B(x) B^T(x) C \simeq \hat{C} B(x), \tag{2.21}$$

where $\hat{C}$ is the $(N+1) \times (N+1)$ product operational matrix defined by

$$\hat{C} = \hat{D} \tilde{C}^T.$$

Proof. Using (2.6) we obtain

$$\begin{aligned} B(x) B^T(x) C &= \left(\hat{D} T_N(x) \right) B^T(x) C \\ &= \hat{D} \left[B^T(x) C, x B^T(x) C, \ldots, x^N B^T(x) C \right]^T \\ &= \hat{D} \left[\sum_{i=0}^N c_i B_i(x), \sum_{i=0}^N c_i x B_i(x), \ldots, \sum_{i=0}^N c_i x^N B_i(x) \right]^T, \end{aligned} \tag{2.22}$$

and using (2.19) and (2.20) yield

$$\begin{aligned}
\sum_{i=0}^{N} c_i x^k B_i(x) &\simeq \sum_{i=0}^{N} c_i \left(B^T(x) e_{k,i}\right) = \sum_{i=0}^{N} c_i \left(\sum_{j=0}^{N} e_j^{k,i} B_j(x) \right) \\
&= B^T(x) \begin{bmatrix} \sum_{i=0}^{N} c_i e_0^{k,i} \\ \sum_{i=0}^{N} c_i e_1^{k,i} \\ \vdots \\ \sum_{i=0}^{N} c_i e_N^{k,i} \end{bmatrix} \\
&= B^T(x)[e_{k,0}, e_{k,1}, \ldots, e_{k,N}]C = B^T(x)\tilde{E}_K.
\end{aligned} \tag{2.23}$$

Combining equations (2.22) and (2.23) gives the result. □

3 Numerical solution of nonlinear VIEs

In this section, we use the operational matrices of the Bernoulli polynomials and a collocation method to numerically solve problem (1.1) with assumption (1.3). So, we consider the following integral equation

$$u(x) = f(x) + \int_0^x k(x,t) u^m(t) dt, \quad x \in \Omega. \tag{3.1}$$

If we approximate functions $f(x)$, $u(x)$, $u^m(x)$ and $k(x,t)$ using Bernoulli polynomials, as described by equations (2.13) and (2.15), then we obtain

$$f(x) \simeq B^T(x) F, \tag{3.2}$$

$$u(x) \simeq B^T(x) U, \tag{3.3}$$

$$u^m(x) \simeq B^T(x) U^{(m)}, \tag{3.4}$$

$$k(x,t) \simeq B^T(x) K B(t), \tag{3.5}$$

where the vectors $F, U, U^{(m)}$ and matrix K are Bernoulli polynomial coefficients of $f(x)$, $u(x)$, $u^m(x)$ and $k(x,t)$ respectively. We again note that $U^{(m)}$ is a column vector whose elements are nonlinear functions of the elements of the unknown vector U. With substituting approximations (3.2)-(3.5) into (3.1), we get

$$\begin{aligned}
B^T(x) U &\simeq B^T(x) F + \int_0^x B^T(x) K B(t) B^T(t) U^{(m)} dt \\
&= B^T(x) F + B^T(x) K \int_0^x B(t) B^T(t) U^{(m)} dt.
\end{aligned}$$

Using (2.21) leads to

$$\begin{aligned} B^T(x)U &\simeq B^T(x)F + B^T(x)K \int_0^x \left(\widehat{U^{(m)}}\right)^T B(t)dt \\ &= B^T(x)F + B^T(x)K \left(\widehat{U^{(m)}}\right)^T \int_0^x B(t)dt. \end{aligned}$$

Now, using (2.16) gives

$$B^T(x)U \simeq B^T(x)F + B^T(x)K \left(\widehat{U^{(m)}}\right)^T PB(x), \tag{3.6}$$

where P is the Bernoulli operational matrix of integration given in (2.17). Collocating equation (3.6) at the $(N+1)$ Newton-Cotes nodes as

$$x_l = \frac{2l+1}{2(N+1)}, \quad l = 0, 1, \ldots, N, \tag{3.7}$$

will result in

$$B^T(x_l)U \simeq B^T(x_l)F + B^T(x_l) \left(K \left(\widehat{U^{(m)}}\right)^T P\right) B(x_l), \quad l = 0, 1, \ldots, N. \tag{3.8}$$

Since $U^{(m)}$ is a column vector whose elements are nonlinear functions of the element of the unknown vector $U = [u_i]_{i=0}^N$, then equation (3.8) is a nonlinear system of $(N+1)$ algebraic equations with $(N+1)$ unknowns u_0, u_1, $\ldots$, u_N. This nonlinear system of algebraic equations can be solved by numerical methods such as Newton's iterative method. If $\bar{U}$ be an approximate solution of this system, then $\bar{U}_m(x) = B^T(x)\bar{U}$ is an approximate solution of equation (3.1).

In the following theorem we shall find an upper bound for the error between the exact solution $u(x)$ and the approximate solution $u_N(x)$ of equation (3.1) with the considered assumptions.

Theorem 10. *Let $u(x)$ be the exact solution and $u_N(x) = B^T(x)\bar{U}$ be the approximated solution of* (3.1) *where the unknown Bernoulli coefficient vector $\bar{U}$ is determined by solving the nonlinear algebraic system of equations* (3.8)*. Moreover assume that*

(1) $|u(x)| \leq \rho, \quad \forall x \in \Omega,$

(2) $|k(x,t)| \leq \tilde{k}, \quad \forall (x,t) \in \Omega \times \Omega,$

(3) $M\big(\tilde{k} + E(k)\big) < 1$ *in which* $M > 0$ *satisfies*

$$|u^m(t) - u_N^m(t)| \leq M|u(t) - u_N(t)|, \quad \forall t \in \Omega. \tag{3.9}$$

Then we have

$$\|u - u_N\|_\infty \leq \frac{E(f) + \rho^m E(k)}{1 - M\big(\tilde{k} + E(k)\big)}.$$

Proof. If we approximate both the driving term $f(x)$ and kernel $k(x,t)$ in terms of Bernoulli polynomials as described by equations (2.13) and (2.15), then the obtained solution is an approximated polynomial; $u_N(x)$ and we have

$$\begin{aligned} |u(x)-u_N(x)| &= \left|f(x)-f_N(x)+\int_0^x \Big(k(x,t)u^m(t)-k_N(x,t)u_N^m(t)\Big)dt\right| \\ &\le \big|f(x)-f_N(x)\big|+\int_0^x \Big|k(x,t)u^m(t)-k_N(x,t)u_N^m(t)\Big|dt. \end{aligned} \tag{3.10}$$

Moreover, using assumptions (1)-(3) we get

$$\begin{aligned} \Big|k(x,t)u^m(t)-k_N(x,t)u_N^m(t)\Big| &= \Big|k(x,t)\big(u^m(t)-u_N^m(t)\big)+\big(k(x,t)-k_N(x,t)\big)u_N^m(t)\Big| \\ &\le |k(x,t)|\,\big|u^m(t)-u_N^m(t)\big|+\big|k(x,t)-k_N(x,t)\big|\big|u_N^m(t)\big| \\ &\le \tilde{k}M\|u-u_N\|_\infty+E(k)\Big(\big|u^m(t)-u_N^m(t)\big|+\big|u_N^m(t)\big|\Big) \\ &\le M\big(\tilde{k}+E(k)\big)\|u-u_N\|_\infty+\rho^m E(k). \end{aligned} \tag{3.11}$$

Substituting (3.11) in (3.10), and noting that $x\in[0,1]$, we obtain

$$\|u-u_N\|_\infty \le E(f)+\big(\tilde{k}+E(k)\big)\|u-u_N\|_\infty+\rho^m E(k).$$

Then, by assumption (3) we get

$$\|u-u_N\|_\infty \le \frac{E(f)+\rho^m E(k)}{1-M\big(\tilde{k}+E(k)\big)},$$

which completes the proof. □

For a given function $f(x)$ if $f'(x)$ is continuous in $[-1,1]$ except for a finite number of bounded jumps, then $f(x)$ can be expanded in a convergent series as [29, pp. 309]

$$f(x)=\frac{1}{2}c_0+\sum_{j=1}^{\infty}c_jT_j(x), \tag{3.12}$$

where

$$c_j=\frac{2}{\pi}\int_{-1}^{1}\frac{f(x)T_j(x)}{(1-x^2)^{\frac{1}{2}}}dx,$$

and $T_n(x)$ denotes the Chebyshev polynomial of the first kind of degree n.

Theorem 11. *[17, Theorem 3.12] When a function f has $r+1$ continuous derivatives on $[-1,1]$, where r is a finite number, then $|f(x)-S_n(x)|=\mathcal{O}(n^{-r})$ as $n\to\infty$ for all $x\in[-1,1]$, in which $S_n(x)=\frac{1}{2}c_0+\sum_{j=1}^{n}c_jT_j(x)$ denotes the partial sum of expansion (3.12).*

We define the residual function $r_N(x)$ on Ω as

$$r_N(x)=u_N(x)-f(x)-\int_0^x k(x,t)u_N^m(t)dt, \tag{3.13}$$

where $u(x)$ is the exact solution of (3.1) and $u_N(x)$ is the approximation of $u(x)$ in terms of Bernoulli polynomials as described by equations (2.13). The next theorem gives an estimation of the residual error.

Theorem 12. *Let r is a finite number and the exact solution $u(x)$ of (3.1) has $r+1$ continuous derivatives on Ω. If $M = \|k(x,t)\|_\infty < \infty$, then $\|r_N\|_\infty = \mathcal{O}(N^{-r})$ as $N \to \infty$.*

Proof. It follows from equations (3.13) and (3.1) that

$$\|r_N\|_\infty \leq (1+M)\|u - u_N\|_\infty. \tag{3.14}$$

Suppose that $u_N(x) = \sum_{n=0}^{N} a_{N,n} B_n(x)$. Since the Bernoulli polynomials can be expressed in terms of Chebyshev polynomials of the first kind [22, Theorem 2.1], then $u_N(x)$ can be expanded as

$$u_N(x) = \sum_{k=0}^{N} b_{N,k} T_k(x),$$

where $b_{N,k}$ can be expressed in terms of $a_{N,n}$, $n, k = 0, \ldots, N$. Therefore, by Theorem 11 we have $\|u - u_N\|_\infty = \mathcal{O}(N^{-r})$ as $N \to \infty$ which along with (3.14) completes the proof. $\square$

4 Expressing $U^{(m)}$ in terms of U

For the numerical implementation of the presented method, we need to express the components of the vector $U^{(m)}$ as nonlinear functions of the elements of the vector U, where $U^{(m)}$ and U are the Bernoulli polynomial coefficients vectors of $u(x)$ and $u^m(x)$ respectively. To do this, we state the following lemma.

Lemma 13. *let m be a positive integer and U and $U^{(m)}$ are respectively the Bernoulli polynomial coefficients vectors of $u(x)$ and $u^m(x)$, that are defined on Ω. Also, let Q be the matrix defined in (2.8). Then, we have*

$$U^{(m)} = Q^{-1}(\hat{U}^T)^m e_1, \tag{4.1}$$

where e_1 denotes the first standard unit vector of order $(N+1)$.

Proof. We have

$$QU^{(m)} = \left(\int_0^1 B(x)B^T(x)dx\right)U^{(m)} = \int_0^1 B(x)B^T(x)U^{(m)}dx.$$

Using relations (3.4), (3.3), (2.2) and (2.21), we can write

$$\begin{aligned}
\int_0^1 B(x)B^T(x)U^{(m)}dx &\simeq \int_0^1 B(x)u^m(x)dx \\
&\simeq \int_0^1 B(x)\Big(B^T(x)U\Big)^m dx \\
&= \int_0^1 \Big(B(x)B^T(x)U\Big)\Big(B^T(x)U\Big)^{m-1} dx \\
&\simeq \hat{U}^T \int_0^1 B(x)\Big(B^T(x)U\Big)^{m-1} dx \\
&= \hat{U}^T \int_0^1 \Big(B(x)B^T(x)U\Big)\Big(B^T(x)U\Big)^{m-2} dx \\
&\simeq \Big(\hat{U}^T\Big)^2 \int_0^1 B(x)\Big(B^T(x)U\Big)^{m-2} dx \\
&\vdots \\
&\simeq \Big(\hat{U}^T\Big)^m \int_0^1 B(x)dx \\
&= \Big(\hat{U}^T\Big)^m e_1.
\end{aligned}$$

Since Q is invertible, we obtain (4.1). □

5 Illustrative examples

To demonstrate the applicability and accuracy of our method, we have applied it to several examples. These examples are solved in different references, so the numerical results obtained here can be compared with those of other numerical methods.

In order to analyze the error of the method we introduce notations

$$e_N(x) = u(x) - u_N(x),$$

and

$$\|e_N\|_\infty = \max\Big\{\big|e_N(x_l)\big|, \quad l = 0, 1, \ldots, N\Big\},$$

where $u_N(x)$ denotes the approximate solution of order N of integral equation, which is obtained by the method presented in Section 3, and $u(x)$ is the exact solution of integral equation. Also, x_l, $l = 0, 1, \ldots, N$, denote the Newton-Cotes nodes defined by (3.7).

Moreover, we define the global error as [26]

$$\epsilon_N = \frac{1}{|u|_{\max}} \sqrt{\frac{1}{N}\sum_{l=0}^{N}\big[e_N(x_l)\big]^2},$$

where $|u|_{\max}$ denotes the maximum absolute value of the exact solution u on Ω.

Experiments were performed on a personal computer using a 2.50 GHz processor and the codes were written in *Mathematica 9*.

Example 14. Consider the nonlinear Volterra integral equation

$$u(x) = 2 - e^x + \int_0^x e^{x-t} u^2(t)dt, \quad x \in [0,1]. \tag{5.1}$$

The exact solution of this equation is $u(x) = 1$. Numerical results obtained by the present method for this example has been shown in the first column of Table 1. Also, Fig. 1 shows the error graph of e_N, for $N = 8$.

Table 1. Computed errors $\|e_N\|_\infty$ for Examples 14-16.

N	Example 14	Example 15	Example 16
1	9.883×10^{-2}	1.138×10^{-2}	3.349×10^{-2}
2	4.703×10^{-2}	5.551×10^{-3}	1.849×10^{-2}
3	9.026×10^{-3}	1.271×10^{-3}	7.851×10^{-3}
4	7.014×10^{-4}	3.847×10^{-4}	2.364×10^{-3}
5	1.938×10^{-4}	1.393×10^{-4}	1.277×10^{-3}
6	1.637×10^{-5}	4.172×10^{-5}	2.925×10^{-4}
7	5.109×10^{-6}	1.397×10^{-5}	1.742×10^{-4}
8	4.010×10^{-7}	4.301×10^{-6}	3.982×10^{-5}

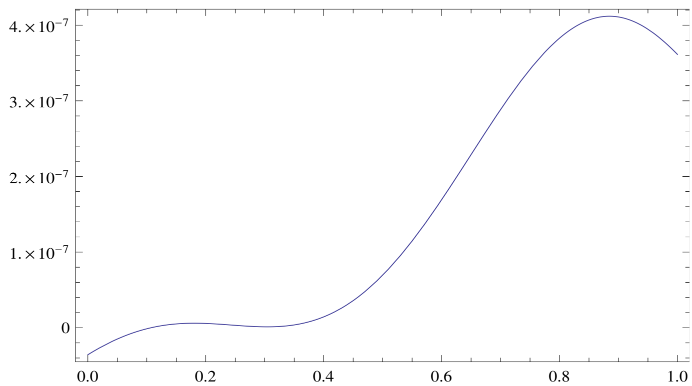

Figure 1. Graph of $e_N(x)$ for Example 14 with $N = 8$.

Example 15. [2, 30] Consider the following nonlinear Volterra integral equation

$$u(x) = \frac{3}{2} - \frac{1}{2}e^{-2x} - \int_0^x \Big(u^2(t) + u(t)\Big)dt, \quad x \in [0,1]. \tag{5.2}$$

The exact solution of this problem is $u(x) = e^{-x}$. The second column of Table 1 illustrates the numerical results obtained by the present method for this example. Also, Fig. 2 shows the error graph of e_N, for $N = 10$.

Integral equation (5.2) is solved in [2] and [30], respectively by Haar wavelets method and triangular functions (TF) method. Comparison of the second column of Table 1 with Fig. 3 of [2] shows better accuracy of our method using fewer number of basis functions and collocation points.

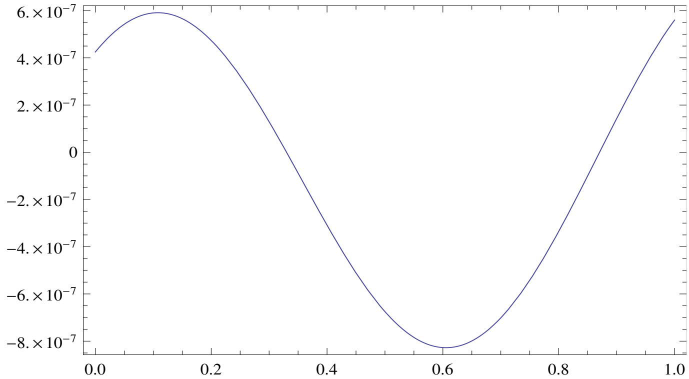

Figure 2. Graph of $e_N(x)$ for Example 15 with $N = 10$.

Example 16. [26] Let us consider the following linear Volterra integral equation

$$u(x) = f(x) + \int_0^x x\cos(t)u(t)dt, \quad x \in [0,1], \tag{5.3}$$

where

$$f(x) = \frac{1}{4}x\cos(2x) + \sin(x) - \frac{1}{2}x.$$

The exact solution of this problem is $u(x) = \sin(x)$. The third column of Table 1 illustrates the numerical results obtained by the present method for this example. Also, Fig. 3 shows the error graph of e_N, for $N = 15$.

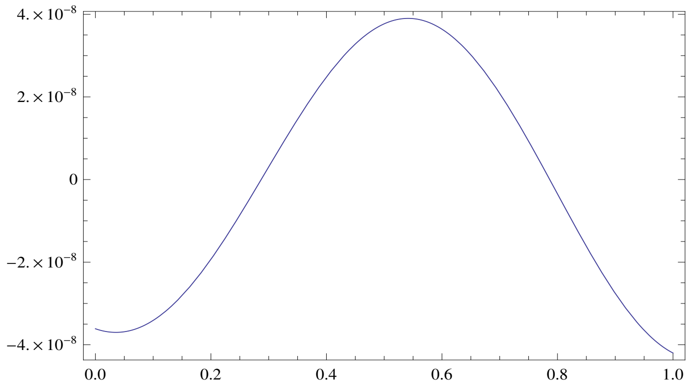

Figure 3. Graph of $e_N(x)$ for Example 16 with $N = 15$.

The random integral quadrature (RIQ) method is used in [26] to approximate the solution of integral equation (5.3) where $0 \le x \le \pi$. In the case of 5 field nodes distributed uniformly and randomly, the global errors obtained by RIQ method are $1.6462E-3$ and $2.4302E-3$ respectively. Also, in the case of 5 collocation points used, the global error obtained by the presented method for Example 16 is $2.0504E-3$ which shows similar accuracies for our method and RIQ method.

Example 17. [3, 33] Consider the nonlinear Volterra integral equation

$$u(x) = x + \cos(x) - 1 + \int_0^x \sin\big(u(t)\big)dt, \quad x \in [0,1], \tag{5.4}$$

with the exact solution $u(x) = x$. Integral equation (5.4) is not in the desired form (1.1), but it can be converted by approximating $\sin(u)$ using a finite number of terms of its Taylor series as

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots + (-1)^d \frac{u^{2d+1}}{(2d+1)!}, \quad d \in \mathbb{Z}^{\geq 0}.$$

Table 2 illustrates the numerical results obtained by the present method for $N = 5$ and different values of d. Also, Fig. 4 shows the error graph of e_N, for $N = 6$ and $d = 3$.

Table 2. Computed errors $\|e_N\|_\infty$ for Example 17.

N	$d = 0$	$d = 1$	$d = 2$	$d = 3$
5	3.482×10^{-2}	8.735×10^{-4}	9.372×10^{-6}	2.960×10^{-6}

Integral equation (5.4) is solved in [33] using cubic B-spline wavelets basis. A comparison between the absolute errors obtained by the present method and the method of [33] is done in Table 3. This table shows that our method needs fewer number of basis functions (and therefore fewer number of collocation points) to achieve the desired accuracy.

Table 3. Comparison of absolute errors for Example 17.

x	Present method with 7 basis functions (N=6) and d=3	Algorithm 1 of [33] with 11 basis functions ($m = 4, s_\mu = 3$)
0	7.13684×10^{-7}	4.14485×10^{-6}
0.1	6.29968×10^{-7}	1.61021×10^{-7}
0.2	3.08742×10^{-7}	4.15844×10^{-7}
0.3	1.22505×10^{-7}	7.48669×10^{-7}
0.4	4.93745×10^{-7}	5.50796×10^{-7}
0.5	6.64531×10^{-7}	4.08869×10^{-7}
0.6	5.71379×10^{-7}	4.95687×10^{-7}
0.7	2.50408×10^{-7}	1.71069×10^{-7}
0.8	1.64765×10^{-7}	1.40256×10^{-6}
0.9	4.69861×10^{-7}	1.62347×10^{-6}
0.9	4.41524×10^{-7}	7.31106×10^{-6}

Example 18. [36] Consider the following second kind linear Volterra integral equation

$$3u(x) - \int_0^x (x+t)^2 u(t)dt = f(x), \quad x \in [0,1]. \tag{5.5}$$

The function $f(x)$ was chosen so that the analytical solution of (5.5) is $u(x) = e^x$. Fig. 5 shows the error graph of e_N, for $N = 12$.

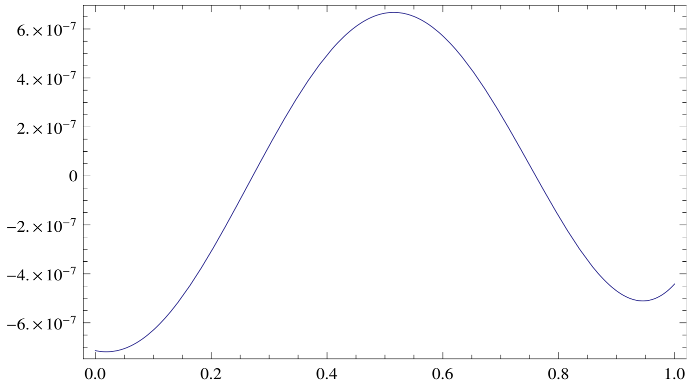

Figure 4. Graph of $e_N(x)$ for Example 17 with $N = 6$ and $d = 3$.

Table 4. Comparison of errors $\|e_N\|_\infty$ for Example 18.

N	Present method	N	Moving least squares (MLS) method [36]	
			Linear ($q = 1$)	Quadratic ($q = 2$)
2	7.09×10^{-2}	5	9.13×10^{-3}	2.39×10^{-4}
4	7.69×10^{-3}	9	2.70×10^{-3}	2.37×10^{-5}
6	4.21×10^{-4}	17	7.84×10^{-4}	7.59×10^{-6}
8	1.93×10^{-5}	33	2.09×10^{-4}	6.14×10^{-6}
10	7.74×10^{-7}	65	5.11×10^{-5}	5.31×10^{-4}
12	2.99×10^{-8}	129	1.37×10^{-5}	2.73×10^{-3}

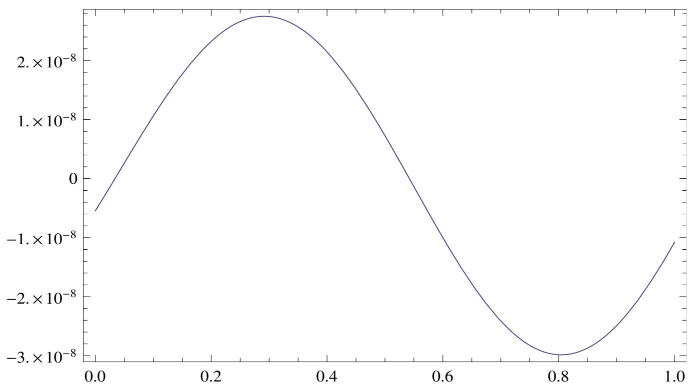

Figure 5. Graph of $e_N(x)$ for Example 18 with $N = 12$.

Integral equation (5.5) was previously considered in [36] by the moving least squares (MLS) method. A comparison between our results and the results of [36] has done in Table 4. The values of N in the first and the third columns of this table show the number of collocation points used for our method and the number of nodal points used for the MLS method, respectively. Based upon the results of Table 4, compared to the

MLS method, our method gives more accurate solutions by solving a very smaller linear system of equations.

Example 19. For our final example we consider the following Volterra integral equation of the second kind

$$u(x) = f(x) + \int_0^x (x-t)u(t)dt, \quad x \in [0,1].$$

The function $f(x)$ was chosen so that the analytical solution of (5.1) is

$$u(x) = \gamma x e^{1-\gamma x},$$

with γ denoting a given (real) parameter. Table 5 illustrates the numerical results obtained by the present method for $\gamma = 1, -1, -2, -3$ and different values of N. In the case of $\gamma = -1$, the numerical results obtained by the present method can be compared with those of Brunner and Yan [10] who used the collocation and iterated collocation methods for the numerical solution of this problem. We see that when γ decreases the total variation of the exact solution $u(x)$ (which is denoted by u_{tv}) increases and the method converges slowly. Also, Fig. 6 shows the error graph of e_N, for $N = 15$ and $\gamma = -1$.

Table 5. Computed errors $\|e_N\|_\infty$ for Example 19.

N	$\gamma = 1(u_{tv} = 1)$	$\gamma = -1(u_{tv} = e^2)$	$\gamma = -2(u_{tv} = 2e^3)$	$\gamma = -3(u_{tv} = 3e^4)$
5	7.204×10^{-4}	2.597×10^{-3}	3.450×10^{-2}	$1.574 \times 10^{+0}$
10	3.126×10^{-7}	1.086×10^{-6}	7.229×10^{-4}	6.114×10^{-2}
15	1.952×10^{-8}	4.549×10^{-9}	5.239×10^{-6}	2.101×10^{-3}

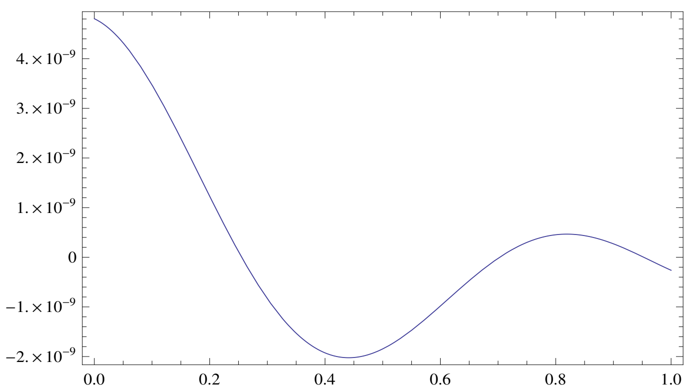

Figure 6. Graph of $e_N(x)$ for Example 19 with $N = 15$ and $\gamma = -1$.

6 Conclusion and comments

In this article we proposed an efficient and simple numerical method for solving a class of nonlinear Volterra integral equations of the form (1.1) and (1.3). For this purpose

the existing functions expanded in terms of Bernoulli polynomials. Then, using the new derived Bernoulli operational matrices and the collocation method, the problem reduced to a nonlinear system of algebraic equations. The obtained results show that this method is competitive with the other ones.
The proposed method has some notable advantages such as:

- The required computational effort to implement the method is small while the accuracy is high (the computations can be carried out on a personal computer).
- As the numerical results show, in the case of smooth solutions, a small number of basis functions ($N \leq 15$) is enough to obtain a high accuracy approximation of the solution $\big($error norm less than $10^{-8}\big)$.

Nevertheless, the method has some limitations and drawbacks, including:

- As we see in Example 19, when the exact solution $u(x)$ of the problem has large total variation, the method converges slowly.
- Since the coefficients of the Bernoulli polynomials grow quite fast in absolute value when N increases, then for large values of N the accuracy of the method is affected badly due to round off errors. This drawback will be also encountered when other classical orthogonal basis such as the shifted Legendre and shifted Chebyshev polynomials is used (since the coefficient of individual terms are greater than the ones of Bernoulli polynomials).

At the end, as it done in Example 17, if the part $N\big(u(x)\big)$ in equation (1.1) is not a polynomial of $u(x)$ but is continuous then the Weierstrass approximation theorem can be used to convert the problem to the desired form.

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On the [p, q]-Order of Meromorphic Solutions of Linear Differential Equations

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Abstract
In this article we study the growth of meromorphic solutions of high order linear differential equations with meromorphic coefficients of $[p, q]$-order. We extend some previous results due to Cao-Xu-Chen, Kinnunen, Liu-Tu-Shi, Li-Cao and others.

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1 Introduction and main results

Consider for $k \geq 2$ the linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.2}$$

where $A_0(z), \cdots, A_{k-1}(z), F(z)$ are meromorphic functions. In [11, 12] Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p, q]$-order and obtained some results about their growth. In [16], in order to maintain accordance with general definitions of the entire function f of iterated p-order [13, 14], Liu-Tu-Shi gave a minor modification of the original definition of the $[p, q]$-order given in [11, 12] . With this new concept of $[p, q]$-order, Liu, Tu and Shi [16] have considered equations (1.1), (1.2) with entire coefficients and obtained different results concerning the growth of their solutions. In this paper, we continue to consider this subject and investigate the complex linear differential equations (1.1) and (1.2) when the coefficients $A_0, A_1, \cdots, A_{k-1}, F$ are meromorphic functions of $[p, q]$ $-$order.

In this paper, it is assumed that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [9, 14, 20]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1. ([13]) Let $p \geq 1$ be an integer. The iterated $p-$order of a meromorphic function $f(z)$ is defined by

$$\rho_p(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f.

Now, we shall introduce the definition of meromorphic functions of $[p,q]$-order, where p, q are positive integers satisfying $p \geq q \geq 1$ or $2 \leq q = p+1$. In order to keep accordance with Definition 1, we will give a minor modification to the original definition of $[p,q]$-order (e.g. see, [11, 12]).

Definition 2. ([15]) Let $p \geq q \geq 1$ or $2 \leq q = p+1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the $[p,q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$. If $f(z)$ is a rational, then $\rho_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By Definition 2, we have that $\rho_{[1,1]}(f) = \rho_1(f) = \rho(f)$, $\rho_{[2,1]}(f) = \rho_2(f)$ and $\rho_{[p+1,1]}(f) = \rho_{p+1}(f)$.

Definition 3. ([15]) A transcendental meromorphic function $f(z)$ is said to have index-pair $[p,q]$ if $0 < \rho_{[p,q]}(f) < \infty$ and $\rho_{[p-1,q-1]}(f)$ is not a nonzero finite number.

Definition 4. ([15]) Let $p \geq q \geq 1$ or $2 \leq q = p+1$ be integers. The $[p,q]$ convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \limsup_{r \to +\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r},$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z : |z| \leq r\}$. Similarly, the $[p,q]$ convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r \to +\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q r},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z : |z| \leq r\}$.

Remark 5. ([15]) If $f(z)$ is a meromorphic function satisfying $0 < \rho_{[p,q]}(f) < \infty$, then
(i) $\rho_{[p-n,q]} = \infty$ $(n < p)$, $\rho_{[p,q-n]} = 0$ $(n < q)$, $\rho_{[p+n,q+n]} = 1$ $(n < p)$ for $n = 1, 2, 3, \cdots$
(ii) If $[p_1, q_1]$ is any pair of integers satisfying $q_1 = p_1 + q - p$ and $p_1 < p$, then $\rho_{[p_1,q_1]} = 0$ if $0 < \rho_{[p,q]} < 1$ and $\rho_{[p_1,q_1]} = \infty$ if $1 < \rho_{[p,q]} < \infty$.
(iii) $\rho_{[p_1,q_1]} = \infty$ for $q_1 - p_1 > q - p$ and $\rho_{[p_1,q_1]} = 0$ for $q_1 - p_1 < q - p$.

Remark 6. ([15]) Suppose that f_1 is a meromorphic function of $[p,q]$-order ρ_1 and f_2 is a meromorphic function of $[p_1,q_1]$-order ρ_2, let $p \leq p_1$. We can easily deduce the result about their comparative growth:
(i) If $p_1 - p > q_1 - q$, then the growth of f_1 is slower than the growth of f_2.
(ii) If $p_1 - p < q_1 - q$, then f_1 grows faster than f_2.

(iii) If $p_1 - p = q_1 - q > 0$, then the growth of f_1 is slower than the growth of f_2 if $\rho_2 \geq 1$, and the growth of f_1 is faster than the growth of f_2 if $\rho_2 < 1$.
(iv) Especially, when $p_1 = p$ and $q_1 = q$ then f_1 and f_2 are of the same index-pair $[p, q]$. If $\rho_1 > \rho_2$, then f_1 grows faster than f_2; and if $\rho_1 < \rho_2$, then f_1 grows slower than f_2. If $\rho_1 = \rho_2$, Definition 2 does not show any precise estimate about the relative growth of f_1 and f_2.

We recall the following definitions. The linear measure of a set $E \subset (0, +\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\,dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt$, where $\chi_H(t)$ is the characteristic function of a set H. The upper density of a set $E \subset (0, +\infty)$ is defined by

$$\overline{dens}E = \limsup_{r \longrightarrow +\infty} \frac{m(E \cap [0, r])}{r}.$$

The upper logarithmic density of a set $F \subset (1, +\infty)$ is defined by

$$\overline{\log dens}(F) = \limsup_{r \longrightarrow +\infty} \frac{lm(F \cap [1, r])}{\log r}.$$

Proposition 7. *For all $H \subset [1, +\infty)$ the following statements hold :*
(*i*) *If $lm(H) = \infty$, then $m(H) = \infty$;*
(*ii*) *If $\overline{dens}H > 0$, then $m(H) = \infty$;*
(*iii*) *If $\overline{\log dens}H > 0$, then $lm(H) = \infty$.*

Proof. (i) Since we have $\frac{\chi_H(t)}{t} \leq \chi_H(t)$ for all $t \in H \subset [1, +\infty)$, then

$$m(H) \geq lm(H).$$

So, if $lm(H) = \infty$, then $m(H) = \infty$. We can easily prove the results (ii) and (iii) by applying the definition of the limit and the properties $m(H \cap [0, r]) \leq m(H)$ and $lm(H \cap [1, r]) \leq lm(H)$. □

Definition 8. ([9, 20]) For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function f is defined as

$$\delta(a, f) = \liminf_{r \to +\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Extensive work in recent years has been concerned with the growth of solutions of $[p, q]$-order of complex linear differential equations in the complex plane and in the unit disc. Many results have been obtained [2, 3, 4, 10, 15, 16, 17, 18, 19]. Examples of such results are the following two theorems:

Theorem 9. *([16]) Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be entire functions satisfying $\max\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\} \leq \alpha$. Suppose that there exists a positive constant β satisfying $\beta < \alpha$ such that for any given ε $(0 < \varepsilon < \alpha - \beta)$, we have*

$$|A_0(z)| \geq \exp_{p+1}\left\{(\alpha - \varepsilon)\log_q r\right\}$$

and

$$|A_j(z)| \leq \exp_{p+1}\left\{\beta \log_q r\right\} \quad (j = 1, \cdots, k-1)$$

for $z \in H$. Then, every solution $f \not\equiv 0$ of equation (1.1) *satisfies $\rho_{[p+1,q]}(f) = \alpha$.*

Theorem 10. *([15]) Let $H \subset (1,\infty)$ be a set satisfying $\overline{\log dens}\{|z| : |z| \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$, $F \not\equiv 0$ be meromorphic functions satisfying $\max\{\rho_{[p,q]}(A_j) : j = 1, 2, \cdots, k-1\} < \alpha$, where α is a constant. Suppose that there exists a constant β satisfying $\beta < \alpha$ such that for any given ε $(0 < \varepsilon < \alpha - \beta)$, we have*

$$|A_0(z)| \geq \exp_{p+1}\left\{(\alpha - \varepsilon)\log_q r\right\}$$

and

$$|A_j(z)| \leq \exp_{p+1}\left\{\beta \log_q r\right\} \quad (j = 1, \cdots, k-1)$$

as $|z| \in H$. Then the following statements hold:
(i) If $\rho_{[p+1,q]}(F) \geq \alpha$, then all meromorphic solutions f whose poles are of uniformly bounded multiplicities of equation (1.2) satisfy $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) If $\rho_{[p+1,q]}(F) < \alpha$, then all meromorphic solutions f whose poles are of uniformly bounded multiplicities of equation (1.2) satisfy $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \alpha$ with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0) < \alpha$.

The main purpose of this paper is to consider the growth of meromorphic solutions of equations (1.1) and (1.2) with meromorphic coefficients of finite $[p,q]$-order in the complex plane. We obtain the following results which generalize and improve Theorem 9 and Theorem 10.

Theorem 11. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be meromorphic functions satisfying $\max\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\} \leq \rho$ $(0 < \rho < \infty)$. Suppose that there exist two real numbers satisfying $0 \leq \beta < \alpha$ such that, we have*

$$|A_0(z)| \geq \exp_p\left\{\alpha\left[\log_{q-1} r\right]^\rho\right\} \tag{1.3}$$

and

$$|A_j(z)| \leq \exp_p\left\{\beta\left[\log_{q-1} r\right]^\rho\right\} \quad (j = 1, \cdots, k-1) \tag{1.4}$$

as $|z| \to +\infty$ for $z \in H$. Then the following statements hold:
(i) If $p \geq q \geq 1$ or $3 \leq q = p+1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho$.
(ii) If $p = 1$, $q = 2$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[2,2]}(f) \geq \rho$.

Theorem 12. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$, and let $A_0(z), \cdots, A_{k-1}(z)$ be meromorphic functions satisfying $\max\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\} \leq \rho$ $(0 < \rho < \infty)$. Suppose that there exist two positive constants α, β such that, we have*

$$m(r, A_0) \geq \exp_{p-1}\left\{\alpha\left[\log_{q-1} r\right]^\rho\right\} \tag{1.5}$$

and

$$m(r, A_j) \leq \exp_{p-1}\left\{\beta\left[\log_{q-1} r\right]^\rho\right\} \quad (j = 1, \cdots, k-1) \tag{1.6}$$

as $|z| \to +\infty$ for $z \in H$. Then the following statements hold:
(i) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho$.

(ii) If $3 \le q = p+1$, $0 \le \beta < \alpha$ *and* $\rho > 1$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.1) *satisfies* $\rho_{[p+1,p+1]}(f) = \rho$.
(iii) If $p = 1$, $q = 2$, $0 \le (k-1)\beta < \alpha$ *and* $\rho > 1$, *then every meromorphic solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies* $\rho_{[2,2]}(f) \ge \rho$.

Corollary 13. *Let* $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ *be meromorphic functions. Suppose that* H, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ *satisfy the hypotheses in Theorem 11. Then we have the following statements:*
(i) Let $p \ge q \ge 1$. *If* $\rho_{[p+1,q]}(F) \le \rho$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.2) *satisfies* $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho$ *with at most one exceptional solution* f_0 *satisfying* $\rho_{[p+1,q]}(f_0) < \rho$; *if* $\rho_{[p+1,q]}(F) > \rho$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.2) *satisfies* $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) Let $3 \le q = p+1$ *and* $\rho > 1$. *If* $\rho_{[p+1,p+1]}(F) \le \rho$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.2) *satisfies* $\overline{\lambda}_{[p+1,p+1]}(f) = \lambda_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(f) = \rho$, *with at most one exceptional solution* f_0 *satisfying* $\rho_{[p+1,p+1]}(f_0) < \rho$; *if* $\rho_{[p+1,p+1]}(F) > \rho$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.2) *satisfies* $\rho_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(F)$.

Corollary 14. *Let* $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ *be meromorphic functions. Suppose that* H, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ *satisfy the hypotheses in Theorem 12. Then we have the following statements:*
(i) Let $p \ge q \ge 2$, $0 \le \beta < \alpha$. *If* $\rho_{[p+1,q]}(F) \le \rho$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.2) *satisfies* $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho$ *with at most one exceptional solution* f_0 *satisfying* $\rho_{[p+1,q]}(f_0) < \rho$; *if* $\rho_{[p+1,q]}(F) > \rho$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.2) *satisfies* $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) Let $3 \le q = p+1$, $0 \le \beta < \alpha$ *and* $\rho > 1$. *If* $\rho_{[p+1,p+1]}(F) \le \rho$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.2) *satisfies* $\overline{\lambda}_{[p+1,p+1]}(f) = \lambda_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(f) = \rho$ *with at most one exceptional solution* f_0 *satisfying* $\rho_{[p+1,p+1]}(f_0) < \rho$; *if* $\rho_{[p+1,p+1]}(F) > \rho$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities or* $\delta(\infty, f) > 0$ *of equation* (1.2) *satisfies* $\rho_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(F)$.

Recently, the author [2, 3, 4], J. Tu and Z. X. Xuan [17] and J. Tu and H. X. Huang [18] have investigated the growth of solutions of differential equations (1.1) and (1.2) with analytic coefficients of $[p, q]$-order in the unit disc. So, it is also interesting to consider the growth of meromorphic solutions of differential equations with coefficients of $[p, q]$-order in the unit disc?

2 Some preliminary lemmas

Our proofs depend mainly upon the following lemmas.

Lemma 15. *([1]) Let* $g : (0, \infty) \to \mathbb{R}$, $h : (0, \infty) \to \mathbb{R}$ *be monotone increasing functions such that* $g(r) \le h(r)$ *outside of an exceptional set* E_1 *of finite linear measure. Then, for any* $\lambda > 1$, *there exists* $r_1 > 0$ *such that* $g(r) \le h(\lambda r)$ *for all* $r > r_1$.

Lemma 16. *([8]) Let $\varphi : [0,+\infty) \to \mathbb{R}$ and $\psi : [0,+\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \leq \psi(r)$ for all $r \notin E_2 \cup [0,1]$, where $E_2 \subset (1,+\infty)$ is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists an $r_2 = r_2(\gamma) > 0$ such that $\varphi(r) \leq \psi(\gamma r)$ for all $r > r_2$.*

Lemma 17. *([9]) Let f be a meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O(\log T(r,f) + \log r)$, possibly outside of an exceptional set $E_3 \subset (0,+\infty)$ with finite linear measure, and if f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 18. *([7]) Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E_4 \subset (1,\infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and i, j $(0 \leq i < j \leq k)$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_4$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left\{\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right\}^{j-i}.$$

Lemma 19. *([5]) Let f be a meromorphic solution of (1.1), assuming that not all coefficients A_j are constants. Given a real constant $\gamma > 1$, and denoting $T(r) = \sum\limits_{j=0}^{k-1} T(r, A_j)$, we have*

$$\log m(r,f) < T(r)\{(\log r)\log T(r)\}^{\gamma}, \text{ if } s = 0,$$
$$\log m(r,f) < r^{2s+\gamma-1}T(r)\{\log T(r)\}^{\gamma}, \text{ if } s > 0$$

outside of an exceptional set E_s with $\int\limits_{E_s} t^{s-1}dt < +\infty$.

Remark 20. We note that in the above lemma, $s = 1$ corresponds to Euclidean measure and $s = 0$ to logarithmic measure.

Lemma 21. *Let $A_0(z), \cdots, A_{k-1}(z)$ be nonconstant meromorphic functions of $[p,q]-$order. Assume the existence of the meromorphic solutions of (1.1). Then the following statements hold:*
(i) If $p \geq q \geq 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) \leq \max\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\}$.
(ii) If $3 \leq q = p+1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,p+1]}(f) \leq \max\{\rho_{[p,p+1]}(A_j)\ (j = 0, 1, \cdots, k-1)\}$.

Proof. We prove only (ii). For the proof of (i) see [15, 19]. From (1.1), we know that the poles of $f(z)$ can only occur at the poles of $A_0(z), \cdots, A_{k-1}(z)$. Since the multiplicities of poles of f are uniformly bounded, we have

$$N(r,f) \leq M_1 \overline{N}(r,f) \leq M_1 \sum_{j=0}^{k-1} \overline{N}(r, A_j)$$

$$\leq M \max \{N(r, A_j) : j = 0, 1, \cdots, k-1\}, \tag{2.1}$$

where M_1 and M are some suitable positive constants. This gives

$$T(r, f) = m(r, f) + O(\max \{N(r, A_j) : j = 0, 1, \cdots, k-1\}). \tag{2.2}$$

Set $\delta(\infty, f) := \eta > 0$, for sufficiently large r, we have

$$m(r, f) \geq \frac{\eta}{2} T(r, f). \tag{2.3}$$

From Lemma 19 and (2.2) or (2.3), we obtain

$$\log T(r, f) \leq \log m(r, f) + O(\log T(r)) \leq O(T(r) \{(\log r) \log T(r)\}^{\gamma}) \tag{2.4}$$

or

$$\log T(r, f) \leq \log \left(\frac{2}{\eta} m(r, f) \right) \leq O(T(r) \{(\log r) \log T(r)\}^{\gamma}) \tag{2.5}$$

outside of an exceptional set E_0 with finite logarithmic measure. From (2.4) or (2.5), we get for $p \geq 2$

$$\log_{p+1} T(r, f) \leq \max \left\{ \log_p T(r), \ \log_{p+1} r \right\} \tag{2.6}$$

outside of an exceptional set E_0 with finite logarithmic measure. If at least one of the coefficients $A_0(z), \cdots, A_{k-1}(z)$ of (1.1) is transcendental, then by using Lemma 16 and (2.6), we obtain

$$\rho_{[p+1,p+1]}(f) \leq \max \left\{ \rho_{[p,p+1]}(A_j) \ \ (j = 0, 1, \cdots, k-1), 1 \right\}$$

$$= \max \left\{ \rho_{[p,p+1]}(A_j) \ \ (j = 0, 1, \cdots, k-1) \right\}.$$

If all the coefficients $A_0(z), \cdots, A_{k-1}(z)$ of (1.1) are rational functions, then by using Lemma 16 and (2.6), we obtain

$$\rho_{[p+1,p+1]}(f) \leq \max \left\{ \rho_{[p,p+1]}(A_j) \ \ (j = 0, 1, \cdots, k-1), 1 \right\} = 1$$

$$= \max \left\{ \rho_{[p,p+1]}(A_j) \ \ (j = 0, 1, \cdots, k-1) \right\}.$$

□

Lemma 22. *([15]) Let $1 \leq q \leq p$ or $2 \leq q = p + 1$ and let f be a meromorphic function with $0 \leq \rho_{[p,q]}(f) = \rho \leq \infty$. Then there exists a set $E_5 \subset [1, +\infty)$ with infinite logarithmic measure such that*

$$\lim_{\substack{r \to +\infty \\ r \in E_5}} \frac{\log_p T(r, f)}{\log_q r} = \rho.$$

Lemma 23. *Let $1 \leq q \leq p$ or $2 \leq q = p+1$ and let f_1 and f_2 be meromorphic functions of $[p, q]$ −order satisfying $\rho_{[p,q]}(f_1) > \rho_{[p,q]}(f_2)$. Then there exists a set $E_6 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_6$, we have*

$$\lim_{r \to \infty} \frac{T(r, f_2)}{T(r, f_1)} = 0.$$

Proof. Set $\rho_1 = \rho_{[p,q]}(f_1)$, $\rho_2 = \rho_{[p,q]}(f_2)$. By using Lemma 22, there exists a set E_6 with infinite logarithmic measure such that for any given $0 < \varepsilon < \frac{\rho_1 - \rho_2}{2}$ and all sufficiently large $r \in E_6$

$$T(r, f_1) > \exp_p\left\{(\rho_1 - \varepsilon)\log_q r\right\}$$

and for all sufficiently large r, we have

$$T(r, f_2) < \exp_p\left\{(\rho_2 + \varepsilon)\log_q r\right\}.$$

From this we can get

$$\frac{T(r, f_2)}{T(r, f_1)} < \frac{\exp_p\left\{(\rho_2 + \varepsilon)\log_q r\right\}}{\exp_p\left\{(\rho_1 - \varepsilon)\log_q r\right\}}$$

$$= \exp\left\{\exp_{p-1}\left\{(\rho_2 + \varepsilon)\log_q r\right\} - \exp_{p-1}\left\{(\rho_1 - \varepsilon)\log_q r\right\}\right\}, \ r \in E_6.$$

Since $0 < \varepsilon < \frac{\rho_1 - \rho_2}{2}$, then we have

$$\lim_{r\to\infty} \frac{T(r, f_2)}{T(r, f_1)} = 0, \ r \in E_6.$$

□

Lemma 24. *Let A_j $(j = 0, \cdots, k-1)$, $F \not\equiv 0$ be meromorphic functions. Then the following statements hold:*
(i) If $p \geq q \geq 1$, then every meromorphic solution f of equation (1.2) *such that* $\max\{\rho_{[p,q]}(A_j)$ $(j = 0, 1, \cdots, k-1), \rho_{[p,q]}(F)\} < \rho_{[p,q]}(f)$ *satisfies* $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f)$.
(ii) If $2 \leq q = p+1$, then every meromorphic solution f of equation (1.2) *such that* $\max\{\rho_{[p,p+1]}(A_j)$ $(j = 0, 1, \cdots, k-1), \rho_{[p,p+1]}(F), 1\} < \rho_{[p,p+1]}(f)$ *satisfies* $\overline{\lambda}_{[p,p+1]}(f) = \lambda_{[p,p+1]}(f) = \rho_{[p,p+1]}(f)$.

Proof. We prove only (ii). For the proof of (i) see [15]. By (1.2), if f has a zero at z_0 of order $\alpha\,(> k)$ and if $A_0, A_1, \cdots, A_{k-1}$ are all analytic at z_0, then F must have a zero at z_0 of order $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f}\right) \leq k\,\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=1}^{k} n\left(r, A_{k-j}\right)$$

and

$$N\left(r, \frac{1}{f}\right) \leq k\,\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=1}^{k} N\left(r, A_{k-j}\right). \tag{2.7}$$

Now (1.2) can be rewritten as

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{2.8}$$

By Lemma 17 and (2.8), we have

$$m\left(r, \frac{1}{f}\right) \leq \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=1}^{k} m\left(r, A_{k-j}\right) + m\left(r, \frac{1}{F}\right) + O(1)$$

$$= \sum_{j=1}^{k} m\left(r, A_{k-j}\right) + m\left(r, \frac{1}{F}\right) + O\left(\log T\left(r,f\right) + \log r\right) \tag{2.9}$$

holds for all r outside a set $E_3 \subset (0, +\infty)$ with a finite linear measure $m(E_3) = \delta < +\infty$. By (2.7) and (2.9), we get

$$T\left(r,f\right) = T\left(r, \frac{1}{f}\right) + O\left(1\right)$$

$$\leq k\overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{k} T\left(r, A_{k-j}\right) + T\left(r, F\right) + O\left(\log T\left(r,f\right) + \log r\right) \quad \left(|z| = r \notin E_3\right). \tag{2.10}$$

Since $\max\{\rho_{[p,p+1]}\left(A_j\right)\ (j = 0, 1, \cdots, k-1), \rho_{[p,p+1]}\left(F\right)\} < \rho_{[p,p+1]}\left(f\right)$, then by Lemma 23, there exists a set $E_6 \subset [1, +\infty)$ with infinite logarithmic measure such that

$$\max\left\{\frac{T\left(r, A_j\right)}{T\left(r,f\right)}\ \left(j = 0, \cdots, k-1\right), \frac{T\left(r,F\right)}{T\left(r,f\right)}\right\} \to 0,\ r \to +\infty,\ r \in E_6. \tag{2.11}$$

Thus, by (2.10) and (2.11), we have for all $r \in E_6 \backslash E_3$

$$\left(1 - o\left(1\right)\right) T\left(r,f\right) \leq k\ \overline{N}\left(r, \frac{1}{f}\right) + O\left(\log T\left(r,f\right) + \log r\right).$$

Then, we obtain $\rho_{[p,p+1]}\left(f\right) \leq \overline{\lambda}_{[p,p+1]}\left(f\right) \leq \lambda_{[p,p+1]}\left(f\right)$. Therefore, by

$$\overline{\lambda}_{[p,p+1]}\left(f\right) \leq \lambda_{[p,p+1]}\left(f\right) \leq \rho_{[p,p+1]}\left(f\right)$$

we have $\overline{\lambda}_{[p,p+1]}\left(f\right) = \lambda_{[p,p+1]}\left(f\right) = \rho_{[p,p+1]}\left(f\right)$. □

Lemma 25. *Let f be a meromorphic function of $[p,q]-$order. Then the following statements hold:*
(i) If $p \geq q \geq 1$, then $\rho_{[p,q]}\left(f\right) = \rho_{[p,q]}\left(f'\right)$.
(ii) If $3 \leq q = p+1$, then $\rho_{[p,p+1]}\left(f'\right) \leq \max\left\{\rho_{[p,p+1]}\left(f\right), 1\right\}$ and $\rho_{[p,p+1]}\left(f\right) \leq \max\left\{\rho_{[p,p+1]}\left(f'\right), 1\right\}$.
(iii) If $p = 1, q = 2$, then $\rho_{[1,2]}\left(f'\right) \leq \max\left\{\rho_{[1,2]}\left(f\right), 1\right\}$ and $\rho_{[1,2]}\left(f\right) \leq 1 + \rho_{[1,2]}\left(f'\right)$.

Proof. (i) – (ii) By Lemma 17, we have

$$T\left(r,f'\right) = m\left(r,f'\right) + N\left(r,f'\right) \leq m\left(r,f\right) + m\left(r, \frac{f'}{f}\right) + 2N\left(r,f\right)$$

$$\leq 2T\left(r,f\right) + m\left(r, \frac{f'}{f}\right) \leq 2T\left(r,f\right) + O\left(\log T\left(r,f\right) + \log r\right) \tag{2.12}$$

holds outside of an exceptional set $E_3 \subset (0, +\infty)$ with finite linear measure. By (2.12) and Lemma 15, it is easy to see $\rho_{[p,q]}\left(f'\right) \leq \rho_{[p,q]}\left(f\right)$ $(p \geq q \geq 1)$ and $\rho_{[p,p+1]}\left(f'\right) \leq \max\left\{\rho_{[p,p+1]}\left(f\right), 1\right\}$ if $3 \leq q = p+1$. On the other hand, [6], ([20], p. 35), we have for $r \to +\infty$

$$T\left(r,f\right) < O\left(T\left(2r,f'\right) + \log r\right). \tag{2.13}$$

Hence, by using (2.13) we obtain $\rho_{[p,q]}\left(f'\right) = \rho_{[p,q]}\left(f\right)$ if $p \geq q \geq 1$ and $\rho_{[p,p+1]}\left(f\right) \leq \max\left\{\rho_{[p,p+1]}\left(f'\right), 1\right\}$ if $3 \leq q = p+1$. We can easily obtain the conclusion (iii) by using (2.12) and (2.13). □

3 Proof of Theorem 11

Proof. (i) Suppose that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1). From the conditions of Theorem 11, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (1.3) and (1.4) as $|z| \to +\infty$. Set $H_1 = \{r = |z| : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then H_1 is a set with $\int_{H_1} \frac{dr}{r} = \infty$. By Lemma 18, we know that there exists a set $E_4 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$, we get

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B\left[T(2r, f)\right]^{j+1} \ (j = 1, \cdots, k). \tag{3.1}$$

By (1.1), we can write

$$|A_0(z)| \leq \left| \frac{f^{(k)}}{f} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}}{f} \right| + \cdots + |A_0(z)| \left| \frac{f'}{f} \right|. \tag{3.2}$$

It follows by (1.3), (1.4), (3.1) and (3.2) that

$$\exp_p \left\{ \alpha \left[\log_{q-1} r \right]^\rho \right\} \leq |A_0(z)| \leq kB \exp_p \left\{ \beta \left[\log_{q-1} r \right]^\rho \right\} \left[T(2r, f)\right]^{k+1} \tag{3.3}$$

holds for all z satisfying $|z| = r \in H_1 \backslash ([0, 1] \cup E_4)$ as $|z| \to +\infty$. If $p \geq q \geq 1$ or $3 \leq q = p + 1$, then by (3.3) and Lemma 16, we obtain $\rho \leq \rho_{[p+1,q]}(f)$. On the other hand, by Lemma 21 (i) – (ii), we have

$$\rho_{[p+1,q]}(f) \leq \max\left\{ \rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1 \right\} \leq \rho,$$

if $p \geq q \geq 1$ or $3 \leq q = p + 1$. Hence every meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho$ if $p \geq q \geq 1$ or $3 \leq q = p + 1$.
(ii) If $p = 1$, $q = 2$, then from (3.3), we have

$$\exp\left\{ \alpha \left[\log r \right]^\rho \right\} \leq |A_0(z)| \leq kB \exp\left\{ \beta \left[\log r \right]^\rho \right\} \left[T(2r, f)\right]^{k+1} \tag{3.4}$$

holds for all z satisfying $|z| = r \in H_1 \backslash ([0, 1] \cup E_4)$ as $|z| \to +\infty$. By (3.4) and Lemma 16, every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[2,2]}(f) \geq \rho$. □

4 Proof of Theorem 12

Proof. (i) Suppose that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1). By (1.1), we can write

$$A_0(z) = -\left(\frac{f^{(k)}}{f} + A_{k-1}(z) \frac{f^{(k-1)}}{f} + \cdots + A_1(z) \frac{f'}{f} \right). \tag{4.1}$$

From the conditions of Theorem 12, there is a set H of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ such that for $z \in H$, we have (1.5) and(1.6) as $|z| \to +\infty$. Set $H_1 = \{r = |z| : z \in H\}$, since $\overline{\log dens}\{|z| : z \in H\} > 0$, then H_1 is a set of r with $\int_{H_1} \frac{dr}{r} = \infty$. It follows by (1.5), (1.6), (4.1) and Lemma 17 that

$$\exp_{p-1} \left\{ \alpha \left[\log_{q-1} r \right]^\rho \right\} \leq m(r, A_0)$$

$$\leq \sum_{j=1}^{k-1} m\left(r, A_j\right) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O\left(1\right)$$

$$\leq \left(k-1\right) \exp_{p-1}\left\{\beta\left[\log_{q-1} r\right]^{\rho}\right\} + O\left(\log T\left(r,f\right) + \log r\right) \tag{4.2}$$

holds for all z satisfying $|z| = r \in H_1 \backslash E_3$ as $|z| \to +\infty$, where $E_3 \subset (0, +\infty)$ is a set with a finite linear measure. If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then by (4.2) and Lemma 15, we obtain $\rho \leq \rho_{[p+1,q]}(f)$. On the other hand, by Lemma 21 (i), we have $\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p,q]}(A_j) : j = 0, 1, \cdots, k-1\right\} \leq \rho$. Hence every meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\rho_{[p+1,q]}(f) = \rho$.

(ii) If $3 \leq q = p+1$, $0 \leq \beta < \alpha$ and $\rho > 1$, by the similar proof in case (i) and Lemma 21 (ii), we can obtain the conclusion.

(iii) If $p = 1$, $q = 2$, $0 \leq (k-1)\beta < \alpha$ and $\rho > 1$, then from (4.2), we have

$$\alpha\left[\log r\right]^{\rho} \leq m\left(r, A_0\right) \leq \left(k-1\right)\beta\left[\log r\right]^{\rho} + O\left(\log T\left(r,f\right) + \log r\right) \tag{4.3}$$

holds for all z satisfying $|z| = r \in H_1 \backslash E_3$ as $|z| \to +\infty$. By (4.3) and Lemma 16, every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{[2,2]}(f) \geq \rho$. □

5 Proof of Corollary 13

Proof. (i) Suppose that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1).

(a) Suppose that $1 \leq q \leq p$ and $\rho_{[p+1,q]}(F) \leq \rho$. We assume that f is a solution of (1.2) and $\{f_1, f_2, \cdots, f_k\}$ is a solution base of the corresponding homogeneous equation (1.1) of (1.2). By Theorem 11, we know that $\rho_{[p+1,q]}(f_j) = \rho$ $(j = 1, 2, \cdots, k)$. Then f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + \cdots + B_k(z) f_k(z), \tag{5.1}$$

where $B_1(z), \cdots, B_k(z)$ are suitable meromorphic functions determined by

$$\begin{aligned} B_1'(z) f_1(z) + B_2'(z) f_2(z) + \cdots + B_k'(z) f_k(z) = 0, \\ B_1'(z) f_1'(z) + B_2'(z) f_2'(z) + \cdots + B_k'(z) f_k'(z) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B_1'(z) f_1^{(k-1)}(z) + B_2'(z) f_2^{(k-1)}(z) + \cdots + B_k'(z) f_k^{(k-1)}(z) = F(z). \end{aligned} \tag{5.2}$$

Since the Wronskian $W(f_1, f_2, \cdots, f_k)$ is a differential polynomial in $f_1, f_2, \cdots, f_k$ with constant coefficients, it is easy by using Theorem 11 to deduce that

$$\rho_{[p+1,q]}(W) \leq \max\left\{\rho_{[p+1,q]}(f_j) : j = 1, 2, \cdots, k\right\} = \rho. \tag{5.3}$$

From (5.2), we get

$$B_j' = F.G_j(f_1, f_2, \cdots, f_k).(W(f_1, f_2, \cdots, f_k))^{-1} \quad (j = 1, 2, \cdots, k), \tag{5.4}$$

where $G_j(f_1, f_2, \cdots, f_k)$ are differential polynomials in $f_1, f_2, \cdots, f_k$ with constant coefficients. Thus

$$\rho_{[p+1,q]}(G_j) \leq \max\left\{\rho_{[p+1,q]}(f_j) : j = 1, 2, \cdots, k\right\} = \rho \quad (j = 1, 2, \cdots, k). \tag{5.5}$$

Since $\rho_{[p+1,q]}(F) \leq \rho$, then by using Lemma 25 (i), (5.3) and (5.5), we have from (5.4) for $j = 1, 2, \cdots, k$

$$\rho_{[p+1,q]}(B_j) = \rho_{[p+1,q]}\left(B_j'\right) \leq \max\left\{\rho_{[p+1,q]}(F), \rho\right\} = \rho. \tag{5.6}$$

Then, by (5.6), we get from (5.1)

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p+1,q]}(f_j), \rho_{[p+1,q]}(B_j) : j = 1, 2, \cdots, k\right\} = \rho. \tag{5.7}$$

Now, we assert that every meromorphic solution f whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of (1.2) satisfies $\rho_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0) < \rho$. In fact, if f^* is another meromorphic solution with $\rho_{[p+1,q]}(f^*) < \rho$ of equation (1.2), then $\rho_{[p+1,q]}(f_0 - f^*) < \rho$. But $f_0 - f^*$ is a meromorphic solution of the corresponding homogeneous equation (1.1) of (1.2). This contradicts Theorem 11. Then $\rho_{[p+1,q]}(f) = \rho$ holds for all meromorphic solutions of (1.2) with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0) < \rho$. By Lemma 24 (i), we know that every meromorphic solution f whose poles are of uniformly bounded multiplicities or $\delta(\infty, f) > 0$ with $\rho_{[p+1,q]}(f) = \rho$ satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho$.

(b) If $\rho < \rho_{[p+1,q]}(F)$, then by using Lemma 25 (i), (5.3) and (5.5), we have from (5.4) for $j = 1, 2, \cdots, k$

$$\begin{aligned} \rho_{[p+1,q]}(B_j) &= \rho_{[p+1,q]}\left(B_j'\right) \\ &\leq \max\left\{\rho_{[p+1,q]}(F), \rho_{[p+1,q]}(f_j) : j = 1, 2, \cdots, k\right\} = \rho_{[p+1,q]}(F). \end{aligned} \tag{5.8}$$

Then from (5.8) and (5.1), we get

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p+1,q]}(f_j), \rho_{[p+1,q]}(B_j) : j = 1, 2, \cdots, k\right\} \leq \rho_{[p+1,q]}(F). \tag{5.9}$$

On the other hand, if $\rho < \rho_{[p+1,q]}(F)$, it follows from equation (1.2) that a simple consideration of $[p, q]$ $-$order implies $\rho_{[p+1,q]}(f) \geq \rho_{[p+1,q]}(F)$. By this inequality and (5.9) we obtain $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.

(ii) For $3 \leq q = p + 1, \rho > 1$, by the similar proof in case (i), we can also obtain that the conclusions of case (ii) hold. □

6 Proof of Corollary 14

Proof. By using the same reasoning of Corollary 13 we can obtain Corollary 14. □

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Some Perturbed Ostrowski Type Inequalities for Absolutely Continuous Functions (I)

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Abstract
In this paper, some two parameters perturbed Ostrowski type inequalities for absolutely continuous functions are established.

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1 Introduction

We start with the following result that generalizes Ostrowski's inequality for real valued differentiable functions whose derivative are bounded.

Theorem 1 (Dragomir, 2003 [20]). *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x \in [a,b]$. Suppose that there exist the functions m_i, $M_i : [a,b] \to \mathbb{R}$ $\left(i = \overline{1,2}\right)$ with the properties:*

$$m_1(x) \le f'(t) \le M_1(x) \quad \text{for a.e. } t \in [a,x] \tag{1.1}$$

and

$$m_2(x) \le f'(t) \le M_2(x) \quad \text{for a.e. } t \in (x,b]. \tag{1.2}$$

Then we have the inequalities:

$$\begin{aligned} &\frac{1}{2(b-a)}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \\ &\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2 - m_2(x)(b-x)^2\right]. \end{aligned} \tag{1.3}$$

The constant $\frac{1}{2}$ is sharp on both sides.

In the case that the derivative is globally bounded on $[a,b]$ by two constants, then we have:

Corollary 2. *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and the derivative $f' : [a,b] \to \mathbb{R}$ is bounded above and below, that is, there exists the constants $M > m$ such that*

$$-\infty < m \le f'(t) \le M < \infty \text{ for a.e. } t \in [a,b], \tag{1.4}$$

then we have the inequality

$$\begin{aligned} &\frac{1}{2(b-a)}\left[m(x-a)^2 - M(b-x)^2\right] \\ &\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{2(b-a)}\left[M(x-a)^2 - m(b-x)^2\right] \end{aligned} \tag{1.5}$$

for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best in both inequalities.

We may rewrite Corollary 2 in the following equivalent manner:

Corollary 3. *With the assumptions on Corollary 2, we have:*

$$\begin{aligned} &\left| f(x) - \left(x - \frac{a+b}{2}\right)\left(\frac{M+m}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \frac{1}{2}(M-m)(b-a)\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \end{aligned} \tag{1.6}$$

for all $x \in [a,b]$.

Remark 4. If we assume that $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < \infty$, then obviously we may choose in (1.5) $m = \|f'\|_\infty$ and $M = \|f'\|_\infty$, obtaining Ostrowski's inequality for absolutely continuous functions whose derivatives are essentially bounded:

$$\begin{aligned} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &\le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] \\ &= \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, \end{aligned}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ here is best.

Remark 5. Ostrowski's inequality for absolutely continuous mappings in terms of $\|f'\|_\infty$ basically states that

$$\begin{aligned} -\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] &\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] \end{aligned} \tag{1.7}$$

for all $x \in [a,b]$.

Now, if we assume that (1.1) and (1.2) hold, then $-\|f'\|_\infty \le m_1(x)$, $m_2(x)$ and $M_1(x)$, $M_2(x) \le \|f'\|_\infty$, which implies:

$$\begin{aligned}
&-\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right] \qquad (1.8)\\
&\le \frac{1}{2(b-a)}\left[m_1(x)(x-a)^2-M_2(x)(b-x)^2\right] \le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&\le \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2-m_2(x)(b-x)^2\right]\\
&\le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right].
\end{aligned}$$

Thus, the inequality (1.3) may also be regarded as a refinement of the classical Ostrowski result.

An important particular case is $x=\frac{a+b}{2}$ providing the following corollary.

Corollary 6. *Assume that the derivative $f':[a,b]\to\mathbb{R}$ satisfy the conditions:*

$$-\infty < m_1 \le f'(t) \le M_1 < \infty \text{ for a.e. } t \in \left[a, \frac{a+b}{2}\right] \qquad (1.9)$$

and

$$-\infty < m_2 \le f'(t) \le M_2 < \infty \text{ for a.e. } t \in \left(\frac{a+b}{2}, b\right]. \qquad (1.10)$$

Then we have the inequalities

$$\begin{aligned}
\frac{1}{8}(m_1-M_2)(b-a) &\le f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \qquad (1.11)\\
&\le \frac{1}{8}(M_1-m_2)(b-a).
\end{aligned}$$

The constant $\frac{1}{8}$ is the best in both inequalities.

Finally, if we know some global bounds for the derivative f' on $[a,b]$, then we may state the following corollary.

Corollary 7. *Under the assumptions of Corollary 2, we have the midpoint inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(M-m)(b-a). \qquad (1.12)$$

The constant $\frac{1}{8}$ is best.

For other Ostrowski type inequalities see [1]-[19] and [21]-[42].

Motivated by the above results, we establish in this paper some perturbed Ostrowski type inequalities for complex valued differentiable functions whose derivatives are either bounded or of bounded variation. Applications for midpoint inequalities are provided as well.

2 Some Identities

We start with the following identity that will play an important role in the following:

Lemma 8. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x \in [a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$
\begin{aligned}
& f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 \lambda_2(x) - (x-a)^2 \lambda_1(x)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \qquad (2.1)\\
&= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - \lambda_1(x)\right]dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - \lambda_2(x)\right]dt,
\end{aligned}
$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Proof. Utilising the integration by parts formula in the Lebesgue integral, we have

$$
\begin{aligned}
& \int_a^x (t-a)\left[f'(t) - \lambda_1(x)\right]dt \qquad (2.2)\\
&= (t-a)\left[f(t) - \lambda_1(x)t\right]\Big|_a^x - \int_a^x \left[f(t) - \lambda_1(x)t\right]dt\\
&= (x-a)\left[f(x) - \lambda_1(x)x\right] - \int_a^x f(t)\,dt + \frac{1}{2}\lambda_1(x)\left(x^2 - a^2\right)\\
&= (x-a)f(x) - \lambda_1(x)x(x-a) - \int_a^x f(t)\,dt + \frac{1}{2}\lambda_1(x)\left(x^2 - a^2\right)\\
&= (x-a)f(x) - \int_a^x f(t)\,dt - \frac{1}{2}(x-a)^2\lambda_1(x)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_x^b (t-b)\left[f'(t) - \lambda_2(x)\right]dt \qquad (2.3)\\
&= (t-b)\left[f(t) - \lambda_2(x)t\right]\Big|_x^b - \int_x^b \left[f(t) - \lambda_2(x)t\right]dt\\
&= (b-x)\left[f(x) - \lambda_2(x)x\right] - \int_x^b f(t)\,dt + \frac{1}{2}\lambda_2(x)\left(b^2 - x^2\right)\\
&= (b-x)f(x) - \int_x^b f(t)\,dt - (b-x)\lambda_2(x)x + \frac{1}{2}\lambda_2(x)\left(b^2 - x^2\right)\\
&= (b-x)f(x) - \int_x^b f(t)\,dt + \frac{1}{2}(b-x)^2\lambda_2(x).
\end{aligned}
$$

If we add the identities (2.2) and (2.3) and divide by $b-a$ we deduce the desired identity (2.1). □

Corollary 9. *With the assumption in Lemma 8, we have for any $\lambda(x) \in \mathbb{C}$ that*

$$
\begin{aligned}
& f(x) + \left(\frac{a+b}{2} - x\right)\lambda(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \qquad (2.4)\\
&= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - \lambda(x)\right]dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - \lambda(x)\right]dt.
\end{aligned}
$$

Remark 10. If we take $\lambda(x) = 0$ in (2.4), then we get Montgomery's identity for absolutely continuous functions, i.e.

$$\begin{aligned} &f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a) f'(t)\,dt + \frac{1}{b-a}\int_x^b (t-b) f'(t)\,dt, \end{aligned} \tag{2.5}$$

for $x \in [a,b]$.

We have the following midpoint representation:

Corollary 11. *With the assumption in Lemma 8, we have for any* $\lambda_1,\ \lambda_2 \in \mathbb{C}$ *that*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_2-\lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-\lambda_1\right]dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_2\right]dt. \end{aligned} \tag{2.6}$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, *then we have the equality*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-\lambda\right]dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda\right]dt. \end{aligned} \tag{2.7}$$

Remark 12. The identity (2.1) has many particular cases of interest.

If $x \in (a,b)$ is a point of differentiability for the absolutely continuous function $f : [a,b] \to \mathbb{C}$, then we have the equality:

$$\begin{aligned} &f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(x)\right]dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'(x)\right]dt. \end{aligned} \tag{2.8}$$

In particular we have

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right]dt \\ &+ \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right]dt \end{aligned} \tag{2.9}$$

provided $f'\left(\frac{a+b}{2}\right)$ exists and is finite.

For $x \in (a,b)$, if we take in (2.1)

$$\lambda_1(x) = \frac{f(x)-f(a)}{x-a} \text{ and } \lambda_2(x) = \frac{f(b)-f(x)}{b-x},$$

then we get, after some elementary calculations,

$$\begin{aligned}
&\frac{1}{2}\left[f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \qquad (2.10)\\
&=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\frac{f(x)-f(a)}{x-a}\right]dt\\
&+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\frac{f(b)-f(x)}{b-x}\right]dt.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \qquad (2.11)\\
&=\frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-\frac{f\left(\frac{a+b}{2}\right)-f(a)}{\frac{b-a}{2}}\right]dt\\
&+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\frac{f(b)-f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}}\right]dt.
\end{aligned}$$

If we assume that the lateral derivatives $f'_+(a)$ and $f'_-(b)$ exist and are finite, then we have from (2.1) for $\lambda_1(x)=f'_+(a)$ and $\lambda_2(x)=f'_-(b)$

$$\begin{aligned}
&f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \qquad (2.12)\\
&=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'_+(a)\right]dt\\
&+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'_-(b)\right]dt,
\end{aligned}$$

for all $x\in[a,b]$.

In particular, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \qquad (2.13)\\
&=\frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-f'_+(a)\right]dt\\
&+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-f'_-(b)\right]dt.
\end{aligned}$$

If we take in (2.1) $\lambda_2(x)=\lambda_2(x)=f'\left(\frac{a+b}{2}\right)$, provided this derivative exists and is finite, then we get

$$\begin{aligned}
&f(x)+\left(\frac{a+b}{2}-x\right)f'\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \qquad (2.14)\\
&=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt\\
&+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt,
\end{aligned}$$

for all $x \in [a,b]$.

If we assume that the derivatives $f'_+(a)$, $f'_-(b)$ and $f'(x)$ exist and are finite, then by taking

$$\lambda_1(x) = \frac{f'_+(a) + f'(x)}{2} \text{ and } \lambda_2(x) = \frac{f'(x) + f'_-(b)}{2}$$

in (2.1) we get

$$\begin{aligned} &f(x) + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &+ \frac{1}{4(b-a)}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right] \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - \frac{f'_+(a) + f'(x)}{2}\right] dt \\ &+ \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - \frac{f'(x) + f'_-(b)}{2}\right] dt. \end{aligned} \tag{2.15}$$

In particular, we have

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) + \frac{1}{16}(b-a)\left[f'_-(b) - f'_+(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \frac{f'_+(a) + f'\left(\frac{a+b}{2}\right)}{2}\right] dt \\ &+ \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \frac{f'\left(\frac{a+b}{2}\right) + f'_-(b)}{2}\right] dt. \end{aligned} \tag{2.16}$$

3 Inequalities for Bounded Derivatives

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\begin{aligned} &\bar{U}_{[a,b]}(\gamma,\Gamma) \\ &:= \left\{ f : [a,b] \to \mathbb{C} \,|\, \operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \geq 0 \text{ for almost every } t \in [a,b] \right\} \end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} \,\middle|\, \left| f(t) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a,b] \right\}.$$

The following representation result may be stated.

Proposition 13. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left|z - \frac{\gamma + \Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 14. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma,\Gamma) = \{f : [a,b] \to \mathbb{C} \mid\ & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \textit{ for a.e. } t \in [a,b]\}. \end{aligned} \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma,\Gamma) := \{f : [a,b] \to \mathbb{C} \mid\ & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a,b]\}. \end{aligned} \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{3.4}$$

Theorem 15. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x \in (a,b)$. Suppose that $\gamma_i, \Gamma_i \in \mathbb{C}$ with $\gamma_i \neq \Gamma_i$, $i = 1,2$ and $f' \in \bar{U}_{[a,x]}(\gamma_1,\Gamma_1) \cap \bar{U}_{[x,b]}(\gamma_2,\Gamma_2)$, then we have*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right. \\ & \left. + \frac{1}{2(b-a)}\left[(b-x)^2\frac{\Gamma_2+\gamma_2}{2} - (x-a)^2\frac{\Gamma_1+\gamma_1}{2}\right]\right| \\ & \leq \frac{1}{4}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^2 + |\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^2\right](b-a) \\ & \leq \frac{1}{4}(b-a) \\ & \times \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\max\{|\Gamma_1-\gamma_1|, |\Gamma_2-\gamma_2|\}. \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[|\Gamma_1-\gamma_1|^q + |\Gamma_2-\gamma_2|^q\right]^{1/q} \\ p > 1, \frac{1}{p}+\frac{1}{q} = 1, \\ \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[|\Gamma_1-\gamma_1| + |\Gamma_2-\gamma_2|\right]. \end{cases} \end{aligned} \tag{3.5}$$

Proof. Since $f' \in \bar{U}_{[a,x]}(\gamma_1, \Gamma_1) \cap \bar{U}_{[x,b]}(\gamma_2, \Gamma_2)$, then by taking the modulus in (2.1) for $\lambda_1(x) = \frac{\Gamma_1+\gamma_1}{2}$ and $\lambda_2(x) = \frac{\Gamma_2+\gamma_2}{2}$ we get

$$\begin{aligned}
&\Bigg| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&+ \frac{1}{2(b-a)}\left[(b-x)^2 \frac{\Gamma_2+\gamma_2}{2} - (x-a)^2 \frac{\Gamma_1+\gamma_1}{2}\right]\Bigg| \\
&\le \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t) - \frac{\Gamma_1+\gamma_1}{2}\right]dt\right| \\
&+ \frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t) - \frac{\Gamma_2+\gamma_2}{2}\right]dt\right| \\
&\le \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - \frac{\Gamma_1+\gamma_1}{2}\right|dt \\
&+ \frac{1}{b-a}\int_x^b (t-b)\left|f'(t) - \frac{\Gamma_2+\gamma_2}{2}\right|dt \\
&\le \frac{1}{b-a}\frac{|\Gamma_1-\gamma_1|}{2}\int_a^x (t-a)\,dt + \frac{1}{b-a}\frac{|\Gamma_2-\gamma_2|}{2}\int_x^b (b-t)\,dt \\
&= \frac{1}{4}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^2 + |\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^2\right](b-a)
\end{aligned}$$

and the first inequality in (3.5) is proved.

The last part follows by Hölder's inequality

$$mn + pq \le (m^\alpha + p^\alpha)^{1/\alpha}\left(n^\beta + q^\beta\right)^{1/\beta},$$

where $m, n, p, q \ge 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. □

Corollary 16. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x \in (a,b)$. Suppose that $\gamma, \Gamma \in \mathbb{C}$ with $\gamma \ne \Gamma$, and $f' \in \bar{U}_{[a,b]}(\gamma, \Gamma)$, then we have*

$$\begin{aligned}
&\left| f(x) + \left(\frac{a+b}{2} - x\right)\frac{\Gamma+\gamma}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \qquad (3.6)\\
&\le \frac{1}{2}|\Gamma-\gamma|\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a).
\end{aligned}$$

In particular, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}|\Gamma-\gamma|(b-a). \qquad (3.7)$$

Remark 17. If the derivative $f' : [a,b] \to \mathbb{R}$ is bounded above and below, that is, there exists the constants $M > m$ such that

$$-\infty < m \le f'(t) \le M < \infty \text{ for a.e. } t \in [a,b],$$

then we recapture from (3.6) the inequality (1.6).

Remark 18. Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$. Suppose that $\gamma_i,\Gamma_i\in\mathbb{C}$ with $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f'\in\bar{U}_{\left[a,\frac{a+b}{2}\right]}(\gamma_1,\Gamma_1)\cap\bar{U}_{\left[\frac{a+b}{2},b\right]}(\gamma_2,\Gamma_2)$, then we have from (3.5) that

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{8}(b-a)\left(\frac{\Gamma_2+\gamma_2}{2}-\frac{\Gamma_1+\gamma_1}{2}\right)\right| \\ &\le\frac{1}{16}\left[|\Gamma_1-\gamma_1|+|\Gamma_2-\gamma_2|\right](b-a).\end{aligned}\tag{3.8}$$

4 Inequalities for Derivatives of Bounded Variation

Assume that the function $f:I\to\mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b]\subset\mathring{I}$. Then, as in (2.15), we have the equality

$$\begin{aligned}&f(x)+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &+\frac{1}{4(b-a)}\left[(b-x)^2f'(b)-(x-a)^2f'(a)\right] \\ &=\frac{1}{b-a}\int_a^x(t-a)\left[f'(t)-\frac{f'(a)+f'(x)}{2}\right]dt \\ &+\frac{1}{b-a}\int_x^b(t-b)\left[f'(t)-\frac{f'(x)+f'(b)}{2}\right]dt,\end{aligned}\tag{4.1}$$

for any $x\in[a,b]$.

Theorem 19. *Let $f:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b]\subset\mathring{I}$. If the derivative $f':\mathring{I}\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)\right. \\ &\left.+\frac{1}{4(b-a)}\left[(b-x)^2f'(b)-(x-a)^2f'(a)\right]\right|\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\le\frac{1}{4}\left[\left(\frac{x-a}{b-a}\right)^2\bigvee_a^x(f')+\left(\frac{b-x}{b-a}\right)^2\bigvee_x^b(f')\right](b-a) \\ &\le\frac{1}{4}(b-a) \\ &\times\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b(f')+\frac{1}{2}\left|\bigvee_a^x(f')-\bigvee_x^b(f')\right|\right], \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x(f')\right]^q+\left[\bigvee_x^b(f')\right]^q\right]^{1/q} \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f'),\end{cases}\end{aligned}$$

for any $x\in[a,b]$.

Proof. Taking the modulus in (4.1) we have

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\ &\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\ &\leq \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t) - \frac{f'(a)+f'(x)}{2}\right] dt\right| \\ &+ \frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t) - \frac{f'(x)+f'(b)}{2}\right] dt\right| \\ &\leq \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| dt \\ &+ \frac{1}{b-a}\int_x^b (b-t)\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| dt. \end{aligned} \tag{4.3}$$

Since $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,x]$ and $[x,b]$, then

$$\begin{aligned} \left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| &= \frac{|f'(t) - f'(a) + f'(t) - f'(x)|}{2} \\ &\leq \frac{1}{2}\left[|f'(t) - f'(a)| + |f'(x) - f'(t)|\right] \\ &\leq \frac{1}{2}\bigvee_a^x (f') \end{aligned}$$

for any $t \in [a,x]$ and, similarly,

$$\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| \leq \frac{1}{2}\bigvee_x^b (f')$$

for any $t \in [x,b]$.

Then

$$\begin{aligned} \int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| dt &\leq \frac{1}{2}\bigvee_a^x (f')\int_a^x (t-a)\,dt \\ &= \frac{1}{4}(x-a)^2\bigvee_a^x (f') \end{aligned}$$

and

$$\begin{aligned} \int_x^b (b-t)\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| dt &\leq \frac{1}{2}\bigvee_x^b (f')\int_x^b (b-t)\,dt \\ &= \frac{1}{4}(b-x)^2\bigvee_x^b (f') \end{aligned}$$

and by (4.3) we get the desired inequality (4.2).

The last part follows by Hölder's inequality

$$mn + pq \leq (m^{\alpha} + p^{\alpha})^{1/\alpha} \left(n^{\beta} + q^{\beta}\right)^{1/\beta},$$

where $m, n, p, q \geq 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. $\square$

Corollary 20. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{16}(b-a)\left[f'(b) - f'(a)\right] \right| \tag{4.4}$$
$$\leq \frac{1}{16}(b-a) \bigvee_a^b (f').$$

Remark 21. If $p \in (a,b)$ is a median point in bounded variation for the derivative, i.e. $\bigvee_a^p (f') = \bigvee_p^b (f')$, then under the assumptions of Theorem 19, we have

$$\left| f(p) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - p\right) f'(p) \right. \tag{4.5}$$
$$\left. + \frac{1}{4(b-a)}\left[(b-p)^2 f'(b) - (p-a)^2 f'(a)\right] \right|$$
$$\leq \frac{1}{8}(b-a)\left[\frac{1}{4} + \left(\frac{p - \frac{a+b}{2}}{b-a}\right)^2\right] \bigvee_a^b (f').$$

5 Inequalities for Lipschitzian Derivatives

We say that $v : [a,b] \to \mathbb{C}$ is *Lipschitzian* with the constant $L > 0$, if it satisfies the condition

$$|v(t) - v(s)| \leq L|t - s| \text{ for any } t, s \in [a,b].$$

Theorem 22. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in (a,b)$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a,x]$ and constant $K_2(x)$ on $[x,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \tag{5.1}$$
$$\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right|$$

$$
\begin{aligned}
&\leq \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^{3} K_{1}(x)+\left(\frac{b-x}{b-a}\right)^{3} K_{2}(x)\right](b-a)^{2} \\
&\leq \frac{1}{8}(b-a)^{2} \\
&\times \begin{cases}
\left[\left(\frac{x-a}{b-a}\right)^{3}+\left(\frac{b-x}{b-a}\right)^{3}\right] \max \left\{K_{1}(x), K_{2}(x)\right\}, \\
\\
\left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[K_{1}^{q}(x)+K_{2}^{q}(x)\right]^{1/q} \\
p>1, \frac{1}{p}+\frac{1}{q}=1, \\
\\
\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^{3}\left[K_{1}(x)+K_{2}(x)\right].
\end{cases}
\end{aligned}
$$

Proof. Since $f^{\prime}: \mathring{I} \rightarrow \mathbb{C}$ is Lipschitzian with the constant $K_{1}(x)$ on $[a, x]$ and constant $K_{2}(x)$ on $[x, b]$, then

$$
\begin{aligned}
\left|f^{\prime}(t)-\frac{f^{\prime}(a)+f^{\prime}(x)}{2}\right| &= \frac{\left|f^{\prime}(t)-f^{\prime}(a)+f^{\prime}(t)-f^{\prime}(x)\right|}{2} \\
&\leq \frac{1}{2}\left[\left|f^{\prime}(t)-f^{\prime}(a)\right|+\left|f^{\prime}(x)-f^{\prime}(t)\right|\right] \\
&\leq \frac{1}{2} K_{1}(x)[|t-a|+|x-t|] \\
&= \frac{1}{2} K_{1}(x)(x-a)
\end{aligned}
$$

for any $t \in[a, x]$ and, similarly,

$$
\begin{aligned}
\left|f^{\prime}(t)-\frac{f^{\prime}(x)+f^{\prime}(b)}{2}\right| &\leq \frac{1}{2} K_{2}(x)[|t-x|+|b-t|] \\
&= \frac{1}{2} K_{2}(x)(b-x)
\end{aligned}
$$

for any $t \in[x, b]$.

Then

$$
\begin{aligned}
\int_{a}^{x}(t-a)\left|f^{\prime}(t)-\frac{f^{\prime}(a)+f^{\prime}(x)}{2}\right| d t &\leq \frac{1}{2} K_{1}(x)(x-a) \int_{a}^{x}(t-a) d t \\
&= \frac{1}{8}(x-a)^{3} K_{1}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{x}^{b}(b-t)\left|f^{\prime}(t)-\frac{f^{\prime}(x)+f^{\prime}(b)}{2}\right| d t &\leq \frac{1}{2} K_{2}(x)(b-x) \int_{x}^{b}(b-t) d t \\
&= \frac{1}{8}(b-x)^{3} K_{2}(x).
\end{aligned}
$$

Making use of the inequality (4.3) we deduce the first bound in (5.1).

The second part is obvious. □

Corollary 23. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant K on $[a,b]$ then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \tag{5.2}$$
$$\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right|$$
$$\leq \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right] K (b-a)^2$$

for any $x \in [a,b]$.

In particular, we have

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{16}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{5.3}$$
$$\leq \frac{1}{32} K (b-a)^2 .$$

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Weyl-Type Theorems For Restrictions Of Closed Linear Unbounded Operators

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Abstract

In this paper, T is a closed linear unbounded operator on an infinite dimensional complex Hilbert space H. We relate the study of Weyl-type theorems and properties for T to the study of Weyl-type theorems and properties for some restriction of T. Sufficient conditions are given for which T satisfies various Weyl-type theorems and properties if and only if $\mathcal{R}(T^n)$ is closed for some n $\in \mathbb{N}$ and some restriction of T satisfies the corresponding theorem or property.

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1 Introduction

Let H be an infinite dimensional complex Hilbert space and let $B(H)$ and $C(H)$ be the set of all bounded linear operators and set of all closed linear operators from domain $\mathcal{D}(\mathrm{T}) \subseteq H$ to H, respectively. By $\mathcal{N}(\mathrm{T})$ and $\mathcal{R}(\mathrm{T})$ we denote the null space and range of T, respectively. We call an operator T $\in C(H)$ *upper semi-Fredholm* (respectively, *lower semi-Fredholm*) if $\mathcal{R}(\mathrm{T})$ is closed and nullity of T, $\alpha(\mathrm{T})$ = dim $\mathcal{N}(\mathrm{T}) < \infty$ (respectively, defect of T, $\beta(\mathrm{T})$ = codim $\mathcal{R}(\mathrm{T}) < \infty$). A *semi-Fredholm operator* is either an upper or lower semi-Fredholm operator. If T is both upper and lower semi-Fredholm, that is, if $\alpha(\mathrm{T})$ and $\beta(\mathrm{T})$ both are finite, then T is called a *Fredholm operator*. By $SF_+(H)$ (respectively, $SF_-(H)$) we denote the class of all upper (respectively, lower) semi-Fredholm operators. For T $\in SF_+(H) \cup SF_-(H)$, index of T is defined as ind(T) = α(T) - β(T). An operator $T \in C(H)$ is called *Weyl* if it is Fredholm of index 0 and the *Weyl spectrum* of T is defined as $\sigma_w(\mathrm{T}) = \{\lambda \in \mathbb{C}$: T - λI is not Weyl$\}$. We have the following notations :

$$SF_+^-(H) = \{T \in C(H)\ :\ T \in SF_+(H) \text{ and } ind(T) \leqslant 0\}$$

$$SF_-^+(H) = \{T \in C(H)\ :\ T \in SF_-(H) \text{ and } ind(T) \geqslant 0\}$$

*corresponding author

and these operators generate the following spectrum

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SF_+^-(H)\}$$

$$\sigma_{SF_-^+}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SF_-^+(H)\}.$$

The *ascent* p(T) and *descent* q(T) of an operator T $\in C(H)$ are defined as follows:

$$p(\mathrm{T}) = inf\{n : \mathcal{N}(\mathrm{T}^n) = \mathcal{N}(\mathrm{T}^{n+1})\}$$

$$q(T) = inf\{n : \mathcal{R}(\mathrm{T}^n) = \mathcal{R}(\mathrm{T}^{n+1})\}.$$

Let σ(T), σ_a(T), σ_s(T) and ρ(T) denote the spectrum, approximate spectrum, surjective spectrum and the resolvent set of T $\in C(H)$, respectively, and let $\sigma_{des}(\mathrm{T}) = \{\lambda \in \mathbb{C}$: q(T - λI) $\not< \infty\}$ denote the *descent spectrum* of T. Evidently, $\sigma_{des}(\mathrm{T}) \subseteq \sigma(\mathrm{T})$.

By isoσ(T) and isoσ_a(T) we denote the isolated points of σ(T) and σ_a(T), respectively. It is well known that the resolvent operator R_λ(T) = (T - λI)$^{-1}$ is an analytic operator-valued function for all $\lambda \in \rho$(T) and the points of isoσ(T) are either poles or essential singularities of R$_\lambda$(T). For T $\in C(H)$, $\lambda \in$ isoσ(T) is said to be a *pole of order p* if p = p(T - λI) $< \infty$ and q(T - λI) $< \infty$ ([6]). Also $\lambda \in \sigma_a$(T) is said to be a *left-pole* if p = p(T - λI) $< \infty$ and $\mathcal{R}$(T - λI)$^{p+1}$ is closed. Let π_o(T) and π_o^a(T) denote the set of all poles of finite multiplicity and left poles of finite multiplicity, respectively.

An important property of closed linear operators in Fredholm theory is the single valued extension property (SVEP). This property was first introduced by Dunford [3]. We mainly concern with the SVEP at a point, localized version of SVEP, introduced by Finch [4], and relate it to the finiteness of the ascent of a closed linear operator. Let $T : \mathcal{D}(T) \subset H \to H$ be a closed linear mapping and let λ_o be a complex number. The operator T has the *single valued extension property* (SVEP) at λ_o, if $f = 0$ is the only solution to $(T - \lambda I)f(\lambda) = 0$ that is analytic in every neighborhood of λ_o. Also T has SVEP, if it has this property at every point λ_o in the complex plane.

Evidently, T $\in C(H)$ has SVEP at every $\lambda \in \rho$(T). Moreover, by identity theorem for analytic functions, it is easily seen that T has SVEP at every boundary point (in particular, every isolated point) of σ(T). Also, from the definition of localized SVEP,

$$\lambda \in \mathrm{iso}\sigma_a(T) \Longrightarrow \text{T has SVEP at } \lambda, \quad \text{and by duality,}$$
$$\lambda \in \mathrm{iso}\sigma_s(T) \Longrightarrow \text{T}^* \text{ has SVEP at } \lambda.$$

The above implications become equivalences whenever T is a bounded semi-Fredholm operator [1, Chapter 3]. For the case T $\in C(H)$, we prove this equivalence in the following theorem. We first give a definition and lemma needed for the proof of the theorem:

Definition 1. [5, Ch IV, §1] Let T $\in C(H)$. Let A be an operator such that $\mathcal{D}$(T) $\subset \mathcal{D}$(A) and $\|$Au$\| \leqslant$ a$\|$u$\|$ + b$\|$Tu$\|$, u $\in \mathcal{D}$(T) where a,b are non-negative constants. Then we say that *A is relatively bounded with respect to T* or *T-bounded* and the *T-bound* of A is inf b.

Lemma 2. *[5, Ch. IV, Theorem 5.31] Let $T \in C(H)$ be semi-Fredholm and let A be a T-bounded operator in H. Then $S = T + \lambda A \in C(H)$, S is semi-Fredholm and $\alpha(S)$ as well as $\beta(S)$ are constant for sufficiently small $|\lambda| > 0$.*

The *reduced minimum modulus* of $\mathrm{T} \in C(H)$ is defined by

$$\gamma(\mathrm{T}) = inf\,\{\|Tx\| : x \in \mathcal{D}(T) \cap \mathcal{N}(T)^{\perp}, \|x\| = 1\}.$$

Then, it is known that $\mathcal{R}(\mathrm{T})$ is closed iff $\gamma(\mathrm{T}) > 0$ for every $\mathrm{T} \in C(H)$.

Theorem 3. *Let $T \in C(H)$ be a semi-Fredholm operator and $\lambda_o \in \mathbb{C}$. Then the following are equivalent:*

(i) T has SVEP at λ_o

(ii) $\sigma_a(T)$ does not cluster at λ_o

(iii) $p(T - \lambda_o I) < \infty$.

Proof. We shall assume that $\lambda_o = 0$.

(i) $\Leftrightarrow$ (iii) follows from [4, Theorem 15]

(i) $\Rightarrow$ (ii) Suppose T has SVEP at zero. Since T is semi-Fredholm operator, so is T^n for all $n \in \mathbb{N}$. Then $\mathcal{R}(\mathrm{T}^n)$ is closed for all n, so that $\mathrm{T}^{\infty}(H) = \bigcap_{n=1}^{\infty} \mathcal{R}(\mathrm{T}^n)$ is closed.

Since T is semi-Fredholm, so that $\mathcal{R}(\mathrm{T})$ is closed, there exists an $\epsilon > 0$ such that $\gamma(\mathrm{T}) > \epsilon$. Consider λ in $0 < |\lambda| < \epsilon$. Then $\|\lambda x\| = |\lambda|\|x\| < \epsilon\|x\| < \gamma(\mathrm{T})\|x\|$, for all $x \in H$. By lemma 2, T - λI is a closed semi-Fredholm operator, so that $\mathcal{R}(\mathrm{T} - \lambda\mathrm{I})$ is closed for all $0 < |\lambda| < \epsilon$. Thus we have that if $0 < |\lambda| < \epsilon$, then $\lambda \in \sigma_a(\mathrm{T})$ iff $\lambda \in \sigma_p(\mathrm{T})$.

If $0 \neq x \in \mathcal{N}(\mathrm{T} - \lambda\mathrm{I})$ then $x = \frac{1}{\lambda}Tx = T(\frac{x}{\lambda}) \in \mathcal{R}(\mathrm{T})$. Also, $\mathrm{T}^2x = \mathrm{T}(\lambda x) = \lambda\mathrm{T}x = \lambda^2 x$. This implies $x = \frac{1}{\lambda^2}\mathrm{T}^2x \in \mathcal{R}(\mathrm{T}^2)$. Continuing like this, we get $x \in \mathrm{T}^{\infty}(H)$. Thus, $\mathcal{N}(\mathrm{T} - \lambda\mathrm{I}) \subseteq \mathrm{T}^{\infty}(H)$ for all $\lambda \neq 0$. This implies that every non-zero eigenvalue of T belongs to $\sigma(\mathrm{T}|_{T^{\infty}(H)})$.

Suppose that 0 is a cluster point of $\sigma_a(\mathrm{T})$. There exists a sequence (λ_n) of non-zero eigenvalues of T such that $\lambda_n \to 0$ as $n \to \infty$. Then $\lambda_n \in \sigma(\mathrm{T}|_{T^{\infty}(H)})$ so that $0 \in \sigma(\mathrm{T}|_{T^{\infty}(H)})$, as the spectrum of an operator is closed. Since T has SVEP at 0, so does $\mathrm{T}|_{T^{\infty}(H)}$. From [1, Lemma 1.9], $\mathrm{T}|_{T^{\infty}(H)}$ is onto. By [4, Theorem 2], $\mathrm{T}|_{T^{\infty}(H)}$ is injective so that $0 \notin \sigma(\mathrm{T}|_{T^{\infty}(H)})$, which is a contradiction. Therefore, $\sigma_a(\mathrm{T})$ does not cluster at 0.

(ii) $\Rightarrow$ (i) holds for all closed linear operators. □

Remark 4. By duality, we have that if $\mathrm{T} \in C(H)$ is semi-Fredholm, then the following statements are equivalent:

(i) T^* has SVEP at λ_o

(ii) $\sigma_s(\mathrm{T})$ does not cluster at λ_o

(iii) q(T - λ_oI) $< \infty$.

Let $\mathrm{E}_o(\mathrm{T})$ and $\mathrm{E}_o^a(\mathrm{T})$ denote the set of all eigenvalues of finite multiplicities in iso$\sigma(\mathrm{T})$ and iso$\sigma_a(\mathrm{T})$, respectively. If $\mathrm{T} \in C(H)$, then T satisfies:

(i) Weyl's theorem if $\sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \mathrm{E}_o(\mathrm{T})$.

(ii) Browder's theorem if $\sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \pi_o(\mathrm{T})$.

(iii) a-Browder's theorem if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) = \pi_o^a(\mathrm{T})$.

(iv) a-Weyl's theorem if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) = \mathrm{E}_o^a(\mathrm{T})$.

(v) property (w) if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) = \mathrm{E}_o(\mathrm{T})$.

(vi) property (b) if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) = \pi_o(\mathrm{T})$.

(vii) property (ab) if $\sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \pi_o^a(\mathrm{T})$.

(viii) property (aw) if $\sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \mathrm{E}_o^a(\mathrm{T})$.

Weyl's theorem and a-Weyl's theorem for restriction of bounded linear operators have been recently studied in [2]. In this paper, it is shown that with certain sufficient conditions on $\mathrm{T} \in C(H)$, several Weyl-type theorems and properties hold for T, if and only if there exists an n such that $\mathcal{R}(\mathrm{T}^n)$ is closed and Weyl type theorems and properties holds for $\mathrm{T}_n = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^n)}$.

In the second section, we consider Weyl's theorem and Browder's theorem and certain conditions have been given for which the study of Weyl's (respectively, Browder's) theorem for $\mathrm{T} \in C(H)$, can be reduced to the study of Weyl's (respectively, Browder's) theorem for some restriction of T. Also, sufficiency theorems are given for the case of a-Weyl's theorem and a-Browder's theorem. In the third section, properties (w), (b), (aw) and (ab) are considered. An example is given at the end of the third section to illustrate all the theorems proved.

Following lemma will be used throughout the paper:

Lemma 5. *[2, Lemma 2.1] Let $T \in C(H)$ and T_n be the restriction of T to the subspace $\mathcal{R}(T^n)$. Then, for all $\lambda \neq 0$, we have:*

(i) $\mathcal{N}((T_n - \lambda I)^m) = \mathcal{N}((T - \lambda I)^m)$, *for all m;*

(ii) $\mathcal{R}((T_n - \lambda I)^m) = \mathcal{R}((T - \lambda I)^m) \cap \mathcal{R}(T^n)$, *for all m;*

(iii) $\alpha(T_n - \lambda I) = \alpha(T - \lambda I)$;

(iv) $p(T_n - \lambda I) = p(T - \lambda I)$;

(v) $\beta(T_n - \lambda I) = \beta(T - \lambda I)$.

2 Weyl-type theorems and Restriction of operators

In this section, we give conditions under which Weyl's theorem (respectively, Browder's theorem) for an operator $\mathrm{T} \in C(H)$ is equivalent to Weyl's theorem (respectively, Browder's theorem) for certain restriction T_n of T. Also, we give certain sufficient conditions in the case of a-Browder's theorem and a-Weyl's theorem. Let $\mathbb{N}$ denote the set of all non-negative integers.

Theorem 6. *Let $T \in C(H)$ and suppose that $0 \notin iso\sigma(T) \cap \sigma_{des}(T)$. Then*

(i) T satisfies Weyl's theorem iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies Weyl's theorem.

(ii) T satisfies Browder's theorem iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies Browder's theorem.

Proof. (i) Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies Weyl's theorem.

Let $\lambda \in \mathrm{E}_o(\mathrm{T})$. Then $\lambda \in \mathrm{iso}\sigma(\mathrm{T})$ so that $\lambda \neq 0$ and and there exists an open disk $\mathbb{D}_\lambda \subseteq \mathbb{C}$ centered at λ such that $\sigma(\mathrm{T}) \cap \mathbb{D}_\lambda = \{\lambda\}$. Since, by Lemma 5, $\sigma(\mathrm{T}) \setminus \{0\}$

$= \sigma(\mathrm{T}_n) \setminus \{0\}$ and $\sigma(\mathrm{T}_n) \subseteq \sigma(\mathrm{T})$, we have that $\sigma(\mathrm{T}_n) \cap \mathbb{D}_\lambda = \{\lambda\}$ so that $\lambda \in \mathrm{iso}\sigma(\mathrm{T}_n)$. Now, $0 \neq \lambda \in \mathrm{E}_o(\mathrm{T}_n) = \sigma(\mathrm{T}_n) \setminus \sigma_w(\mathrm{T}_n)$ and thus, $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T})$.

Now, suppose $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T})$. If $\lambda = 0$, then since $0 \notin \sigma_{des}(\mathrm{T})$, [1, Theorem 3.4(iv)] implies $0 \in \mathrm{iso}\sigma(\mathrm{T})$ which is a contradiction to the hypothesis. Therefore, $\lambda \neq 0$ and $\alpha(\mathrm{T}_n - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) = \beta(\mathrm{T} - \lambda\mathrm{I}) = \beta(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ so that $\lambda \in \sigma(\mathrm{T}_n) \setminus \sigma_w(\mathrm{T}_n) = \mathrm{E}_o(\mathrm{T}_n)$ and thus $\lambda \in \mathrm{iso}\sigma(\mathrm{T}_n)$. Then T_n and T_n^* have SVEP at λ and by Theorem 3, $\mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) = \mathrm{q}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$. Hence, $\mathrm{q}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) < \infty$ and $\lambda \in \mathrm{E}_o(\mathrm{T})$.

Conversely, assume T satisfies Weyl's theorem, then for $n = 0$, $\mathcal{R}(\mathrm{T}^0) = \mathrm{H}$ is closed and $\mathrm{T}_o = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = \mathrm{T}$ satisfies Weyl's theorem.

(ii) Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies Browder's theorem.

Let $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T})$. As proved above, $0 \neq \lambda \in \sigma(\mathrm{T}_n) \setminus \sigma_w(\mathrm{T}_n) = \sigma(\mathrm{T}_n) \setminus \sigma_b(\mathrm{T}_n)$, and thus $\mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) = \mathrm{q}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$. Then, $\mathrm{q}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) < \infty$. Therefore, $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_b(\mathrm{T})$. Hence, $\sigma_b(\mathrm{T}) \subseteq \sigma_w(\mathrm{T})$ and since the reverse inclusion holds for every operator in $C(H)$, T satisfies Browder's theorem.

Now, suppose T satisfies Browder's theorem, then for $n = 0$, $\mathcal{R}(\mathrm{T}^0) = \mathrm{H}$ is closed and $\mathrm{T}_o = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = \mathrm{T}$ satisfies Browder's theorem.

□

Recently, in [7], *property* (w_1) for bounded linear operators, as a variant of Weyl's theorem, was introduced and studied. Further in [8], property (aw_1) was introduced as a variant of a-Weyl's theorem, where we say that an operator T satisfies *property* (aw_1) if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) \subset \mathrm{E}_o^a(\mathrm{T})$.

Similarly, we introduce a variant of a-Browder's theorem, viz. *property* (aB_1), where an operator $\mathrm{T} \in C(H)$ satisfies property (aB_1) if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) \subset \pi_o^a(\mathrm{T})$.

For the case of a-Browder's theorem and a-Weyl's theorem, we do not have necessary and sufficient conditions similar to Theorem 6. However, we give the following sufficiency theorems:

Theorem 7. *Let $T \in C(H)$ and suppose that $0 \notin \mathrm{iso}\sigma(T) \cap \sigma_{des}(T)$. Then T satisfies property (aB_1) if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies a-Browder's theorem.*

Proof. Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies a-Browder's theorem.

If $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$, then $0 < \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$, $\mathcal{R}(\mathrm{T} - \lambda\mathrm{I})$ is closed and $ind(\mathrm{T} - \lambda\mathrm{I}) \leqslant 0$. Suppose $\lambda = 0$, then $ind(\mathrm{T}) \leqslant 0$ and by hypothesis, $0 \notin \sigma_{des}(\mathrm{T})$ implies $\mathrm{q}(\mathrm{T}) < \infty$ so that $ind(\mathrm{T}) \geqslant 0$. Now $ind(\mathrm{T}) = 0$ together with [1, Theorem 3.4(iv)] implies $\mathrm{p}(\mathrm{T}) = \mathrm{q}(\mathrm{T}) < \infty$ which is a contradiction since $0 \notin \mathrm{iso}\sigma(\mathrm{T})$. Therefore, $\lambda \neq 0$ and by Lemma 5, $\lambda \in \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n) = \pi_o^a(\mathrm{T}_n)$. Thus, $\mathrm{p} = \mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$. Also, since $\mathrm{T} - \lambda\mathrm{I}$ is semi-fredholm, so is $(\mathrm{T} - \lambda\mathrm{I})^{p+1}$. Thus, $\mathcal{R}(\mathrm{T} - \lambda\mathrm{I})^{p+1}$ is closed and $\lambda \in \pi_o^a(\mathrm{T})$. Hence, T satisfies property (aB_1). □

Theorem 8. *Let $T \in C(H)$ and suppose that $0 \notin \mathrm{iso}\sigma_a(T) \cap \sigma_{des}(T)$. Then T satisfies property (aw_1) if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies a-Weyl's theorem.*

Proof. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies a-Weyl's theorem.

Suppose $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$. If $\lambda = 0$, then $ind(\mathrm{T}) \leqslant 0$ and $0 \notin \sigma_{des}(\mathrm{T})$ implies $\mathrm{q}(\mathrm{T}) < \infty$ and thus, $ind(\mathrm{T}) = 0$. Now, $\mathrm{p}(\mathrm{T}) = \mathrm{q}(\mathrm{T}) < \infty$ and $0 \in \mathrm{iso}\sigma(\mathrm{T})$ which is a contradiction to the hypothesis. Therefore, $0 \neq \lambda \in \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n) = \mathrm{E}_o^a(\mathrm{T}_n)$. Since $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T}_n)$, by Theorem 3, $\mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ and now $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T})$. Thus, $\lambda \in \mathrm{E}_o^a(\mathrm{T})$ so that T satisfies property (aw_1). □

3 Extended Weyl-type theorems and Restriction of operators

In this section, we give conditions under which property (w) (respectively, property (b), property (aw) and property (ab)) for an operator $\mathrm{T} \in C(H)$ is equivalent to property (w) (respectively, property (b), property (aw) and property (ab)) for certain restriction T_n of T.

Theorem 9. *Let $T \in C(H)$ and suppose that $0 \notin iso\sigma(T) \cap \sigma_{des}(T)$. Then*

(i) T satisfies property (w) iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (w).

(ii) T satisfies property (b) iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (b).

(iii) T satisfies property (ab) iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (ab).

Proof. (i) Assume that there exists an $\mathrm{n} \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies property (w).

Let $\lambda \in \mathrm{E}_o(\mathrm{T})$. Then proceeding as in Theorem 6(i), $0 \neq \lambda \in \mathrm{E}_o(\mathrm{T}_n) = \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n)$. By Theorem 3, since $\lambda \in \mathrm{iso}\sigma(\mathrm{T}_n)$, $\mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) = \mathrm{q}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$. Then, [1, Theorem 3.4(iii)] implies $\beta(\mathrm{T}_n - \lambda\mathrm{I}) = \alpha(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ so that $0 < \beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ and hence $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$.

Now, suppose $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$. Then, $0 \neq \lambda \in \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n) = \mathrm{E}_o(\mathrm{T}_n)$. Using SVEP for T_n and T_n^* at λ, $\mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) = \mathrm{q}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ and thus, $\beta(\mathrm{T}_n - \lambda\mathrm{I}) = \alpha(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$. Now, $\beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ together with $\mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ imply $\mathrm{q}(\mathrm{T} - \lambda\mathrm{I}) < \infty$ and thus $\lambda \in \mathrm{E}_o(\mathrm{T})$. Hence, T satisfies property (w).

Conversely, if T satisfies property (w), then for $n = 0$, $\mathcal{R}(\mathrm{T}^0) = \mathrm{H}$ is closed and $\mathrm{T}_o = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = \mathrm{T}$ satisfies property (w).

(ii) Assume that there exists an $\mathrm{n} \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies property (b).

Let $\lambda \in \pi_o(\mathrm{T})$. Then $\mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{q}(\mathrm{T} - \lambda\mathrm{I}) < \infty$ together with $\alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ imply $0 < \beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ and thus, $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$.

Now, suppose $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$. As in the proof of Theorem 7, we get that $0 \neq \lambda \in \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n) = \pi_o(\mathrm{T}_n)$. Then, $\mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) = \mathrm{q}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ so that $\beta(\mathrm{T}_n - \lambda\mathrm{I}) = \alpha(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$. Now, $\beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ together with $\mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ imply $\mathrm{q}(\mathrm{T} - \lambda\mathrm{I}) < \infty$ and thus $\lambda \in \pi_o(\mathrm{T})$. Hence, T satisfies property (b).

Conversely, if T satisfies property (b), then for $n = 0$, $\mathcal{R}(\mathrm{T}^0) = \mathrm{H}$ is closed and $\mathrm{T}_o = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = \mathrm{T}$ satisfies property (b).

(iii) Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies property (ab).

Let $\lambda \in \pi_o^a(\mathrm{T})$. Then, proceeding as in Theorem 6, $0 \neq \lambda \in \pi_o^a(\mathrm{T}_n) = \sigma(\mathrm{T}_n) \setminus \sigma_w(\mathrm{T}_n)$. and thus $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T})$.

Now, suppose $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T})$. If $\lambda = 0$, then by [1, Theorem 3.4(iv)], $0 < \alpha(\mathrm{T}) = \beta(\mathrm{T}) < \infty$ and $0 \notin \sigma_{des}(\mathrm{T})$ together imply that $0 \in \mathrm{iso}\sigma(\mathrm{T})$, which is a contradiction to the hypothesis. Therefore, $0 \neq \lambda \in \sigma(\mathrm{T}_n) \setminus \sigma_w(\mathrm{T}_n) = \pi_o^a(\mathrm{T}_n)$ so that $\lambda \in \sigma_a(\mathrm{T}_n)$, $\mathrm{p} = \mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$, $\alpha(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ and $\mathcal{R}(\mathrm{T}_n - \lambda\mathrm{I})^{p+1}$ closed. Now, $\lambda \in \sigma_a(\mathrm{T})$, $\mathrm{p} = \mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) < \infty$ and $\alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$. Also, $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T})$ so that $\mathrm{T} - \lambda\mathrm{I}$ and hence $(\mathrm{T} - \lambda\mathrm{I})^{p+1}$ is a Fredholm operator. Then, $\mathcal{R}(\mathrm{T} - \lambda\mathrm{I})^{p+1}$ is closed and so $\lambda \in \pi_o^a(\mathrm{T})$. Hence, T satisfies property (ab).

Conversely, suppose T satisfies property (ab), then for $n = 0$, $\mathcal{R}(\mathrm{T}^0) = \mathrm{H}$ is closed and $\mathrm{T}_o = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = \mathrm{T}$ satisfies property (ab).

□

Theorem 10. *Let $T \in C(H)$ and suppose $0 \notin iso\sigma_a(T) \cap \sigma_{des}(T)$. Then T satisfies property (aw) iff there exists an $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (aw).*

Proof. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies property (aw).

Suppose $\lambda \in \mathrm{E}_o^a(\mathrm{T})$. Then $\lambda \neq 0$, $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T})$ and $0 < \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ so that $0 < \alpha(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ and thus $\lambda \in \sigma_a(\mathrm{T}_n)$. Since $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T})$, there exists a disk $\mathbb{D}_\lambda \subseteq \mathbb{C}$ such that $\mathbb{D}_\lambda \cap \sigma_a(\mathrm{T}) = \{\lambda\}$. Then $\mathbb{D}_\lambda \cap \sigma_a(\mathrm{T}_n) \subseteq \{\lambda\}$. If $\mathbb{D}_\lambda \cap \sigma_a(\mathrm{T}_n) = \phi$, then $\lambda \notin \sigma_a(\mathrm{T}_n)$ which is a contradiction. Thus $\mathbb{D}_\lambda \cap \sigma_a(\mathrm{T}_n) = \{\lambda\}$ and so $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T}_n)$. Now, $\lambda \in \mathrm{E}_o^a(\mathrm{T}_n) = \sigma(\mathrm{T}_n) \setminus \sigma_w(\mathrm{T}_n)$. Since $\lambda \neq 0$, we get that $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T})$.

Now, let $\lambda \in \sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T})$. Then, $0 \neq \lambda \in \sigma(\mathrm{T}_n) \setminus \sigma_w(\mathrm{T}_n) = \mathrm{E}_o^a(\mathrm{T}_n)$. Since $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T}_n)$, by Theorem 3, $\mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{p}(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ and $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T})$. Therefore, $\lambda \in \mathrm{E}_o^a(\mathrm{T})$ and hence, T satisfies property (aw).

Conversely, suppose T satisfies property (aw), then for $n = 0$, $\mathcal{R}(\mathrm{T}^0) = \mathrm{H}$ is closed and $\mathrm{T}_o = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = \mathrm{T}$ satisfies property (aw). □

The following example illustrates all the theorems of this paper:

Example 11. Let $H = l^2$ and let T be defined as follows:

$$T(x_1, x_2, x_3, \dots) = (x_1, 2x_2, 3x_3, 4x_4, 5x_5, \dots)$$
$$\text{where} \quad \mathcal{D}(T) = \left\{(x_1, x_2, x_3, \dots) \in l^2 : \sum_{j=1}^{\infty} |jx_j|^2 < \infty\right\}.$$

Then $c_{oo} = \{(x_n) : x_n \neq 0$ for only finitely many $n\} \subseteq \mathcal{D}(\mathrm{T})$. Since c_{oo} is dense in l^2, so is $\mathcal{D}(\mathrm{T})$. Also $\mathrm{T} = \mathrm{T}^*$, so that T is a closed linear operator.

Now, $\sigma(\mathrm{T}) = \sigma_a(\mathrm{T}) = \sigma_p(\mathrm{T}) = \{j : j \in \mathbb{N}\}$ and $0 \notin \mathrm{iso}\sigma(\mathrm{T}) \cap \sigma_{des}(\mathrm{T})$.

For $\lambda = j$, $j \in \mathbb{N}$,

$$\mathcal{N}(T - \lambda I) = span\{e_j\} \Longrightarrow \alpha(T - \lambda I) = 1 < \infty \quad \text{and}$$
$$\mathcal{R}(T - \lambda I) = span\{e_j\}^{\perp} \Longrightarrow \mathcal{R}(T - \lambda I) \text{ is closed and } \beta(T - \lambda I) = 1 < \infty.$$

Therefore, T - λI is Fredholm operator of index 0 and thus $\sigma_w(\mathrm{T}) = \sigma_{SF_+^-}(\mathrm{T}) = \phi$.

Since $\lambda = j$, for $j \in \mathbb{N}$ is isolated in $\sigma(\mathrm{T}) = \sigma_a(\mathrm{T})$ and $\alpha(\mathrm{T} - \lambda\mathrm{I}) = 1 < \infty$, thus $\mathrm{E}_o(\mathrm{T}) = \mathrm{E}_o^a(\mathrm{T}) = \{\ j : j \in \mathbb{N}\ \}$. Also $\mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{q}(\mathrm{T} - \lambda\mathrm{I}) = 1$, $\pi_o(\mathrm{T}) = \pi_o^a(\mathrm{T}) = \{j : j \in \mathbb{N}\}$.

Hence:

$$\sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \{j : j \in \mathbb{N}\} = E_o(T),$$

i.e., *Weyl's theorem* and *property (w)* hold for T.

$$\sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \{j : j \in \mathbb{N}\} = \pi_o(T),$$

i.e., *Browder's theorem* and *property (b)* hold for T.

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_w(T) = \{j : j \in \mathbb{N}\} = E_o^a(T),$$

i.e., *property* (aw_1) and *property (aw)* hold for T.

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_w(T) = \{j : j \in \mathbb{N}\} = \pi_o^a(T),$$

i.e., *property* (aB_1) and *property (ab)* hold for T.

Infact, $\mathcal{R}(\mathrm{T}^n)$ is closed and $\mathrm{T}_n = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^n)}$ satisfies the corresponding Weyl-type theorems and properties for all $n \in \mathbb{N}$.

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Certain results on a class of Entire functions represented by Dirichlet series having complex frequencies

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Abstract

Consider F to be a class of entire functions represented by Dirichlet series with complex frequencies for which $(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_k|$ is bounded. A study on certain results has been made for this set that is F is proved to be an algebra with continuous quasi-inverse, commutative Banach algebra with identity etc. Moreover, the conditions for the elements of F to possess an inverse, quasi-inverse and the form of spectrum of F are also established.

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1 Introduction

Consider a Dirichlet series of the form

$$f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k, z>}, \qquad z \in \mathbb{C}^n \tag{1.1}$$

where $\{\lambda^k\}$; $\lambda^k = (\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k),\ k = 1, 2, \ldots$ be a sequence of complex vectors in $\mathbb{C}^n$. Then $< \lambda^k, z >= \lambda_1^k z_1 + \lambda_2^k z_2 + \ldots + \lambda_n^k z_n$. If $a_k{}'s \in \mathbb{C}$ and $\{\lambda^k\}'s$ satisfy the condition $|\lambda^k| \to \infty$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \frac{\log |a_k|}{|\lambda^k|} = -\infty \tag{1.2}$$

$$\limsup_{k\to\infty} \frac{\log\, k}{|\lambda^k|} = D < \infty \tag{1.3}$$

then from [1] the Dirichlet series (1.1) represents an entire function. In this paper let F be the set of series (1.1) for which $(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_k|$ is bounded where $c_1, c_2 \geq 0$ and

c_1, c_2 are simultaneously not zero. Then every element of F represents an entire function. If

$$f(z) = \sum_{k=1}^{\infty} a_k \, e^{<\lambda^k, z>} \qquad \text{and} \qquad g(z) = \sum_{k=1}^{\infty} b_k \, e^{<\lambda^k, z>}$$

define binary operations that is addition and scalar multiplication in F as

$$f(z) + g(z) = \sum_{k=1}^{\infty} (a_k + b_k) \, e^{<\lambda^k, z>},$$

$$\alpha . f(z) = \sum_{k=1}^{\infty} (\alpha . a_k) \, e^{<\lambda^k, z>},$$

$$f(z).g(z) = \sum_{k=1}^{\infty} \{(k!)^{c_1} \, e^{c_2 k |\lambda^k|} \, a_k \, b_k\} \, e^{<\lambda^k, z>}.$$

The norm in F is defined as follows

$$\|f\| = \sup_{k \geq 1} (k!)^{c_1} \, e^{c_2 k |\lambda^k|} \, |a_k|. \tag{1.4}$$

If $c_1 = c_2 = 1$ we get the norm as defined in [3] for a class of entire functions represented by Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad s = \sigma + it, \qquad (\sigma, t \in \mathbb{R}) \tag{1.5}$$

whose coefficients belonged to a commutative Banach algebra with identity and $\lambda_n{}'s \in \mathbb{R}$ satisfied the condition $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_n \ldots ; \lambda_n \to \infty$ as $n \to \infty$. Further in the same paper authors proved the above class to be a complex FK-space and a Fréchet space. Several other results for a different class of entire Dirichlet series (1.5) may be found in [2].

In the present paper the weighted norm is generalized and various results based on the notions of Banach algebra, Quasi-inverse, Algebra with continuous quasi-inverse, Spectrum of a set have been established.

In the sequel following definitions are required to prove the main results.

Definition 1. A function $g(z) \in F$ is said to be a quasi-inverse of $f(z) \in F$ if $f(z) * g(z) = 0$ where

$$f(z) * g(z) = f(z) + g(z) + f(z).g(z).$$

Definition 2. A topological algebra F is said to be an algebra with continuous quasi-inverse if there exists a neighbourhood of the zero element, every point f of which has a quasi-inverse f' and the mapping $f \to f'$ is continuous.

Definition 3. The set $\sigma(A)$ defined as

$$\sigma(A) = \{k \in K : A - kI \text{ is not invertible}\}$$

is called the spectrum of A.

2 Main Results

In this section main results are proved. For the definitions of terms used refer [4] and [5].

Theorem 4. *An element* $f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k, z>} \in F$ *is quasi-invertible if and only if*

$$\inf_{k \geq 1} \{|1 + (k!)^{c_1}\, e^{c_2 k |\lambda^k|}\, a_k|\} > 0. \tag{2.1}$$

The quasi-inverse of $f(z)$ *is the function* $g(z) = \sum_{k=1}^{\infty} b_k\, e^{<\lambda^k, z>}$ *where*

$$b_k = \frac{-a_k}{1 + (k!)^{c_1}\, e^{c_2 k |\lambda^k|}\, a_k}. \tag{2.2}$$

Proof. Let $f(z) \in F$ be quasi-invertible. By Definition 1, there exists $g(z) \in F$ such that $f(z) * g(z) = 0$. This implies

$$a_k + b_k + (k!)^{c_1}\, e^{c_2 k |\lambda^k|}\, a_k\, b_k = 0$$

for all $k \geq 1$. Let (2.1) does not hold that is

$$\inf_{k \geq 1} \{|1 + (k!)^{c_1}\, e^{c_2 k |\lambda^k|}\, a_k|\} = 0. \tag{2.3}$$

There exists a subsequence $\{k_t\}$ of a sequence of indices $\{k\}$ such that $\|f_t\| = 1$ that is

$$(k_t!)^{c_1}\, e^{c_2 k_t |\lambda^{k_t}|}\, |a_{k_t}| = 1 \ \text{ as } \ t \to \infty. \tag{2.4}$$

Thus

$$(k_t!)^{c_1}\, e^{c_2 k_t |\lambda^{k_t}|}\, |b_{k_t}| = \frac{(k_t!)^{c_1}\, e^{c_2 k_t |\lambda^{k_t}|}\, |a_{k_t}|}{|1 + (k_t!)^{c_1}\, e^{c_2 k_t |\lambda^{k_t}|}\, a_{k_t}|}$$

Using (2.3) and (2.4),

$$\|g_t\| \to \infty \ \text{ as } \ t \to \infty$$

which is a contradiction.
Conversely let (2.1) be fulfilled. The function $g(z)$ defined by (2.2) obviously belongs to F. Thus

$$\begin{aligned} f(z) * g(z) &= \sum_{k=1}^{\infty} \{a_k + b_k + (k!)^{c_1}\, e^{c_2 k |\lambda^k|}\, a_k\, b_k\}\, e^{<\lambda^k, z>} \\ &= 0. \end{aligned}$$

Thus $f(z)$ is quasi-invertible which completes the proof of the theorem. □

Theorem 5. *F is an algebra with continuous quasi-inverse.*

Proof. Let $N_\epsilon(0)$ be an ϵ-neighbourhood of 0 where $0 < \epsilon < 1$. Let $p(z) \in N_\epsilon(0)$ where $p(z) = \sum_{k=1}^{\infty} p_k\, e^{<\lambda^k, z>}$. This implies $\|p\| < \epsilon$. Then

$$(k!)^{c_1}\, e^{c_2 k |\lambda^k|}\, |p_k| < \epsilon \ \text{ for all } \ k \geq 1$$

which further implies

$$\inf_{k\geq 1}\{|1+(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, p_k|\}\geq 1-\epsilon>0.$$

Hence by Theorem 4, $p(z)$ possesses a quasi-inverse say $q(z)=\sum_{k=1}^{\infty} q_k\, e^{<\lambda^k,z>}$ where

$$q_k=\frac{-p_k}{1+(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, p_k}.$$

Now

$$\begin{aligned}
\|q\| &= \sup_{k\geq 1}(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|q_k| \\
&= \sup_{k\geq 1}\frac{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|p_k|}{|1+(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, p_k|} \\
&< \frac{\epsilon}{1-\epsilon}.
\end{aligned}$$

Hence the mapping $p(z)\rightarrow q(z)$ is continuous. Thus by Definition 2, F is an algebra with continuous quasi-inverse. Thus the theorem is proved. □

Theorem 6. *F is a commutative Banach algebra with identity.*

Proof. To prove the theorem we first show that F is complete under the norm defined by (1.4). Let $\{f_{m_1}\}$ be a cauchy sequence in F. For given $\epsilon>0$ we find m such that

$$\|f_{m_1}-f_{m_2}\|<\epsilon \text{ where } m_1,m_2\geq m.$$

This implies that

$$\sup_{k\geq 1}(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|a_{m_{1k}}-a_{m_{2k}}|<\epsilon \text{ where } m_1,m_2\geq m.$$

Clearly $\{a_{m_{1k}}\}$ forms a cauchy sequence in the set of complex numbers for all $k\geq 1$ and thus converges to a_k. Therefore $f_{m_1}\rightarrow f$. Also

$$\begin{aligned}
\sup_{k\geq 1}(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|a_k| \leq& \sup_{k\geq 1}(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|a_{m_{1k}}-a_k| \\
&+\sup_{k\geq 1}(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|a_{m_{1k}}|
\end{aligned}$$

Hence $f(z)\in F$. Thus F is complete. Now if $f(z),g(z)\in F$ then

$$\begin{aligned}
\|f.g\| &= \sup_{k\geq 1}(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|\,(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k\, b_k| \\
&\leq \sup_{k\geq 1}(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|a_k|\,.\sup_{k\geq 1}(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\,|b_k| \\
&= \|f\|.\|g\|
\end{aligned}$$

The identity element in F is

$$e(z)=\sum_{k=1}^{\infty}(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, e^{<\lambda^k,z>}.$$

Hence the theorem. □

Theorem 7. *The function* $f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k,z>}$ *is invertible in* F *if and only if*

$$\{|\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|\}$$

is a bounded sequence.

Proof. Let $f(z)$ be invertible and $g(z) = \sum_{k=1}^{\infty} b_k\, e^{<\lambda^k,z>}$ be its inverse. Then

$$(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k\, b_k = (k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}$$

Equivalently

$$(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |b_k| = |\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|$$

Clearly since $g(z) \in F$ hence

$$\{|\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|\}$$

is a bounded sequence.

Conversely suppose $\{|\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|\}$ be a bounded sequence. Define $g(z)$ such that

$$g(z) = \sum_{k=1}^{\infty} (k!)^{-2c_1}\, e^{-2c_2 k|\lambda^k|}\, a_k^{-1}\, e^{<\lambda^k,z>}.$$

Obviously $g(z) \in F$. Moreover

$$\begin{aligned} f(z).g(z) &= \sum_{k=1}^{\infty} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, \{a_k\,(k!)^{-2c_1}\, e^{-2c_2 k|\lambda^k|}\, a_k^{-1}\}\, e^{<\lambda^k,z>} \\ &= e(z). \end{aligned}$$

Hence the proof of the theorem is completed. □

Theorem 8. *The spectrum* $\sigma(f)$ *where* $f(z) \in F$ *is precisely of the form*

$$\sigma(f) = cl\{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k : k \geq 1\}.$$

Proof. In Theorem 7, $f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k,z>} \in F$ is invertible if and only if

$$\{|\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|\}$$

is a bounded sequence. Thus $\{f(z) - \lambda.e(z)\}$ is not invertible if and only if

$$\{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |\, a_k - \lambda\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, |\}^{-1}$$

is not bounded. Therefore by Definition 3, this is possible if and only if there exists a subsequence $\{k_n\}$ of a sequence of indices $\{k\}$ such that

$$|(k_n!)^{c_1}\, e^{c_2 k_n|\lambda^{k_n}|}\, a_{k_n} - \lambda\,|$$

tends to zero as $n \to \infty$. Equivalently

$$\lambda \in\, cl\{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k :\, k \geq 1\}$$

which proves the theorem. □

The results proved in this section would further be useful in the study of the spaces like FK-space, Fréchet space, Montel space, C^*-algebra etc. and in the study of functions preserving the asymptotic equivalence of functions and sequences that is Pseudo-regularly varying (PRV) functions. Also these results have significant applications in the fields of topology, functional analysis, modern analysis etc.

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On Stability and Boundedness of Solutions of Certain Non Autonomous Fourth-Order Delay Differential Equations

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Abstract
This paper establishes sufficient conditions to ensure the boundedness and asymptotic stability of solutions of a certain fourth order nonlinear non-autonomous differential equation by constructing Lyapunov functionals.

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1 Introduction

In applied science, some practical problems are associated with higher-order nonlinear differential equations, such as , electronic theory [22], biological model and other models [8] and [18] . Many results concerning the stability and boundedness of solutions of fourth order differential equations without delay have been obtained in view of various methods, especially, Lyapunov's method, see, the book of Reissig et al. [26] as a survey and the papers of Abou-El-Ela and Sadek [1], Adesina and Ogundare [3], Cartwright [9], Chukwu [10], Ezeilo [12], [14] Ezeilo and Tejumola [15], Harrow [16], Hu [17], Tejumola [30], Tunc [36], [37], [38], [39], Wu and Xiong [44] and the references cited therein. Besides, it should be noted that there are only a few results on the same problem for nonlinear differential equations of fourth order with delay, it have been discussed by a few authors, see, Bereketoglu [4], Abou-El-Ela et al. [2], Kang and Si [20], Sadek [27], Sinha [28], Tejumola [31], and Tunc [40], [41], [42]. The most efficient tool for the study of the stability and boundedness of solutions of a given nonlinear differential equation is provided by Lyapunov theory. But the construction of such functions which are positive definite with negative definite derivatives for higher-order differential equations with delay, is is in general a difficult task.

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In [7], Chin has tried to use a new technique (the intrinsic method) to construct new Lyapunov functions for the following fourth-order differential equations

$$x'''' + a_1x''' + a_2x'' + a_3x' + f(x) = 0, \tag{1.1}$$
$$x'''' + a_1x''' + \psi(x')x'' + a_3x' + a_4x = 0, \tag{1.2}$$
$$x'''' + a_1x''' + f(x, x')x'' + a_3x' + a_4x = 0, \tag{1.3}$$

in which a_1, a_2, a_3 and a_4 are constants. In [44], Wu and Xiong also investigated the asymptotic stability of the zero solution of the differential equations (1.1) and (1.3). Later, in 2004, Sadek [27] considered the fourth-order nonlinear delay differential equations of the form

$$x'''' + a_1x''' + a_2x'' + a_3x' + f(x(t-r)) = 0,$$

and he derived sufficient conditions for the asymptotic stability of the zero-solution of these equations by constructing new Lyapunov functional.
In [33], Tunc investigated the asymptotic stability of zero solution of the fourth order non-linear differential equations with delay as follows

$$x'''' + \varphi(x'')x''' + a_2x'' + a_3x' + f(x(t-r)) = 0.$$

In this article, we establish the uniform asymptotic stability of the differential equation of the form

$$\begin{aligned}\Big(g\big(x(t)\big)x''(t)\Big)'' + a(t)\Big(p\big(x(t)\big)x''(t)\Big)' \quad &+ b(t)\Big(q\big(x(t)\big)x'(t)\Big)' + c(t)f\big(x(t)\big)x'(t) \\ &+ d(t)h\big(x(t-r)\big) = 0,\end{aligned} \tag{1.4}$$

where $g(x) > 0$ and r is a positive constant to be determined later; the primes in (1.4) denote differentiation with respect to t; the functions $a, b, c, d,$ are continuously differentiable functions. The functions $f, g, h, p, q,$ are continuous. It is also supposed that the derivatives, $g'(x)$, $p'(x)$, $q'(x)$, $f'(x)$ and $h'(x)$ exist and are continuous.
Equation (1.4) is equivalent to the system

$$\begin{cases} x' = y, \\ y' = \dfrac{1}{g(x)}z, \\ z' = w, \\ w' = -a(t)\dfrac{p(x)}{g(x)}w + \left(a(t)p(x)\theta_1(t) - b(t)\dfrac{q(x)}{g(x)} - a(t)g(x)\theta_2(t)\right)z \\ \qquad -\Big(b(t)g^2(x)\theta_3(t) + c(t)f(x)\Big)y - d(t)h(x) + d(t)\displaystyle\int_{t-r}^{t} y(s)h'(x(s))\,ds, \end{cases} \tag{1.5}$$

where
$\theta_1(t) = \dfrac{g'(x(t))}{g^2(x(t))}x'(t)$, $\theta_2(t) = \dfrac{p'(x(t))}{g^2(x(t))}x'(t)$, and $\theta_3(t) = \dfrac{q'(x(t))}{g^2(x(t))}x'(t)$.
The continuity of the functions $a, b, c, d, f, g, g', h, p, p', q,$ and q' guarantees the existence of the solutions of (1.4) (see [11], pp.15). It is assumed that the right hand side of the system (1.5) satisfies a Lipschitz condition in $x(t), y(t), z(t), w(t)$ and $x(t-r)$. This assumption guarantees the uniqueness of solutions of (1.4) ([11], pp.15). The motivation for the present paper comes from the results mentioned above. Our purpose is to

extend and improve the result established by Tunc [33], and Sadek [27] to the equation (1.4). Clearly the equation discussed in [27] and is a special case of equation (1.1) when $g(x) = p(x) = q(x) = 1$, and $a(t) = a$, $b(t) = b$, $c(t) = c$. Our approach is based on Lyapunov's second (direct) method. We shall use appropriate Lyapunov function and impose suitable conditions on the functions $g(x)$, $p(x)$, and $q(x)$.

2 Preliminaries

In this section, we shall state and prove certain results useful in the proof of our main result. Consider the functional differential equation

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta)\ , \quad -r \le \theta \le 0, \quad t \ge 0, \tag{2.1}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t,0) = 0$, $C_H := \{\phi \in (C[-r,0],\ \mathbb{R}^n) : \|\phi\| \le H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 1. [6] An element $\psi \in C$ is in the $\omega - limit$ set of ϕ, say $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,+\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n+\theta, 0, \phi)\ for\ \ -r \le \theta \le 0$.

Definition 2. [6] A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.

Lemma 3. *[5] If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2,1) with $x_0(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \le H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \quad as \quad t \to \infty.$$

Lemma 4. *[5] Let $V(t,\phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition, $V(t,0) = 0$, and wedges W_i such that:*
(i) $W_1(\|\phi\|) \le V(t,\phi) \le W_2(\|\phi\|)$.
(ii) $V'_{(2,1)}(t,\phi) \le -W_3(\|\phi\|)$.
Then, the zero solution of (2.1) is uniformly asymptotically stable.

3 Assumptions and main results

First, we state some assumptions on the functions that appeared in (1.4), and suppose that there are positive constants $a_0, b_0, c_0, d_0, f_0, g_0, p_0, q_0, a_1, b_1, c_1, d_1, f_1, g_1, p_1, q_1, h_0, m$, $M, \delta, \delta_0, \eta_1$ and η_2 such that the following conditions are satisfied

i) $0 < a_0 \le a\,(t) \le a_1;\ 0 < b_0 \le b\,(t) \le b_1;\ 0 < c_0 \le c\,(t) \le c_1;$
$0 < d_0 \le d\,(t) \le d_1$ for $t \ge 0$.

ii) $0 < f_0 \le f\,(x) \le f_1;\ g_0 \le g\,(x) \le g_1;\ 0 < p_0 \le p\,(x) \le p_1;\quad 0 < q_0 \le q\,(x) \le q_1$
for $x \in \mathbb{R}$ and $0 < m < \min\{f_0, p_0, g_0, 1\}$, $M > \max\{f_1, , g_1, p_1, 1\}$.

iii) $\dfrac{h(x)}{x} \ge \delta > 0 \quad (\text{ for } x \ne 0)\,;\ h\,(0) = 0.$

iv) $\dfrac{h_0}{m} - \dfrac{a_0 m \delta_0}{d_1} \le h'\,(x) \le \dfrac{h_0}{2M}$ for $x \in \mathbb{R}$,

v) $b_0 q_0 > \max(\kappa_1, \kappa_2)$ where

$$\begin{cases} \kappa_1 = \dfrac{a_1 h_0 d_1 M^2}{c_0 m^3} + \dfrac{M^3(c_1+\delta_0)}{a_0 m^2} + a_0 a_1 m\,(M-1)\,, \\[2ex] \kappa_2 = \dfrac{2d_1 h_0 a_0}{c_0\,(M-1)}\left(\dfrac{1}{m}-\dfrac{1}{M}\right)^2 + 2\dfrac{c_0 M}{a_0} + 2a_1\dfrac{d_1 h_0 M}{c_0 m^3} + \dfrac{c_0 c_1 (M^2+2)mM}{d_1 h_0}. \end{cases}$$

vi) $\displaystyle\int_0^{+\infty} \left(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right) dt < \eta_1 < +\infty.$

vii) $\displaystyle\int_{-\infty}^{+\infty} \left(|g'(s)| + |p'(s)| + |q'(s)| + |f'(s)|\right) ds < \eta_2 < \infty.$

Now we dispose of the following lemma which will be required in the proof of next theorem.

Lemma 5. *[19] Let* $h(0)=0,\ xh(x)>0\ (x\neq 0)$ *and* $\delta(t)-h'(x)\geq 0\ (\delta(t)>0)$, *then*

$$2\delta(t)H(x)\geq h^2(x) \qquad \textit{where} \quad H(x)=\int_0^x h(s)ds.$$

The main result of this paper is the following theorem.

Theorem 6. *Suppose that assumptions i) $\sim$ vii) hold. Then, every solution $x(t)$ of (1.4) and their derivatives $x'(t), x''(t)$ and $x'''(t)$ are uniformly asymptotically stable, provided that*

$$r < \frac{1}{\lambda}\min\left\{\epsilon c_0 m,\ \epsilon\frac{a_0 m}{M},\ \frac{m^2(b_0 q_0-\kappa_1)-\epsilon M^2(a_1+c_1 mM)}{Mm^2}\right\}, \tag{3.1}$$

where

$$\lambda = d_1\lambda_0(\alpha+\beta+1),\ \lambda_0=\max\left\{\frac{h_0}{2M},\ \left|\frac{h_0}{m}-\frac{a_0 m\delta_0}{d_1}\right|\right\}, \quad \textit{and} \tag{3.2}$$

$$\epsilon < \min\left\{\frac{M}{a_0 m}\ ,\ \frac{d_1 h_0}{c_0 m}\ ,\ \frac{m^2(b_0 q_0-\kappa_1)}{M^2(a_1+mMc_1)}\right\}. \tag{3.3}$$

Proof. The proof depend on some fundamental properties of a continuously differentiable Lyapunov functional $W=W(t,x_t,y_t,z_t,w_t)$ defined by

$$W = e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} V, \tag{3.4}$$

where

$$\gamma(t) = |a'(t)|+|b'(t)|+|c'(t)|+|d'(t)|+|\theta_1(t)|+|\theta_2(t)|+|\theta_3(t)|+|\theta_4(t)|,$$

and

$$V = V_0(t,x_t,y_t,z_t,w_t)+\lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)\,d\theta ds,$$

such that

$$\begin{aligned}
\theta_4(t) &= \frac{f'(x(t))}{g^2(x(t))}x'(t), \\
2V_0 &= 2\beta d(t)H(x) + c(t)g(x)f(x)y^2 + \alpha b(t)\frac{q(x)}{g(x)}z^2 + a(t)\frac{p(x)}{g(x)}z^2 \\
&\quad +2\beta a(t)\frac{p(x)}{g(x)}yz + [\beta b(t)q(x) - \alpha h_0 d(t)]y^2 - \beta\frac{1}{g(x)}z^2 + \alpha w^2 \\
&\quad +2d(t)g(x)h(x)y + 2\alpha d(t)h(x)z + 2\alpha c(t)f(x)yz + 2\beta yw + 2zw,
\end{aligned}$$

with $H(x) = \int_0^x h(s)ds$, $\alpha = \dfrac{M}{a_0 m} + \epsilon$, $\beta = \dfrac{d_1 h_0}{c_0 m} + \epsilon$ and η is positive constant to be determined later in the proof. $2V_0$ can be rearranged as the following

$$\begin{aligned}
2V_0 &= a(t)p(x)\left(\frac{w}{a(t)p(x)} + z + \beta\frac{1}{g(x)}y\right)^2 + c(t)f(x)\left(\frac{d(t)h(x)}{c(t)f(x)} + y + \alpha z\right)^2 \\
&\quad +c(t)f(x)\left[\left(g(x)-1\right)y + \frac{d(t)h(x)}{c(t)f(x)}\right]^2 + 2\epsilon d(t)H(x) + V_1 + V_2 + V_3,
\end{aligned}$$

where

$$\begin{aligned}
V_1 &= 2d(t)\int_0^x h(s)\left(\frac{d_1 h_0}{c_0 m} - 2\frac{d(t)}{c(t)f(x)}h'(s)\right)ds, \\
V_2 &= \left(\alpha b(t)\frac{q(x)}{g(x)} - \beta\frac{1}{g(x)} - \alpha^2 c(t)f(x) + a(t)p(x)\left(\frac{1}{g(x)} - 1\right)\right)z^2,
\end{aligned}$$

and

$$\begin{aligned}
V_3 &= \left(\beta b(t)q(x) - \alpha h_0 d(t) - \beta^2 a(t)\frac{p(x)}{g^2(x)} - c(t)f(x)\left(g^2(x) - 3g(x) + 2\right)\right)y^2 \\
&\quad + \left(\alpha - \frac{1}{a(t)p(x)}\right)w^2 + 2\beta\left(1 - \frac{1}{g(x)}\right)yw.
\end{aligned}$$

To prove that V is positive definite it suffice to show that V_1, V_2 and V_3 are positive. Using conditions i) $\sim$ v), and inequality (3.3) we obtain the following

$$\begin{aligned}
V_1 &\geq 2d(t)\int_0^x 2h(s)\frac{d_1}{c_0 m}\left(\frac{h_0}{2} - h'(s)\right)ds \\
&\geq 4d_0\frac{d_1}{c_0 m}\int_0^x h(s)\left(\frac{h_0}{2M} - h'(s)\right)ds \geq 0.
\end{aligned}$$

Since (3.3) we get

$$\frac{M}{a_0 m} < \alpha < 2\frac{M}{a_0 m}, \qquad \frac{d_1 h_0}{c_0 m} < \beta < 2\frac{d_1 h_0}{c_0 m}. \tag{3.5}$$

From (3.5) and rearrange V_2 we obtain

$$\begin{aligned}
V_2 &= \alpha\Bigg(b\left(t\right)\frac{q(x)}{g\left(x\right)} - \beta\frac{a\left(t\right)}{g\left(x\right)} - \alpha c\left(t\right)f\left(x\right) - \frac{a\left(t\right)p(x)}{\alpha}\Bigg(1-\frac{1}{g\left(x\right)}\Bigg)\Bigg)z^2 \\
&\quad + \beta\Bigg(\alpha\frac{a\left(t\right)}{g\left(x\right)} - \frac{1}{g\left(x\right)}\Bigg)z^2 \\
&\geq \alpha\Bigg(\frac{b_0q_0}{M} - (\frac{d_1h_0}{c_0m}+\epsilon)\frac{a_1}{m} - (\frac{M}{a_0m}+\epsilon)c_1M - a_1\frac{a_0m}{M}(M-1)\Bigg)z^2 \\
&\quad + \beta\Bigg(\alpha\frac{a_0}{M} - \frac{1}{m}\Bigg)z^2 \\
&\geq \alpha\Bigg(\frac{b_0q_0}{M} - \frac{d_1h_0a_1}{c_0m^2} - \frac{c_1M^2}{a_0m} - a_1\frac{a_0m}{M}(M-1) - \frac{\epsilon}{m}(a_1+c_1mM)\Bigg)z^2 \\
&\geq \frac{\alpha}{Mm}\Bigg(m(b_0q_0-\kappa_1) - \epsilon M\left(a_1+c_1mM\right)\Bigg)z^2 \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
V_3 &\geq \beta\Bigg(b_0q_0 - \frac{\alpha}{\beta}h_0d_1 - a_1\beta\frac{M}{g^2\left(x\right)} - \frac{c_1M(M^2+2)}{\beta}\Bigg)y^2 + \Bigg(\frac{M-1}{a_0m}\Bigg)w^2 \\
&\quad +2\beta\Bigg(1-\frac{1}{g\left(x\right)}\Bigg)yw \\
&\geq \beta\Bigg(b_0q_0 - 2\frac{M}{a_0}c_0 - 2a_1\frac{d_1h_0M}{c_0m^3} - \frac{c_0c_1(M^2+2)mM}{d_1h_0}\Bigg)y^2 + \Bigg(\frac{M-1}{a_0m}\Bigg)w^2 \\
&\quad +2\beta\Bigg(1-\frac{1}{g\left(x\right)}\Bigg)yw \\
&\geq \psi(y,\omega),
\end{aligned}$$

such that

$$\psi(y,\omega) = \beta\frac{2d_1h_0a_0}{c_0\left(M-1\right)}\left(\frac{1}{m}-\frac{1}{M}\right)^2 y^2 + \Bigg(\frac{M-1}{a_0m}\Bigg)w^2 + 2\beta\Bigg(1-\frac{1}{g\left(x\right)}\Bigg)yw.$$

Observe that $\psi(y,\omega)$ is positive definite. Indeed by calculating the discriminant

$$\triangle = \beta^2\Bigg(1-\frac{1}{g\left(x\right)}\Bigg)^2 - \beta\frac{2d_1h_0}{c_0m}\left(\frac{1}{m}-\frac{1}{M}\right)^2,$$

since

$$\frac{1}{M} < \frac{1}{g(x)} < \frac{1}{m}, and \;\; \frac{1}{M} < 1 < \frac{1}{m},$$

we get

$$\left|1-\frac{1}{g\left(x\right)}\right| < \frac{1}{m} - \frac{1}{M}.$$

Using (3.5) we obtain

$$\triangle \leq \beta\left[\frac{2d_1h_0}{c_0m}\left(\frac{1}{M}-\frac{1}{m}\right)^2 - \frac{2d_1h_0}{c_0m}\left(\frac{1}{M}-\frac{1}{m}\right)^2\right] = 0.$$

Using the fact that the integral $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ is positive, we deduce that there exists positive number D_0 such that

$$2V \geq D_0\left(y^2 + z^2 + w^2 + H(x)\right). \tag{3.6}$$

By lemma 5 and conditions iii) and iv) we conclude that there exists a positive number D_1 such that

$$2V \geq D_1\left(x^2 + y^2 + z^2 + w^2\right), \tag{3.7}$$

thus V is positive definite which implies that W is also positive definite.
Then, we can find positive definite functions $U_1(\|X\|)$ and $U_2(\|X\|)$ such that $U_1(\|X\|) \leq V \leq U_2(\|X\|)$. By (ii) and (vii), we get

$$\begin{aligned}\int_0^t \Big(\sum_{i=1}^{4}|\theta_i(s)|\Big)ds &= \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|g'(u)| + |p'(u)| + |q'(u)| + |f'(u)|}{g^2(u)}du \\ &\leq \frac{1}{m^2}\int_{-\infty}^{+\infty}\Big(|g'(u)| + |p'(u)| + |q'(u)| + |f'(u)|\Big)du < \infty, \end{aligned}\tag{3.8}$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$, and $\alpha_2(t) = \max\{x(0), x(t)\}$. By inequalities (3.4), (3.7), and (3.8), we have

$$W \geq D_2(x^2 + y^2 + z^2 + w^2), \text{ where } D_2 = \frac{D_1}{2}e^{-\frac{1}{\eta}(\eta_1 + \frac{\eta_2}{m^2})}. \tag{3.9}$$

Therefore we can find positive definite functions $W_1(\|X\|)$ and $W_2(\|X\|)$ such that

$$W_1(\|X\|) \leq W \leq W_2(\|X\|).$$

Now we prove that $\dot{W}$ is negative definite functional.
The derivative of V along any solution $(x(t), y(t), z(t), w(t))$ of system (1.5), we have

$$\begin{aligned}2\dot{V}_{(1.5)} &= -2\epsilon c(t) f(x)y^2 + V_4 + V_5 + V_6 + V_7 + 2\frac{\partial V_0}{\partial t} + \lambda r y^2(t) - \lambda\int_{t-r}^{t} y^2(u)\,du \\ &\quad +2\alpha w d(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds + 2\beta y d(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds \\ &\quad +2z d(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds,\end{aligned}$$

where

$$\begin{aligned}V_4 &= -2\left(\frac{d_1 h_0}{c_0 m}c(t) f(x) - d(t)\, g(x)\, h'(x)\right)y^2 - 2\alpha d(t)\left(\frac{h_0}{g(x)} - h'(x)\right)yz, \\ V_5 &= -2\left(\frac{b(t)\, q(x)}{g(x)} - \alpha c(t)\frac{f(x)}{g(x)} - \beta a(t)\frac{p(x)}{g^2(x)}\right)z^2, \\ V_6 &= -2\left(\alpha\frac{a(t)\, p(x)}{g(x)} - 1\right)w^2\end{aligned}$$

and

$$\begin{aligned}V_7 &= \theta_1\Big(a(t)\, p(x)z^2 - \alpha b(t)\, q(x)z^2 + c(t)\, f(x)\, g^2(x)\, y^2 + \beta z^2 + 2d(t)\, g^2(x)\, h(x)\, y \\ &\quad +2\alpha a(t)\, p(x)zw\Big) - b(t)\theta_3 g(x)\Big(\alpha z^2 + 2\alpha g(x)zw + \beta g(x)y^2 + 2g(x)yz\Big) \\ &\quad -a(t)\theta_2 g(x)\Big(z^2 + 2\alpha zw\Big) + \theta_4\Big(c(t)\, g^3(x)\, y^2 + 2\alpha c(t)\, g^2(x)\, yz\Big).\end{aligned}$$

By conditions i), ii), iv), v) and inequalities (3.2), (3.3) and (3.5) we get

$$\begin{aligned}
V_4 &\leq -2\Big(d(t)\,h_0 - d(t)\,g(x)\,h'(x)\Big)y^2 - 2\alpha d(t)\left(\frac{h_0}{g(x)} - h'(x)\right)yz \\
&\leq -2d(t)\,g(x)\left(\frac{h_0}{g(x)} - h'(x)\right)y^2 - 2\alpha d(t)\left(\frac{h_0}{g(x)} - h'(x)\right)yz \\
&\leq -2d(t)\,m\left(\frac{h_0}{g(x)} - h'(x)\right)\left[\left(y + \frac{\alpha}{2m}z\right)^2 - \left(\frac{\alpha}{2m}z\right)^2\right] \\
&\leq -2d(t)\,m\left(\frac{h_0}{M} - h'(x)\right)\left(y + \frac{\alpha}{2m}z\right)^2 + 2d(t)m\left(\frac{h_0}{m} - h'(x)\right)\left(\frac{\alpha}{2m}z\right)^2 \\
&\leq \frac{\alpha^2}{2m}d(t)\left(\frac{h_0}{m} - h'(x)\right)z^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
V_4 + V_5 &\leq -2\left[b(t)\,\frac{q(x)}{g(x)} - \alpha c(t)\,\frac{f(x)}{g(x)} - \beta a(t)\,\frac{p(x)}{g^2(x)} - \frac{\alpha^2}{4m}d(t)\left(\frac{h_0}{m} - h'(x)\right)\right]z^2 \\
&\leq -2\left[\frac{b_0q_0}{M} - \Big(\frac{M}{a_0m} + \epsilon\Big)\frac{c_1M}{m} - \Big(\frac{d_1h_0}{c_0m} + \epsilon\Big)\frac{a_1M}{m^2} - \frac{\alpha^2}{4m}(a_0\delta_0)\right]z^2 \\
&\leq -2\left[\frac{b_0q_0}{M} - \frac{M^2}{a_0m^2}c_1 - \frac{d_1h_0a_1M}{c_0m^3} - \frac{M^2\delta_0}{a_0m^2} - \epsilon\frac{M}{m}\left(\frac{a_1}{m} + c_1\right)\right]z^2 \\
&\leq -\frac{2}{Mm^2}\Big(m^2(b_0q_0 - \kappa_1) - \epsilon M^2\,(a_1 + c_1m)\Big)z^2 \leq 0.
\end{aligned}$$

We have also,

$$V_6 \leq -2\left(\alpha\frac{a_0m}{M} - 1\right)w^2 = -2\epsilon\frac{a_0m}{M}w^2 \leq 0.$$

Putting $\lambda_1 = \min\left\{\epsilon c_0m,\ \epsilon\frac{a_0m}{M},\ \frac{1}{Mm^2}\Big(m^2(b_0q_0 - \kappa_1) - \epsilon M^2\,(a_1 + c_1mM)\Big)\right\}$ and using Cauchy Schwartz inequality we have

$$\begin{aligned}
&-2\epsilon c(t)\,f(x)y^2 + V_4 + V_5 + V_6 + 2\alpha wd(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds - \lambda\int_{t-r}^{t} y^2(s)\,ds \\
&+ 2\beta yd(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds + 2zd(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds + \lambda ry^2(t) \\
&\leq -2\lambda_1\left(y^2 + z^2 + w^2\right) + \alpha d_1\lambda_0\left(w^2r + \int_{t-r}^{t} y^2(s)\,ds\right) + \beta d_1\lambda_0\left(y^2r + \int_{t-r}^{t} y^2(s)\,ds\right) \\
&\quad + d_1\lambda_0\left(z^2r + \int_{t-r}^{t} y^2(s)\,ds\right) + \lambda ry^2(t) - \lambda\int_{t-r}^{t} y^2(s)\,ds \\
&\leq -2\lambda_1\left(y^2 + z^2 + w^2\right) + d_1\lambda_0r\left(\beta y^2 + z^2 + \alpha w^2\right) + \left(d_1\lambda_0(\alpha + \beta + 1) - \lambda\right)\int_{t-r}^{t} y^2(s)\,ds \\
&\leq -2\lambda_1\left(y^2 + z^2 + w^2\right) + d_1\lambda_0(\alpha + \beta + 1)r\left(y^2 + z^2 + w^2\right) \\
&\leq -2D_3\left(y^2 + z^2 + w^2\right).
\end{aligned} \tag{3.10}$$

Where $D_3 = \lambda_1 - \lambda r$. It can be seen that if $r < \frac{\lambda_1}{\lambda}$, then $D_2 > 0$.
Now by the inequalities (3.6), (3.10), the lemma 5 and the Cauchy Schwartz inequality

we get the following.

$$\begin{aligned}
V_7 \leq |\theta_1|\Big(& a\,(t)\,p(x)z^2 + \alpha b\,(t)\,q(x)z^2 + c\,(t)\,f\,(x)\,g^2\,(x)\,y^2 + \beta z^2 + d\,(t)\,g^2\,(x)\,(h^2\,(x) + y^2) \\
&+ \ \alpha a\,(t)\,p(x)(z^2 + w^2)\Big) + |\theta_4|\Big(c\,(t)\,g^3\,(x)\,y^2 + \alpha c\,(t)\,g^2\,(x)\,(y^2 + z^2)\Big) \\
&+ \ b(t)|\theta_3|g(x)\Big(\alpha z^2 + \alpha g(x)(z^2 + w^2) + \beta g(x)y^2 + g(x)(y^2 + z^2)\Big) \\
&+ \ a(t)|\theta_2|g(x)\Big(z^2 + \alpha(z^2 + w^2)\Big) \\
\leq \lambda_2&\big(|\theta_1| + |\theta_2| + |\theta_3| + |\theta_4|\big)\big(y^2 + z^2 + w^2 + H(x)\big) \\
\leq \frac{2\lambda_2}{D_0}&\big(|\theta_1| + |\theta_2| + |\theta_3| + |\theta_4|\big)V.
\end{aligned}$$

such that,

$$\begin{aligned}
\lambda_2 = \max\big\{&d_1 h_0 M, \alpha M(a_1 + M b_1), c_1 M^3 + (d_1 + \alpha c_1 + \beta b_1 + b_1)M^2, \\
&\beta + a_1 M(\alpha + 1) + b_1 M(2\alpha + 1) + \alpha c_1 M^2\big\}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
2\frac{\partial V_0}{\partial t} \ &= \ d'\,(t)\,\big[2\beta H\,(x) - \alpha h_0 y^2 + 2g\,(x)\,h\,(x)\,y + 2\alpha h\,(x)\,z\big] \\
&\quad + c'\,(t)\,\big[g\,(x)\,f(x)y^2 + 2\alpha f(x)yz\big] + b'\,(t)\left[\alpha\frac{q(x)}{g\,(x)}z^2 + \beta q(x)y^2\right] \\
&\quad + a'\,(t)\left[\frac{p(x)}{g\,(x)}z^2 + 2\beta\frac{p(x)}{g\,(x)}yz\right].
\end{aligned}$$

There exist positive constant λ_3 such that

$$\begin{aligned}
2\left|\frac{\partial V_0}{\partial t}\right| \ &\leq \ |d'\,(t)\,|\Big(2\beta H\,(x) + \alpha h_0 y^2 + g\,(x)\,\big(h^2\,(x) + y^2\big) + \alpha\,\big(h^2\,(x) + z^2\big)\Big) \\
&\quad + |c'\,(t)\,|f(x)\Big(g\,(x)\,y^2 + \alpha\,\big(y^2 + z^2\big)\Big) + |b'\,(t)\,|q(x)\Big(\alpha\frac{1}{g\,(x)}z^2 + \beta y^2\Big) \\
&\quad + |a'\,(t)\,|\frac{p(x)}{g\,(x)}\Big(z^2 + \beta\,\big(y^2 + z^2\big)\Big) \\
&\leq \ \lambda_3\Big(|a'\,(t)| + |b'\,(t)| + |c'\,(t)| + |d'\,(t)|\Big)\big(y^2 + z^2 + w^2 + H(x)\big) \\
&\leq \ 2\frac{\lambda_3}{D_0}\Big(|a'\,(t)| + |b'\,(t)| + |c'\,(t)| + |d'\,(t)|\Big)V,
\end{aligned}$$

where

$$\lambda_3 = \max\big\{2\beta + h_0(\alpha + 1), M^2 + \beta\frac{M}{m} + \alpha(h_0 + M), \frac{M}{m}(\alpha + \beta + 1)\big\}.$$

Thus for $\dfrac{1}{\eta} = \dfrac{1}{D_0}\max\big\{\lambda_2, \lambda_3\big\}$ we have

$$\begin{aligned}
\dot{V}_{(1.5)} \ &\leq \ -D_3(y^2 + z^2 + w^2) + \frac{1}{\eta}\Big(|a'\,(t)| + |b'\,(t)| + |c'\,(t)| + |d'\,(t)| \\
&\quad + |\theta_1| + |\theta_2| + |\theta_3| + |\theta_4|\Big)V. \qquad (3.11)
\end{aligned}$$

By conditions vi), vii) and inequalities (3.8), (3.11) we have

$$\begin{aligned}\dot{W}_{(1.5)} &= \left(\dot{V}_{(1.5)} - \frac{1}{\eta}\gamma(t)\,V\right) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\leq -D_3\left(y^2+z^2+w^2\right) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\leq -D_4\left(y^2+z^2+w^2\right),\end{aligned}$$

where $D_4 = D_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$. From (1.5, $W_3(\|X\|) = D_4(y^2+z^2+w^2)$ is positive definite function. Thus, we conclude that the solutions of system (1.5) are uniformly asymptotically stable. Now, it is evident from (1.5) that

$$\begin{cases} |x'(t)| = |y(t)|, \\ |x''(t)| = |\dfrac{z(t)}{g(x)}| \leq \dfrac{|z(t)|}{m}, \\ |x'''(t)| = |\dfrac{w(t)}{g(x)} - \dfrac{g'(x)x'(t)}{g^2(x)}| \leq \dfrac{|w(t)|}{m} + \dfrac{|g'(x)||x'(t)|}{m^2}. \end{cases}$$

Clearly, from the above discussion

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} x'(t) = 0, \;\; \lim_{t\to\infty} x''(t) = 0, \;\; and \;\; \lim_{t\to\infty} x'''(t) = 0.$$

This fact completes the proof of the Theorem. □

4 Example

We consider the following fourth order non-autonomous delay differential equation

$$\begin{aligned}&\left(\left(\frac{x^2(t)\sin x(t) + 5x^4(t) + 5}{5\left(1+x^4(t)\right)}\right) x''(t)\right)'' \\ &+ \left(e^{-t}\sin t + 2\right)\left(\left(\frac{x(t) + 4e^{x(t)} + 4e^{-x(t)}}{4\left(e^{x(t)} + e^{-x(t)}\right)}\right) x''(t)\right)' \\ &+ \left(\frac{\cos t + 7t^2 + 7}{1+t^2}\right)\left(\left(\frac{\sin x(t) + 6e^{x(t)} + 6e^{-x(t)}}{e^{x(t)} + e^{-x(t)}}\right) x'(t)\right)' \\ &+ \left(e^{-2t}\sin^3 t + 2\right)\left(\frac{x(t)\cos x(t) + 5x^4(t) + 5}{5\left(1+x^4(t)\right)}\right) x'(t) \\ &+ \left(\frac{\cos^2 t + t^2 + 1}{10\left(1+t^2\right)}\right)\left(\frac{x(t-r)}{x^2(t-r)+1}\right) = 0. \qquad (4.1)\end{aligned}$$

By taking $g(x) = \dfrac{x^2\sin x + 5x^4 + 5}{5\left(1+x^4\right)}$, $p(x) = \dfrac{x + 4e^x + 4e^{-x}}{4\left(e^x + e^{-x}\right)}$, $q(x) = \dfrac{\sin x + 6e^x + 6e^{-x}}{e^x + e^{-x}}$, $f(x) = \dfrac{x\cos x + 5x^4 + 5}{5\left(1+x^4\right)}$, $h(x) = \dfrac{x}{x^2+1}$, $a(t) = e^{-t}\sin t + 2$, $b(t) = \dfrac{\cos t + 7t^2 + 7}{1+t^2}$, $c(t) = e^{-2t}\sin^3 t + 2$ and $d(t) = \dfrac{\cos^2 t + t^2 + 1}{10\left(1+t^2\right)}$.

We have

$m = \dfrac{9}{10}$, $M = \dfrac{11}{10}$, $q_0 = \dfrac{11}{2}$, $q_1 = \dfrac{13}{2}$, $h_0 = \dfrac{5}{2}$, $\delta_0 = \dfrac{5}{3}$, $a_0 = 1$, $a_1 = 3$,
$b_0 = 6$, $b_1 = 8$, $c_0 = 1$, $c_1 = 3$, $d_0 = \dfrac{1}{10}$, $d_1 = \dfrac{1}{5}$, and

$$\begin{aligned}
\frac{h_0}{m} - \frac{a_0 m \delta_0}{d_1} &\leq -4.55 \leq h'(x) \leq 1.1 \leq \frac{h_0}{2M}, \\
\kappa_1 &= \frac{a_1 h_0 d_1 M^2}{c_0 m^3} + \frac{M^3(c_1+\delta_0)}{a_0 m^2} + a_0 a_1 m (M-1) \leq 11, \\
\kappa_2 &= \frac{2 d_1 h_0 a_0}{c_0 (M-1)} \left(\frac{1}{m} - \frac{1}{M} \right)^2 + 2\frac{c_0 M}{a_0} + 2a_1 \frac{d_1 h_0 M}{c_0 m^3} + \frac{c_0 c_1 (M^2+2) m M}{d_1 h_0} \\
&\leq 27, \\
\epsilon &< \min \left\{ \frac{M}{a_0 m} , \frac{d_1 h_0}{c_0 m} , \frac{m^2(b_0 q_0 - \kappa_1)}{M^2(a_1 + m M c_1)} \right\} = \frac{5}{9}.
\end{aligned}$$

By choosing $\epsilon = \dfrac{1}{4}$ we get $\alpha = \dfrac{M}{a_0 m} + \epsilon = \dfrac{53}{36}$, $\beta = \dfrac{d_1 h_0}{c_0 m} + \epsilon = \dfrac{29}{36}$,

$$\begin{aligned}
\lambda_0 &= \max \left\{ \frac{h_0}{2M}, \left| \frac{h_0}{m} - \frac{a_0 m \delta_0}{d_1} \right| \right\} = \frac{85}{18}, \\
\lambda &= d_1 \lambda_0 (\alpha + \beta + 1) = \frac{1003}{324}, \\
r &< \frac{1}{\lambda} \min \left\{ \epsilon c_0 m, \ \epsilon \frac{a_0 m}{M}, \ \frac{m^2(b_0 q_0 - \kappa_1) - \epsilon M^2 (a_1 + c_1 m M)}{M m^2} \right\} = \frac{729}{11033}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{-\infty}^{+\infty} |g'(x)| \, dx &= \frac{1}{5} \int_{-\infty}^{+\infty} \left| \frac{-4x^5 \sin x + \left(2x \sin x + x^2 \cos x\right)\left(x^4+1\right)}{\left(x^4+1\right)^2} \right| dx \leq \frac{3}{5}\sqrt{2}\pi, \\
\int_{-\infty}^{+\infty} |p'(x)| \, dx &= \frac{1}{4} \int_{-\infty}^{+\infty} \left| \frac{1}{e^x + e^{-x}} + x \frac{e^{-x} - e^x}{\left(e^x + e^{-x}\right)^2} \right| dx \leq \frac{\pi}{4},
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} |q'(x)| \, dx &= \frac{1}{5} \int_{-\infty}^{+\infty} \left| \frac{\left(e^x + e^{-x}\right) \cos x - \left(e^x - e^{-x}\right) \sin x}{\left(e^x + e^{-x}\right)^2} \right| dx \leq \frac{\pi}{5}, \\
\int_{-\infty}^{+\infty} |f'(x)| \, dx &= \frac{1}{5} \int_{-\infty}^{+\infty} \left| \frac{\left(\cos x - x \sin x\right)\left(x^4+1\right) - 4x^4 \cos x}{\left(x^4+1\right)^2} \right| dx \\
&\leq \frac{1}{5} \int_{-\infty}^{+\infty} \frac{5 + x^2}{x^4+1} dx = \frac{9}{10}\sqrt{2}\pi,
\end{aligned}$$

then,

$$\int_{-\infty}^{+\infty} \left(|g'(s)| + |p'(s)| + |q'(s)| + |f'(s)| \right) ds < \infty.$$

We have also

$$\begin{aligned}
\int_0^{+\infty} |a'(t)|\, dt &= \int_0^{+\infty} \left| (\cos t)\, e^{-t} - (\sin t)\, e^{-t} \right| dt \le \int_0^{+\infty} 2e^{-t} dt = 2,\\
\int_0^{+\infty} |b'(t)|\, dt &= \int_0^{+\infty} \left| -\frac{\sin t}{t^2+1} - 2t\frac{\cos t}{(t^2+1)^2} \right| dt \le \int_0^{+\infty} \left(\frac{1}{t^2+1} + \frac{2t}{(t^2+1)^2} \right) dt\\
&\le \int_0^{+\infty} \frac{2}{t^2+1} dt = \pi,\\
\int_0^{+\infty} |c'(t)|\, dt &= \int_0^{+\infty} \left| 3\left(\cos t \sin^2 t\right) e^{-2t} - 2\left(\sin^3 t\right) e^{-2t} \right| dt \le \int_0^{+\infty} 5e^{-2t} dt = \frac{5}{2},\\
\int_0^{+\infty} |d'(t)|\, dt &= \int_0^{+\infty} \left| -2(\cos t)\frac{\sin t}{t^2+1} - 2t\frac{\cos^2 t}{(t^2+1)^2} \right| dt \le \int_0^{+\infty} \frac{3}{t^2+1} dt = \frac{3\pi}{2}.
\end{aligned}$$

Hence

$$\int_0^{+\infty} \left(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right) dt < +\infty.$$

Thus all the assumptions of Theorem (6) hold, this shows that every solution $x(t)$ of (4.1) and their derivatives $x'(t), x''(t)$ and $x'''(t)$ are uniformly asymptotically stable.

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Contents

G. Monozsová and A. Haviar
Professor Ľubomír Snoha, 60 years old ... 3

A. Rosa
Maximal designs and configurations - a survey ... 9

Ş. Altınkaya and S. Yalçın
Initial Coefficient Bounds for a Comprehensive Subclass of Sakaguchi Type Functions ... 27

S. Bazm and M.R. Azimi
Numerical solution of a class of nonlinear Volterra integral equations using Bernoulli operational matrix of integration ... 35

B. Belaïdi
On the [p, q]-Order of Meromorphic Solutions of Linear Differential Equations ... 57

S.S. Dragomir
Some Perturbed Ostrowski Type Inequalities for Absolutely Continuous Functions (I) ... 71

A. Gupta and K. Mamtani
Weyl-Type Theorems For Restrictions Of Closed Linear Unbounded Operators ... 87

N. Kumar and G. Manocha
Certain results on a class of Entire functions represented by Dirichlet series having complex frequencies ... 95

M. Rahmane and M. Remili
On Stability and Boundedness of Solutions of Certain Non Autonomous Fourth-Order Delay Differential Equations ... 101