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A note and a short survey on supporting lines of compact convex sets in the plane

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Abstract

After surveying some known properties of compact convex sets in the plane, we give two rigorous proofs of the general feeling that supporting lines can be *slide-turned* slowly and continuously. Targeting a wide readership, our treatment is elementary on purpose.

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1 Motivation

Nowadays, there is a growing interest in the combinatorial properties of convex sets, usually, in compact convex sets. A large part of the papers belonging to this field go back to Erdős and Szekeres [15]; see, for example, Dobbins, Holmsen, and Hubard [12] and [13], Pach and Tóth [24] and [25], and their references. Recently, besides combinatorists and geometers, algebraists are also interested in compact convex sets; see, for example, Adaricheva [1], Adaricheva and Bolat [2], Adaricheva and Nation [4], Czédli [8], [9], and [10], Czédli and Kincses [11], and Richter and Rogers [26]. The interest of algebraists is explained by the fact that *antimatroids*, introduced by Korte and Lovász [17] and [18], and the dual concept of *abstract convex geometries*, introduced by Edelman and Jamison [14], have close connections to lattice theory. These connections are surveyed in Adaricheva and Czédli [3], Adaricheva and Nation [4], Czédli [7], and Monjardet [21]. Finally, there are other types of combinatorial investigations of convex sets; the most recent is, perhaps, Novick [23].

One of the most important concepts related to planar convex sets is that of *supporting lines.* Most of the papers mentioned above rely, explicitly or implicitly, on the properties of these lines. We guess that not only the experts of advanced analysis of convex sets and functions are interested in the above papers; at least, this is surely true in case of the first author of the present paper. However, it is quite difficult to explain to or understand by all the interested readers in a short, easy-to-follow, but rigorous way that why one of the most useful property of compact convex sets holds. This property, which seems to

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be absent in the literature, will be formulated in Theorem 1. This theorem is the "note" occurring in the title.

This motivates the aim of this short paper: even if Theorem 1 could be proved in a shorter way by using advanced tools of Analysis and even if it states what is expected by geometric intuition, we are going to give a rigorous proof for it. Actually, we give two different proofs. We believe that if other statements for planar compact convex sets like (2.6) deserve proofs that are easy to reference, then so does this theorem. Note that Czédli [10] exemplifies why the present paper is expected to be useful in further research: while the first version, `arXiv:1611.09331v1`, of [10] spends a dozen of pages on properties of supporting lines, its second version needs only few lines and a reference to the present paper. Also, we exemplify the use of Theorem 1 by an easy corollary, which is a well known but we have not found a rigorous proof for it.

2 A short survey

A *compact* subset of the plane $\mathbb{R}^2$ is a topologically closed bounded subset. The *boundary* of H will be denoted by ∂H. A subset H of $\mathbb{R}^2$ is *convex*, if for any two points $X, Y \in \mathbb{R}^2$, the closed line segment $[X, Y]$ is a subset of H. In this section, H will stand for a compact convex set. Even if this is not always repeated, we always assume that a convex set is nonempty. Each line ℓ gives rise to two *closed halfplanes*; their intersection is ℓ. Usually, unless otherwise is stated explicitly, we assume that ℓ is a *directed line*; then we can speak of the *left and right halfplanes* determined by ℓ. Points or sets in the left halfplane are *on the left* of ℓ; being on the right is defined analogously. If H is on the left of ℓ such that $H \cap \ell = \emptyset$, then H is *strictly on the left* of ℓ. The *direction* of a directed line ℓ will be denoted by $\mathrm{dir}(\ell) \in [-\pi, \pi)$. It is understood modulo 2π, whence we could also consider $\mathrm{dir}(\ell)$ an element of $[0, 2\pi)$. Furthermore, denoting the unit circle $\{\langle x, y\rangle : x^2 + y^2 = 1\}$ by C_{unit}, we will often say that $\mathrm{dir}(\ell) \in C_{\mathrm{unit}}$. Following the convention of Yaglom and Boltyanskiĭ [31], if H is on the left of ℓ and $\ell \cap H \neq \emptyset$, then ℓ is a *supporting line* of H. Clearly, for a supporting line ℓ of H, $\ell \cap H = \ell \cap \partial H \neq \emptyset$. We know from Yaglom and Boltyanskiĭ [31, page 8] that parallel to each line ℓ, a compact convex set with nonempty interior has exactly two supporting lines. Hence, without any stipulation on the interior,

$$\begin{array}{l}\text{for every } \alpha \in C_{\mathrm{unit}}, \text{ a compact convex set has}\\ \text{exactly one supporting line of direction } \alpha.\end{array} \tag{2.1}$$

Note at this point that, by definition, a *curve* is the range $\mathrm{Range}(g)$ of a continuous function g from an interval I of positive length to $\mathbb{R}^n$ for some $n \in \{2, 3, 4, \dots\}$. If $x_1 \neq x_2 \Rightarrow g(x_1) \neq g(x_2)$ except possibly for the endpoints of I, then $\mathrm{Range}(g)$ is a *simple curve*. A *Jordan curve* is a homeomorphic *planar* image of a circle of nonzero radius, that is, a Jordan curve is a *simple closed curve in the plane*. A curve is *rectifiable* if the lengths of its inscribed polygons form a bounded subset of $\mathbb{R}$. The following statement is known, say, from Latecki, Rosenfeld, and Silverman [19, Thm. 32] and Topogonov [30, page 15]; see also [32].

$$\begin{array}{l}\text{For a compact convex } H \subseteq \mathbb{R}^2 \text{ with nonempty}\\ \text{interior, } \partial H \text{ is a rectifiable Jordan curve.}\end{array} \tag{2.2}$$

For $P \in \partial H$, there are two possibilities; see, for example, Yaglom and Boltyanskiĭ [31, page 12]. First, if there is exactly one supporting line through P,

$$\text{then } P \text{ is a \textit{regular point} of } \partial H \text{ and the curve } \partial H \text{ is \textit{smooth} at } P. \tag{2.3}$$

Second, if there are at least two distinct supporting lines ℓ_1 and ℓ_2 through P, then P is a *corner* of ∂H (or of H). In both cases, a supporting line ℓ containing P is called the *last semitangent* of H through P if for every small positive ε, there is an $\varepsilon' \in (0, \varepsilon)$ such that the line obtained from ℓ by rotating it around P *forward* (that is, counterclockwise) by ε' degree is not a supporting line. The *first semitangent* is defined similarly. The first and the last semitangents coincide iff $P \in \partial H$ is a regular point. For $P \in \partial H$,

ℓ_P^- and ℓ_P^+ will denote the first semitangent and the last semitangent through P, respectively. When they coincide, $\ell_P := \ell_P^- = \ell_P^+$ will stand for the *tangent line* through P. (2.4)

Let us emphasize that no matter if $P \in \partial H$ is a regular point or a vertex,

there exists a supporting line through P; in particular, both ℓ_P^- and ℓ_P^+ exist and they are uniquely determined. (2.5)

Besides Yaglom and Boltyanskiĭ [31], this folkloric fact is also included, say, in Boyd and Vanderberghe [6, page 51]. We note but will not use the fact that every line separating P and the interior of H is a supporting line through P. As an illustration for (2.5), some supporting lines of H are given in Figure 1. If ℓ_i is the supporting line denoted by i in the figure, then $\ell_1 = \ell_{P_1}$ is a tangent line, $\ell_{P_2}^- = \ell_2$ is the first semitangent through P_2, and $\ell_{P_2}^+ = \ell_4$ is the last semitangent through the same point. We know from, say, Borwein and Vanderwerff [5, 2.2.15 in page 42], Yaglom and Boltyanskiĭ [31, page 110], or even from [32], that the boundary ∂H of a compact convex set $H \subseteq \mathbb{R}^2$ can have $\aleph_0$ many corners. This possibility, which is not so easy to imagine, also justifies that we are going to give a rigorous proof for our theorem. Next, restricting ourselves to the compact case and to the plane, we recall the *strict separation theorem* as follows.

If $H_1, H_2 \subseteq \mathbb{R}^2$ are *disjoint* compact convex set, then there exists a directed line ℓ such that H_1 is strictly on the left and H_2 is strictly on the right of ℓ. (2.6)

This result follows, for example, from Subsection 2.5.1 in Boyd and Vandenberghe [6] plus the fact that the *distance* $\mathrm{dist}(H_1, H_2)$ of H_1 and H_2 is positive in this case.

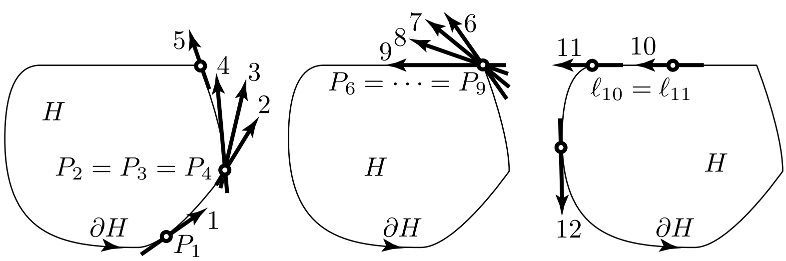

Figure 1. Supporting lines

3 A note and its corollary

Given a compact convex set H, visual intuition tells us that any supporting line can be continuously transformed to any other supporting line. We think of this transformation as a *slow*, *continuous* progression in time. For example, in Figure 1, ℓ_{i+1} comes, after some time, later than ℓ_i, for $i \in 1, \ldots, 11$. While continuity makes a well-known mathematical sense, a comment on slowness is appropriate here. By *slowness* we shall mean rectifiability, because this is what guarantees that running the process with a constant speed, it will terminate. Therefore, since rectifiability is an adjective of curves, we are going to associate a simple closed rectifiable curve with H such that the progression is described by moving along this curve forward. The only problem with this initial idea is that, say, ℓ_{11} cannot follow ℓ_{10}, because they are the same supporting lines. Therefore, we consider pointed supporting lines. A *pointed supporting line* of H is a pair $\langle P, \ell\rangle$ such that $P \in \partial H$ and ℓ is a supporting line of H through P. The transition from ℓ_i to ℓ_{i+1} will be called *slide-turning.* Of course, the $\langle P_i, \ell_i\rangle$, for $i \in \{1, \dots, 12\}$, represent only twelve snapshots of a continuous progression. In order to capture the progression mathematically, note that each pointed supporting line $\langle P, \ell\rangle$ of H is determined uniquely by the point $\langle P, \mathrm{dir}(\ell)\rangle \in \mathbb{R}^4$. To be more precise, define the following *cylinder*

$$\mathrm{Cyl} := \mathbb{R}^2 \times C_{\mathrm{unit}} = \{\langle x, y, z, t\rangle \in \mathbb{R}^4 : z^2 + t^2 = 1\} \subseteq \mathbb{R}^4. \tag{3.1}$$

As the crucial concept of this section, the *slide curve* of H is

$$\mathrm{Sli}(H) := \{\langle P, \mathrm{dir}(\ell)\rangle : \langle P, \ell\rangle \text{ is a pointed supporting line of } H\}; \tag{3.2}$$

it is a subset of Cyl. Although $\mathrm{Sli}(H)$ looks only a set at present, it will soon turn out that it is a curve. Actually, the main result of the paper says the following.

Theorem 1. *For every nonempty compact convex set $H \subseteq \mathbb{R}^2$, $\mathrm{Sli}(H)$ is a* rectifiable *simple closed curve.*

In order to exemplify the usefulness of this theorem, we state a corollary. Although it is well known, we have not found a rigorous proof for it.

Corollary 2. *If $H_1, H_2 \subseteq \mathbb{R}^2$ are disjoint compact convex sets with nonempty interiors, then they have exactly four non-directed supporting lines in common.*

The stipulation on the interior above can be relaxed but then we have to speak of *directed* supporting lines.

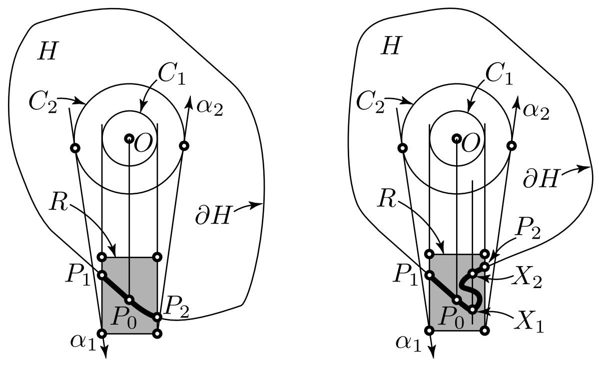

Figure 2. Reducing the problem to functions

4 Proofs

First proof of Theorem 1. We can assume that the interior of H is nonempty, because otherwise H is a line segment, possibly a singleton segment, and the statement trivially holds. In order to reduce the task to functions rather than convex sets, let P_0 be an arbitrary point of ∂H. Pick a point O in the interior of H, and choose a coordinate system such that both P_0 and O are on the y-axis and O is above P_0; see on the left of Figure 2. For a positive u, let C_1 and C_2 be the circles of radii u and $2u$ around O; we can assume that u is so small that C_2 is in the interior of H. Let A be the intersection of ∂H and the closed strip S between the two vertical tangent lines of C_1. In the figure, A is the thick arc of ∂H between P_1 and P_2. Let

$$\begin{aligned}&\mathrm{Sli}_A(H) := \{\langle P, \mathrm{dir}(\ell)\rangle : P \in A,\ \langle P, \mathrm{dir}(\ell)\rangle \in \mathrm{Sli}(H)\},\\ &\text{and similarly for future other arcs of } \partial H.\end{aligned} \tag{4.1}$$

Since the distance of O and the complement set of H is positive, we can assume that u is so small that the grey-filled rectangle containing A in the figure is strictly below C_2. (We have some freedom to choose the upper and lower edges of this rectangle.) Let $\alpha_1, \alpha_2 \in C_{\mathrm{unit}}$ be the directions of the external common supporting lines of C_2 and this rectangle, see the figure. Note that if we consider C_{unit} the interval $[-\pi, \pi)$, then $\alpha_1 = -\alpha_2$. The presence of C_2 within H guarantees the second half of the following observation:

$$\begin{aligned}&0 < \alpha_2 < \pi \text{ and for every supporting line } \ell \text{ of } H\\ &\text{that contains a point of } A,\ -\alpha_2 \le \mathrm{dir}(\ell) \le \alpha_2.\end{aligned} \tag{4.2}$$

We claim that

$$A \text{ is the graph of a convex function } f\colon [-u, u] \to \mathbb{R}. \tag{4.3}$$

By the convexity of H and (2.2), every vertical line in the strip S intersects A. Suppose, for a contradiction, that U is not the graph of a function. Then a vertical line in S intersects A in at least two distinct points, X_1 and X_2. Let, say, X_2 be above X_1; see on the right of the figure. Then X_2 is in the interior of the convex hull of $\{X_1\} \cup C_2$, whereby it is in the interior rather than on the boundary of H. This contradiction shows that f is a function. It is convex, since so is H. This proves (4.3). Clearly, the same consideration shows that

$$\text{each ray starting from } O \text{ intersects } \partial H \text{ exactly once.} \tag{4.4}$$

For a real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ and x_0 in the interior of its domain, the left derivative $\lim_{x\to x_0-}(f(x)-f(x_0))/(x-x_0)$ and the right derivative of f at x_0 are denoted by $f'_-(x_0)$ and $f'_+(x_0)$, respectively. By a theorem of Stolz [29], see also Niculescu and Persson [22, Theorem 1.3.3], if f is convex in the open interval $(-u, u)$, then

$$\begin{aligned}&\text{for all } x, x_1, x_2 \in (-u, u), \text{ both } f'_-(x) \text{ and } f'_+(x) \text{ exist,}\\ &f'_-(x) \le f'_+(x), \text{ and } x_1 < x_2 \text{ implies that } f'_+(x_1) \le f'_-(x_2).\end{aligned} \tag{4.5}$$

Recall that a function g from a subset of $\mathbb{R}^k$ to $\mathbb{R}^n$ is *Lipschitz* (or f is a *Lipschitz function* or f is *Lipschitzian*) if there exists a positive constant L such that $\mathrm{dist}(g(x), g(x')) \le L \cdot \mathrm{dist}(x, x')$ holds for all x and x' in the domain of g. Since f is convex, we know from Rockafellar [27, Theorems 10.1, 10.4, and 24.1] that

$$\begin{aligned}&\text{in } (-u, u),\ f \text{ is Lipschitz, } f'_- \text{ is continuous from}\\ &\text{the left, and } f'_+ \text{ is continuous from the right.}\end{aligned} \tag{4.6}$$

Note that if a function is Lipschitz in an interval, then it is uniformly continuous there. From now on, we consider f only in the open interval $(-u, u)$ and we fix a positive $v \in (0, u)$, For $x_0 \in (-u, u)$, the *subdifferential* is defined as the interval

$$\begin{aligned}f^{(\mathrm{sub})}(x_0) &= \{d \in \mathbb{R} : \forall x \in (-u, u),\ f(x) \ge f(x_0) + d(x - x_0)\}\\ &= [f'_-(x_0), f'_+(x_0)];\end{aligned} \tag{4.7}$$

see Niculescu and Persson [22, Section 1.5]. As a consequence of (4.5), the subdifferential is a *dissipative* set-valued function, that is,

$$\begin{array}{l}\text{for } x_1, x_2 \in (-u,u), \text{ if } x_1 < x_2,\ d_1 \in f^{(\mathrm{sub})}(x_1),\\ \text{and } d_2 \in f^{(\mathrm{sub})}(x_2), \text{ then } d_1 \le d_2.\end{array} \tag{4.8}$$

Consider the set

$$D := \{\langle x, d\rangle : x \in [-v, v] \text{ and } d \in f^{(\mathrm{sub})}(x)\} \subseteq \mathbb{R}^2 \tag{4.9}$$

with the (strict) lexicographic ordering

$$\langle x_1, d_1\rangle <^{\mathrm{lex}} \langle x_2, d_2\rangle \overset{\mathrm{def}}{\iff} (x_1 < x_2,\ \text{or } x_1 = x_2 \text{ and } d_1 < d_2). \tag{4.10}$$

We define a function

$$t\colon D \to \mathbb{R} \ \text{ by } \ t(x,d) = x + d. \tag{4.11}$$

Note that $t(x,d)$ is a short form of $t(\langle x,d\rangle)$. Recall that the *Manhattan distance* of $\langle x_1, d_1\rangle$ and $\langle x_2, d_2\rangle$ in $\mathbb{R}^2$ is defined as $\mathrm{d_M}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle) := |x_1 - x_2| + |d_1 - d_2|$. It has the usual properties of a distance function. It follows from (4.5) that, for $\langle x_1, d_1\rangle$ and $\langle x_2, d_2\rangle$ in D (rather than in $\mathbb{R}^2$),

$$\text{if } \langle x_1, d_1\rangle \le^{\mathrm{lex}} \langle x_2, d_2\rangle, \text{ then } \mathrm{d_M}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle) = t(x_2, d_2) - t(x_1, d_1); \tag{4.12}$$

that is, for points of D, the Manhattan distance is derived from the function t. Let $\mathrm{dist}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle)$ stand for the Euclidean distance $((x_1 - x_2)^2 + (d_1 - d_2)^2))^{1/2}$; in $\mathbb{R}^4$, it is understood analogously. For the sake of a later reference, we note in advance that for $x^{(i)}, d^{(i)} \in \mathbb{R}^2$, the *Manhattan distance* in $\mathbb{R}^4$ is understood as

$$\mathrm{d_M}(\langle x^{(1)}, d^{(1)}\rangle, \langle x^{(2)}, d^{(2)}\rangle) := \mathrm{dist}(x^{(1)}, x^{(2)}) + \mathrm{dist}(d^{(1)}, d^{(2)}). \tag{4.13}$$

It is well known and easy to see that, for all $\langle x_1, d_1\rangle, \langle x_1, d_1\rangle$ in $\mathbb{R}^2$, and even in $\mathbb{R}^4$ if $x_1, x_2, d_1, d_2 \in \mathbb{R}^2$,

$$\mathrm{dist}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle) \le \mathrm{d_M}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle) \le 2 \cdot \mathrm{dist}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle). \tag{4.14}$$

It follows from (4.12) and the second half of (4.14) that t is a Lipschitz function (with Lipschitz constant 2). Since $\mathrm{d_M}(-,-)$ is a distance function, (4.12) yields that t is injective. Actually, it is bijective as a $D \to \mathrm{Range}(t)$ function. Thus, it has an inverse function, $t^{-1}\colon \mathrm{Range}(t) \to D$, which is also bijective. In order to see that the function t^{-1} is also a Lipschitz function, let $y_i = t(x_i, d_i) = x_i + d_i \in \mathrm{Range}(t)$, for $i \in \{1, 2\}$. Since $\mathrm{dist}(-,-)$ is a symmetric function, we can assume that $\langle x_1, d_1\rangle \le^{\mathrm{lex}} \langle x_2, d_2\rangle$. We can also assume that $d_1 \le d_2$; either because $x_1 = x_2$ and then we can interchange the subscripts 1 and 2, or because $x_1 < x_2$ and (4.8) applies. With these assumptions, let us compute:

$$\begin{aligned}\mathrm{dist}(y_1, y_2) &= |y_2 - y_1| = |x_2 + d_2 - (x_1 + d_1)| = |x_2 - x_1 + d_2 - d_1|\\ &= x_2 - x_1 + d_2 - d_1 = |x_1 - x_2| + |d_1 - d_2| = \mathrm{d_M}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle).\end{aligned}$$

Hence, using the second part of (4.14), it follows that the function t^{-1} is Lipschitz (with Lipschitz constant 2). So, we can summarize that

$$\begin{array}{l}t\colon D \to \mathrm{Range}(t) \text{ and } t^{-1}\colon \mathrm{Range}(t) \to D \text{ are reciprocal bijections}\\ \text{and both of them are Lipschitz; in short, } t \text{ is } \textit{bi-Lipschitzian}.\end{array} \tag{4.15}$$

Next, let $w_1 = t(-v, f'_-(-v))$ and $w_2 = t(v, f'_+(v))$. We claim that

$$\mathrm{Range}(t) = [w_1, w_2]. \tag{4.16}$$

In order to see the easier inclusion, assume that $\langle x, d\rangle \in D$. Using (4.8) and (4.10), we obtain that $\langle -v, f'_-(-v)\rangle \leq^{\mathrm{lex}} \langle x, d\rangle \leq^{\mathrm{lex}} \langle v, f'_+(v)\rangle$. Thus, since (4.12) yields that t is order-preserving, we conclude that $w_1 \leq t(x, d) \leq w_2$, that is, $\mathrm{Range}(t) \subseteq [w_1, w_2]$. In order to show the converse inclusion, assume that $s \in [w_1, w_2]$. We need to find an $\langle x_0, d_0\rangle \in D$ such that $s = t(x_0, d_0)$, that is, $s = x_0 + d_0$. Define

$$\begin{aligned} x^- &:= \sup\{x : \text{there is a } d \text{ such that } \langle x, d\rangle \in D \text{ and } x + d \leq s\},\\ x^+ &:= \inf\{x : \text{there is a } d \text{ such that } \langle x, d\rangle \in D \text{ and } x + d \geq s\}. \end{aligned} \tag{4.17}$$

Since $t(-v, f'_-(-v)) = w_1 \leq s \leq w_2 = t(v, f'_+(v))$, the sets occurring in (4.17) are nonempty. Hence, both x^- and x^+ exist and we have that $x^-, x^+ \in [-v, v]$. Suppose, for a contradiction, that $x^+ < x^-$. Then $x^- = 3\varepsilon + x^+$ for a positive ε. By (4.17), which defines x^- and x^+, we can pick $\langle x^\dagger, d^\dagger\rangle, \langle x^\ddagger, d^\ddagger\rangle \in D$ such that $x^\dagger \in (-\varepsilon + x^-, x^-]$, $t(x^\dagger, d^\dagger) = x^\dagger + d^\dagger \leq s$, $x^\ddagger \in [x^+, \varepsilon + x^+)$, and $t(x^\ddagger, d^\ddagger) = x^\ddagger + d^\ddagger \geq s$. In particular, $x^\dagger + d^\dagger \leq x^\ddagger + d^\ddagger$. However, since $x^\ddagger < x^\dagger$, the dissipative property from (4.8) gives that $d^\ddagger \leq d^\dagger$, whereby $x^\dagger + d^\dagger \geq x^\dagger + d^\ddagger > x^\ddagger + d^\ddagger$, contradicting $x^\dagger + d^\dagger \leq x^\ddagger + d^\ddagger$. This proves that $x^- \leq x^+$. Next, suppose for a contradiction that $x^- < x^+$. Let $x^* := (x^- + x^+)/2$, and pick a $d^* \in f^{(\mathrm{sub})}(x^*)$. Since $x^* + d^* \leq s$ would contradict the definition of x^-, we have that $x^* + d^* > s$, which contradicts the definition of x^+. This excludes the case $x^- < x^+$. So we have that $x^- = x^+$, and we let $x_0 := x^- = x^+$. Clearly, for all x and the corresponding d in the upper line of (4.17), $x + f'_-(x) \leq x + d \leq s$. Hence, the left continuity formulated in (4.6) gives that $t(x_0, f'_-(x_0)) = x_0 + f'_-(x_0) = x^- + f'_-(x^-) \leq s$. Similarly, $t(x_0, f'_+(x_0)) = x_0 + f'_+(x_0) = x^+ + f'_+(x^+) \geq s$. So $x_0 + f'_-(x_0) \leq s \leq x_0 + f'_+(x_0)$, whereby (4.7) gives a $d_0 \in [f'_-(x_0), f'_+(x_0)]$ such that $s = x_0 + d_0 = t(x_0, d_0)$. This proves (4.16).

It is well known (and evident) that, with self-explanatory domains,

> the composition of two bi-Lipschitzian functions is bi-Lipschitzian. Thus, a bi-Lipschitzian function maps a rectifiable simple curve to a rectifiable simple curve. (4.18)

Before utilizing (4.18), we need some preparations. Let $Q_1 = \langle -v, f(-v)\rangle$ and $Q_2 = \langle v, f(v)\rangle$; they are points on the arc A before and after P_0, respectively. Let B be the sub-arc of A (and of ∂H) from Q_1 to Q_2, and note that P_0 is in the interior of B. Let $f^*\colon [-v, v] \to B$ be the function defined by $f^*(x) := \langle x, f(x)\rangle$. Using (4.6) and the relation between the Euclidean and the Manhattan distance functions, see (4.14), it follows that f^* is Lipschitz. This fact implies trivially that f^* is bi-Lipschitzian. So is the arctangent function on $[-v, v]$. Therefore, it follows in a straightforward way from (4.14) that the Cartesian (or categorical) product function

> $\langle f^*, \arctan\rangle\colon D \to \mathrm{Sli}_B(H)$, defined by $\langle x, d\rangle \mapsto \langle f^*(x), \arctan(d)\rangle$, where $\mathrm{Sli}_B(H)$ is defined in (4.1), is bi-Lipschitzian. (4.19)

The line segment $[w_1, w_2]$ is clearly a simple rectifiable curve. So is D by (4.11), (4.15), (4.16), and (4.18). Hence, (4.18) and (4.19) yield that $\mathrm{Sli}_B(H)$ is a simple rectifiable curve. Finally, since $P_0 \in \partial H$ was arbitrary and since the endpoints of B can be omitted from B, we obtain that ∂H can be covered by a set $\{B_i : i \in I\}$ of open arcs such that the $\mathrm{Sli}_{B_i}(H) \subseteq \mathrm{Cyl}$ are simple rectifiable curves. Clearly, the $\mathrm{Sli}_{B_i}(H)$ cover $\mathrm{Sli}(H)$. Since

∂H is compact, we can assume that I is finite. Therefore, $\mathrm{Sli}(H)$ is covered by finitely many open simple rectifiable curves. Furthermore, (4.4) yields that each of these open curves overlaps with its neighbors. Thus, we conclude the validity of Theorem 1. $\square$

In the following proof, the argument leading to (4.20) can be extracted from the more general approach of Kneser [16] and Stachó [28]. For the planar case and for the reader's convenience, it is more convenient to prove (4.20) directly.

Second proof of Theorem 1. Define $H^{+1} := \{P \in \mathbb{R}^2 : \mathrm{dist}(P, H)) \le 1\}$. First, we prove that H^{+1} is a compact convex set. Let Q be a limit point of H^{+1} and suppose, for a contradiction, that $Q \notin H^{+1}$. This means that $\mathrm{dist}(Q, H) = 1 + 3\varepsilon$ for a positive $\varepsilon \in \mathbb{R}$. Take a sequence $(P_n : n \in \mathbb{N})$ of points in H^{+1} such that $\lim_{n\to\infty} P_n = P$. For each $n \in N$, pick a point $Q_n \in H$ such that $\mathrm{dist}(P_n, Q_n) \le 1$. Since H is compact, the sequence $(Q_n : n \in \mathbb{N})$ has a convergent subsequence. Deleting members if necessary, we can assume that $(Q_n : n \in \mathbb{N})$ itself converges to a point Q of H. Take a sufficiently large $n \in \mathbb{N}$ such that $\mathrm{dist}(P, P_n) < \varepsilon$ and $\mathrm{dist}(Q_n, Q) < \varepsilon$. Then $1 + 3\varepsilon = \mathrm{dist}(P, Q) \le \mathrm{dist}(P, P_n) + \mathrm{dist}(P_n, Q_n) + \mathrm{dist}(Q_n, Q) \le \varepsilon + 1 + \varepsilon = 1 + 2\varepsilon$ is a contradiction. Hence, H^{+1} is closed, whereby it is obviously compact. In order to show that it is convex, let $X, Y \in H^{+1}$ and let $\lambda \in (0, 1)$; we need to show that $Z := (1 - \lambda)X + \lambda Y \in H^{+1}$. The containments $X \in H^{+1}$ and $Y \in H^{+1}$ are witnessed by some $X_0, Y_0 \in H$ such that $\mathrm{dist}(X, X_0) \le 1$ and $\mathrm{dist}(Y, Y_0) \le 1$. Since H is convex, $Z_0 := (1 - \lambda)X_0 + \lambda Y_0 \in H$. The vectors $\vec{a} := X - X_0$ and $\vec{b} := Y - Y_0$ are of length at most 1, and it suffices to show that so is $\vec{c} := Z - Z_0$. Since $(\vec{a}, \vec{b}) \le ||\vec{a}|| \cdot ||\vec{b}|| \le 1$, we have that

$$\begin{aligned}
(\vec{c}, \vec{c}) &= ((1 - \lambda)\vec{a} + \lambda\vec{b}, (1 - \lambda)\vec{a} + \lambda\vec{b}) \\
&= (1 - \lambda)^2(\vec{a}, \vec{a}) + \lambda^2(\vec{b}, \vec{b}) + 2\lambda(1 - \lambda)(\vec{a}, \vec{b}) \\
&\le (1 - \lambda)^2 + \lambda^2 + 2\lambda(1 - \lambda) = 1.
\end{aligned}$$

Hence, $\mathrm{dist}(Z, Z_0) = ||\vec{c}|| \le 1$, and H^{+1} is convex. Thus, (2.2) gives that

$$\partial H^{+1} \text{ is rectifiable Jordan curve.} \tag{4.20}$$

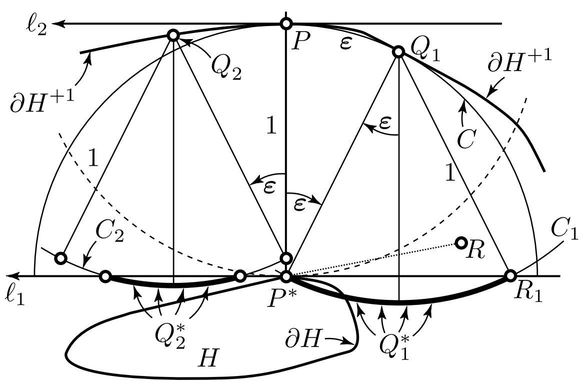

Figure 3. Illustration for the second proof

Clearly, $\partial H^{+1} = \{X : \mathrm{dist}(X, H) = 1\} = \{X : \mathrm{dist}(X, \partial H) = 1\}$. Define the following relation

$$\rho := \{\langle P, P^* \rangle \in \partial H^{+1} \times \partial H : \mathrm{dist}(P, P^*) = 1\}$$

between ∂H^{+1} and ∂H; see Figure 3. Let $\langle P, P^*\rangle \in \rho$ as in the figure. The coordinate system is chosen so that P and P^* determine a vertical line and P is above P^*. Through P^* and P, let ℓ_1 and ℓ_2 be the lines of direction π; they are perpendicular to $[P^*, P]$. We claim that

$$\ell_1 \text{ is a supporting line of } H. \tag{4.21}$$

Suppose to the contrary that ℓ_1 is not a supporting line and pick a point $R \in H$ strictly on the right of ℓ_1; see the figure. Since $P \in \partial H^{+1}$, $\mathrm{dist}(P, H) = 1$, whereby R cannot be inside the dotted circle of radius 1 around P. However, since this circle touches ℓ_1 at P^*, the line segment $[P^*, R]$, which is a subset of H by convexity, has a point inside the dotted circle. This contradicts $\mathrm{dist}(P, H) = 1$ and proves (4.21). From (4.21), it follows that if $\langle P, Q\rangle \in \rho$, then $Q = P^*$. Hence,

$$\begin{gathered} f\colon \partial H^{+1} \to \mathrm{Sli}(H), \text{ defined by } f(P) = \langle P^*, \mathrm{dir}(\ell^*)\rangle \in \mathrm{Sli}(H) \iff \\ \langle P, P^*\rangle \in \rho,\ \ell^* \text{ is a supporting line, and } \ell^* \text{ is perpendicular to } [P, P^*] \end{gathered}$$

is a mapping. Trivially,

$$\begin{gathered} g\colon \mathrm{Sli}(H) \to \partial H^{+1}, \text{ defined by } g(\langle P^*, \mathrm{dir}(\ell^*)\rangle) = P \iff \\ \mathrm{dir}([P^*, P]) = \mathrm{dir}(\ell^*) - \pi/2 \text{ and } \mathrm{dist}(P, P^*) = 1, \end{gathered} \tag{4.22}$$

is also a mapping. Moreover f and g are reciprocal bijections. Recall from Luukkainen [20, Definition 2.14] that a function $\tau\colon X \to Y$ is *Lipschitz in the small* if there are $\delta > 0$ and $L \ge 0$ such that $\mathrm{dist}(\tau(x_1), \tau(x_2)) \le L \cdot \mathrm{dist}(x_1, x_2)$ for all $x_1, x_2 \in X$ with $\mathrm{dist}(x_1, x_2) \le \delta$. We know from [20, 2.15] that every bounded function with this property is Lipschitz. We are going to show that f and g are Lipschitz in the small, witnessed by $\delta = 1/5$ and $L = 9$, because then $g = f^{-1}$, (4.18), and (4.20) will imply the theorem. (Note that $\delta = 1/5$ and $L = 9$ are convenient but none of them is optimal.)

First, we deal with f. Assume that $Q_1 \in \partial H^{+1}$ such that $\gamma := \mathrm{dist}(P, Q_1) < \delta = 1/5$; see Figure 3. The angle $\boldsymbol{\varepsilon} := \angle(PP^*Q_1)$, which is the length of the circular arc from P to Q_1, is close to γ in the sense that

$$\text{both } \boldsymbol{\varepsilon}/\gamma \text{ and } \gamma/\boldsymbol{\varepsilon} \text{ are in the interval } (99/100, 101/100); \tag{4.23}$$

this is shown by easy trigonometry since both $\sin(1/5)/(1/5)$ and $(1/5/)\sin(1/5)$ are in the open interval on the right of (4.23). Let C and C_1 be the circles of radius 1 around P^* and Q_1, respectively. Since $\mathrm{dist}(Q_1, H) = 1$, Q_1 is not in the interior of (the disk determined by) C. Also, since ℓ_1 is a supporting line of H, we have that ℓ_2 is a supporting line of H^{+1} and Q_1 cannot be strictly on the right (that is, above) ℓ_2. So either Q_1 is on the circle C, or it is above C but not above ℓ_2 (but then we write Q_2 instead of Q_1 in the figure). Denote $f(Q_1)$ by $\langle Q_1^*, \mathrm{dir}(\ell_1^*)\rangle$. Clearly, Q_1^* is on the thick arc of C_1 from P^* to R_1, as indicated in the figure. The length of this arc is $2\boldsymbol{\varepsilon}$, whence $\mathrm{dist}(P^*, Q_1^*) \le 2\boldsymbol{\varepsilon}$. Since ℓ_1^* is perpendicular to $[Q_1, Q_1^*]$ and Q_1^* is on the thick arc of C_1, we have that $\mathrm{dist}(\mathrm{dir}(\ell^*), \mathrm{dir}(\ell_1^*)) \le \boldsymbol{\varepsilon} \le 2\boldsymbol{\varepsilon}$. So the Manhattan distance $\mathrm{d_M}(\langle P^*, \mathrm{dir}(\ell^*)\rangle, \langle Q_1^*, \mathrm{dir}(\ell_1^*)\rangle)$, see (4.13), is at most $4\boldsymbol{\varepsilon}$. Hence, (4.14) and (4.23) yield that $\mathrm{dist}(f(P), f(Q_1) \le 9 \cdot \mathrm{dist}(P, Q_1)$. The other case, represented by Q_2, follows from the fact that $\mathrm{dist}(P^*, Q_2^*)$ and $\mathrm{dist}(\mathrm{dir}(\ell^*), \mathrm{dir}(\ell_2^*))$ are smaller than the respective distances in the previous case. This shows that f is Lipschitz in the small.

Next, we deal with g. Assume that $\langle P^*, \mathrm{dir}(\ell^*)\rangle$ and $\langle P_1^*, \mathrm{dir}(\ell_1^*)\rangle$ are in $\mathrm{Sli}(H)$ and their distance, γ, is less than δ. With the auxiliary point $\langle P^*, \mathrm{dir}(\ell_1^*)\rangle \in \mathbb{R}^4$, which need not be in $\mathrm{Sli}(H)$, we have that $\mathrm{dist}(\langle P^*, \mathrm{dir}(\ell^*)\rangle, \langle P^*, \mathrm{dir}(\ell_1^*)\rangle) \le \gamma$ and $\mathrm{dist}(\langle P^*, \mathrm{dir}(\ell_1^*)\rangle, \langle P_1^*, \mathrm{dir}(\ell_1^*)\rangle) \le \gamma$. Although the auxiliary point is not in the domain of g in general, we can extend the domain of g to this point by (4.22). Since the secants of the unit circles are shorter than the corresponding circular arcs, whose lengths equal the corresponding central angles, it follows that $\mathrm{dist}(g(\langle P^*, \mathrm{dir}(\ell^*)\rangle), g(\langle P^*, \mathrm{dir}(\ell_1^*)\rangle)) \le \gamma$. Since parallel shifts are distance-preserving, $\mathrm{dist}(g(\langle P^*, \mathrm{dir}(\ell_1^*)\rangle), g(\langle P_1^*, \mathrm{dir}(\ell_1^*)\rangle)) = \gamma$. Hence, the triangle inequality yields that $\mathrm{dist}(g(\langle P^*, \mathrm{dir}(\ell^*)\rangle), g(\langle P_1^*, \mathrm{dir}(\ell_1^*)\rangle)) \le 2\gamma \le 9\delta$. Thus, g is also Lipschitz in the small, as required. This completes the second proof of Theorem 1. $\square$

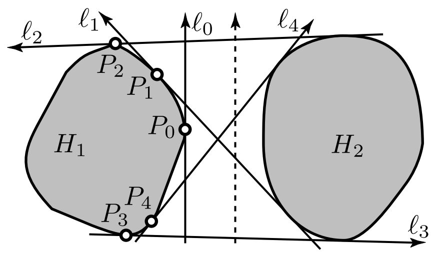

Figure 4. Illustration for Corollary 2

Proof of Corollary 2. By (2.6), we have a directed line, the dotted one in Figure 4, such that H_1 is strictly in the left and H_2 is strictly on the right of this line. By (2.1), we can take a $\langle P_0, \mathrm{dir}(\ell_0)\rangle \in \mathrm{Sli}(H_1)$ such that ℓ_0 and the dotted line have the same direction. For $0 < L \in \mathbb{R}$, let

$$L \cdot C_{\mathrm{unit}} \text{ denote the circle } \{\langle x, y\rangle : x^2 + y^2 = (L/(2\pi))^2\} \text{ of perimeter } L.$$

Since $\mathrm{Sli}(H_1)$ is a rectifiable simple closed curve by Theorem 1, we can let L be its perimeter. Let

$$\{h(t) : t \in L \cdot C_{\mathrm{unit}}\} \text{ be a parameterization of } \mathrm{Sli}(H_1) \tag{4.24}$$

such that $\langle P_0, \mathrm{dir}(\ell_0)\rangle = h(t_0)$. We think of the parameter t as the *time* measured in seconds. While the time t is slowly passing, $\langle P(t), \mathrm{dir}(\ell(t))\rangle$ is slowly and continuously moving forward along $\mathrm{Sli}(H_1)$, and the directed supporting line $\langle P(t), \ell(t)\rangle$ is *slide-turning* forward, slowly and continuously. Since H_2 is compact, the distance $\mathrm{dist}(\ell(t), H_2)$ is always witnessed by a pair of points in $\ell(t) \times H_2$, and this distance is a continuous function of t. At $t = t_0$, this distance is positive and H_2 is on the right of $\ell_0 = \ell(t_0)$. Slide-turn this pointed supporting line around H_1 forward during L seconds; that is, make a full turn around $\mathrm{Sli}(H_1)$. By continuity, in the chronological order listed below, there are

1. a last $t = t_1$ such that H_2 is still on the right of $\ell(t)$ (this t_1 exists, because it is the first value of t where $\mathrm{dist}(\ell(t), H_2) = 0$),

2. a first $t = t_2$ such that H_2 is on the left of $\ell(t)$,

3. a last $t = t_3$ such that H_2 is still on the left of $\ell(t)$,

4. a first $t = t_4$ such that H_2 is on the right of $\ell(t)$.

In Figure 4, $h(t_i) = \langle P(t_i), \mathrm{dir}(\ell(t_i))\rangle$ is represented by $\langle P_i, \ell_i\rangle$. Clearly, $\ell_1, \ldots, \ell_4$ is the list of all common supporting lines and these lines are pairwise disjoint. $\square$

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Roughness in MV-modules

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Abstract
In this paper, we consider an MV-module M over a PMV-algebra A as a universal set and we introduce the notion of a rough A-ideal with respect to an A-ideal of an A-module M, which is an extended notion of an A-ideal in an MV-module M. We also give some properties of the lower and the upper approximations in an A-module. In particular, we study the lower and the upper approximations with respect to fuzzy congruences in MV-modules.

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1 Introduction and Preliminaries

The notion of rough sets was introduced by Pawlak [20]. The relations between rough sets and algebraic systems have been already considered by many mathematicians.

Some authors, for example, Iwinski [16], and Pomykala [22] have studied algebraic properties of rough sets. The lattice theoretical approach has been suggested by Iwinski [16]. Pomykala [22] showed that the set of rough sets forms a stone algebra. Comer [4] presented an interesting discussion of rough sets and various algebras related to the study of algebraic logic, such as Stone algebras and relation algebras. If we substitute an algebraic system instead of the universe set, then a natural question is what will happen. Biswas and Nanda [1] introduced the notion of rough subgroups. Kuroki in [17], introduced the notion of a rough ideal in a semigroup, also see [25]. Davvaz in [5] introduced the notion of rough subrings with respect to an ideal of a ring, also see [6]. Rough modules have been investigated by Davvaz and Mahdavipour [7]. Also Rasouli and Davvaz introduced the notion of roughness in MV-algebras [23].

In 1958, Chang defined the MV-algebras and in 1959 he also proved the completeness theorem which stated the real unit interval [0,1] as a standard model of this logic [2].

In 2003, Di Nola, et.al. introduced the notion of MV-modules over a PMV-algebra and A-ideals in MV-modules [10]. These are structures that naturally correspond to lu-modules over lu-ring [24]. We recall that an lu-ring is a pair (R,u) where $(R, \oplus, \cdot, 0, \leq)$ is an l-ring and u is a strong unit of R (i.e, u is a strong unit of the underlying l-group), with $u \cdot u \leq u$ and l-ring is a structure $(R, +, \cdot, 0, \leq)$ that $(R, +, 0, \leq)$ is an l-group and for any $x, y \in R$, $x \geq 0$ and $y \geq 0$ implies $x \cdot y \geq 0$. Fixing an lu-ring

(R,v), they proved equivalence between the category of lu-modules over (R,v) and the category of MV-modules over $\Gamma(R, v)$. They also proved the natural equivalence between MV-modules and truncated modules [10].

In the present paper, we consider an MV-module over PMV-algebra A as a universal set and we shall introduce the notion of rough A-ideal with respect to an A-ideal of an MV-module, which is an extended notion of an A-ideal in an MV-module. We give some properties of the lower and the upper approximations in an MV-module.

1.1 MV-modules

In this section, we summarize the basic concepts on MV-algebras and MV-modules. For more details on these concepts, we refer the reader to [2], [3]-[12] and [21].

Definition 1. [2] An MV-algebra is a structure $(M, \oplus, ^*, 0)$ where $\oplus$ is a binary operation, *, is a unary operation and 0 is a constant such that the following conditions are satisfied for any $a, b \in M$:

1. $(M, \oplus, 0)$ is an abelian monoid,
2. $(a^*)^* = a$,
3. $0^* \oplus a = 0^*$,
4. $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Define the constant $1 = 0^*$ and the auxiliary operations $\odot, \vee$ and $\wedge$ by:

$$a \odot b = (a^* \oplus b^*)^*, \quad a \vee b = a \oplus (b \odot a^*), \quad a \wedge b = a \odot (b \oplus a^*).$$

It is shown that $(M, \odot, 1)$ is an abelian monoid and the structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice [21].

In an MV-algebra M, the Chang distance function is

$$d : M \times M \longrightarrow M, \quad d(a, b) := (a \odot b^*) \oplus (b \odot a^*).$$

Note. An element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. We denote the set of complemented of A by $B(A)$.

Lemma 2. *[21] Let M be an MV-algebra. If $x, y, z, t \in M$ and d is a Chang distance function, then*

1. $x \leq y$ *iff* $y^* \leq x^*$,
2. $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,
3. $(x \vee y)^* = x^* \wedge y^*$, $(x \wedge y)^* = x^* \vee y^*$,
4. $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,
5. *If* $x \leq y$ *and* $z \leq t$, *then* $x \oplus z \leq y \oplus t$,
6. *If* $x \in B(A)$, *then* $x \odot x = x$, $x \oplus x = x$ *and* $x \wedge y = x \odot y$,
7. $x \leq y^* \oplus z$ *if and only if* $x \odot y \leq z$,
8. $d(x, 0) = x$, $d(x, 1) = x$,
9. $d(x^*, y^*) = d(x, y)$,

10. $d(x, y) \leq d(x, z) \oplus d(z, y)$,

11. $d(x \oplus u, y \oplus v) \leq d(x, y) \oplus d(u, v)$.

Lemma 3. *[3] Let M be an MV-algebra. For $x, y \in M$, the following conditions are equivalent*

1. $x^* \oplus y = 1$,

2. $x \odot y^* = 0$.

For any two elements $x, y \in M$, $x \leq y$ iff x and y satisfy the equivalent conditions (1)-(2) in the above lemma.

Definition 4. [2] An ideal of an MV-algebra M is a nonempty subset I of M satisfying the following conditions:

1. If $x \in I$, $y \in M$ and $y \leq x$ then $y \in I$,

2. If $x, y \in I$, then $x \oplus y \in I$.

We denote by $Id(M)$ the set of all ideals of an MV-algebra M.

Definition 5. [3] Let A and B be two MV-algebras. A function $h : A \to B$ is a morphism of MV-algebras if and only if it satisfies the following conditions, for every $x, y \in A$:

1. $h(0) = 0$

2. $h(a \oplus b) = h(a) \oplus h(b)$,

3. $h(a^*) = h(a)^*$.

Lemma 6. *Let M be a linearly ordered MV-algebra and I be an ideal of M. If $x \leq y$ and $[x]_I \neq [y]_I$, for $x, y \in A$, then for each $t \in [x]_I$ and $s \in [y]_I$, $t \leq s$.*

Definition 7. [9] A product MV-algebra (or PMV-algebra, for short) is a structure (A, $\oplus$, *,$\cdot$, 0), where (A, $\oplus$, *, 0) is an MV-algebra and $\cdot$ is a binary associative operation on A such that the following property is satisfied if $x + y$ is defined, then $x \cdot z + y \cdot z$ and $z \cdot x + z \cdot y$ are defined and

$$(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y,$$

where, $+$ is a partial addition on an MV-algebra A as follows For any $x, y \in M$, $x + y$ is defined if and only if $x \leq y^*$ and in this case,

$$x + y := x \oplus y,$$

the partial addition was defined in [11].

Note. If A is PMV-algebra, then a unit for the product is an element $e \in A$ such that $e \cdot x = x \cdot e = x$ for any $x \in A$. A PMV-algebra that has unity for the product will be called unital.

Theorem 8. *[9] A finite MV-algebra A admits a product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in A$ if and only if A is a Boolean algebra, i.e., $a \oplus a = a$ for any $a \in A$. If it is the case, then $a \cdot b = a \wedge b \in A$.*

Definition 9. [10] Let $(A, \oplus, {}^*, \cdot, 0)$ be a PMV-algebra and $(M, \oplus, {}^*, 0)$ an MV-algebra. We say that M is a (left) MV-module over A (or, simply, A-module) if there is an external operation:

$$\varphi : A \times M \longrightarrow M, \quad \varphi(\alpha, x) = \alpha x,$$

such that the following properties hold for any $x, y \in M$ and $\alpha, \beta \in A$:

1. If $x + y$ is defined in M, then $\alpha x + \alpha y$ is defined and

$$\alpha(x + y) = \alpha x + \alpha y,$$

2. If $\alpha + \beta$ is defined in A then $\alpha x + \beta x$ is defined in M and

$$(\alpha + \beta)x = \alpha x + \beta x,$$

3. $(\alpha \cdot \beta)x = \alpha(\beta x)$.

We say that M is a unital MV-module if A is a unital PMV-algebra and M is an MV-module over A such that $1_A x = x$ for any $x \in M$.

We will refer to [9, 19] for the basic properties of PMV-algebras. Obviously, a PMV-algebra homomorphism will be an MV-algebra homomorphism which also commutes with the product operation. We shall denote by $\mathcal{PMV}$ the category of product MV-algebras with the corresponding homomorphisms.

In the sequel, an lu-ring will be a pair (R, u) where $(R, \oplus, \cdot, \leq)$ is an l-ring and u is a strong unit of R such that $u \cdot u \leq u$. We imply that the interval $[0, u]$ of an lu-ring (R, u) is closed under the product of R. Thus, if we consider the restriction of $\cdot$ to $[0, u] \times [0, u]$, then the interval $[0, u]$ has a canonical PMV-algebra structure:

$$x \oplus y := (x + y) \wedge u, \quad x^* := u - x, \quad x \cdot y := x \cdot y,$$

for any $0 \leq x, y \leq u$. We shall denote this structure by $[0, u]_R$.

If $\mathcal{UR}$ is the category of lu-rings, whose objects are pairs (R, u) as above and whose morphisms are l-rings homomorphisms which preserve the strong unit, then we get a functor

$$\Gamma : \mathcal{UR} \to \mathcal{PMV},$$

$$\Gamma(R, u) := [0, u]_R, \text{ for any lu-ring } (R, u),$$

$$\Gamma(h) := h \mid_{[0,u]} \text{ for any lu-rings homomorphism h.}$$

In [9] it is proved that Γ establishes a categorical equivalence between $\mathcal{UR}$ and $\mathcal{PMV}$.

Definition 10. [10] Let M and N be two MV-modules over a PMV-algebra A. An A-module homomorphism is an MV-algebra homomorphism $h : M \to N$ such that $h(\alpha x) = \alpha h(x)$, for any $\alpha \in A$ and $x \in M$.

Definition 11. [10] Let M be an A-module. Then ideal $I \subseteq M$ is called an A-ideal if it satisfies the following condition: if $x \in I$ and $\alpha \in A$, then $\alpha x \in I$.

Lemma 12. *[10] If M is an A-module, then the following properties hold for any $x, y \in M$ and $\alpha, \beta \in A$,*

(a) $\alpha x^* \leq (\alpha x)^*$,

(b) $(\alpha x) \odot (\alpha y)^* \leq \alpha(x \odot y^*)$,

(c) $\alpha(x \oplus y) \leq \alpha x \oplus \alpha y$,

(d) If $x \leq y$, *then* $\alpha x \leq \alpha y$,

(e) $(\alpha x)^* = \alpha^* x + (1x)^*$,

(f) $d(\alpha x, \alpha y) \leq \alpha d(x, y)$.

Proposition 13. *[13] Let M be an A-module.*

1. *If $N \subseteq M$ is a nonempty set, then we have $(N] = \{x \in M : x \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 y_1 \oplus \ldots \oplus \alpha_m y_m \quad \text{for some} \quad x_1, \ldots, x_n, y_1, \ldots y_m \in N, \alpha_1, \ldots \alpha_m \in A\}$, where by $(N]$, we mean the ideal generated by N.*

   In particular, for $a \in M$,

   $$(a] = \{x \in M : x \leq na \oplus m(\alpha a) \quad \text{for some integer} \quad n, m \geq 0\},$$

2. *If $I_1, I_2 \in Id_A(M)$, then*
   $I_1 \vee I_2 = (I_1 \cup I_2] = \{a \in M : a \leq x_1 \oplus x_2 \quad \text{for some} \quad x_1 \in I_1 \quad \text{and} \quad x_2 \in I_2\}$,

3. *If $x, y \in A$, then $(x \wedge y] \subseteq (x] \cap (y]$.*

1.2 Pawlak approximation spaces

Let θ be an equivalence relation on a set U. The set of the elements of U that are related to $x \in U$, is called the equivalence class of x, and is denoted by $[x]_\theta$. In addition U/θ denote the family of all equivalence classes induced on U by θ. For any $X \subseteq U$, we write X^c to denote the complement of X in U, that is the set $U \setminus X$.

Definition 14. A pair (U, θ) where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an approximation space. The interpretation of rough set is that our knowledge of the objects in U extends only up to a membership in the class of θ, and our knowledge about a subset X of U is limited to the class of θ and their unions.

This leads to the following definition.

Definition 15. For an approximation space (U, θ), by a rough approximation in (U, θ) we mean a mapping $Apr : P(U) \to P(U) \times P(U)$ defined for every $X \in P(U)$ by

$$Apr(X) = (\underline{Apr}(X), \overline{Apr}(X)),$$

where $\underline{Apr}(X) = \{x \in U | [x]_\theta \subseteq X\}$, $\overline{Apr}(X) = \{x \in U | [x]_\theta \cap X \neq \emptyset\}$. $\overline{Apr}(X)$ is called an upper rough approximation of X in (U, θ), while $\underline{Apr}(X)$ is called a lower rough approximation of X in $U, \theta)$.

Definition 16. Given an approximation space (U, θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough subset in (U, θ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$. Note that a rough subset is also called a rough set.

The reader will find a deep study of rough set theory in [1, 5, 6, 7, 16, 17, 20, 23].

Definition 17. Let $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ and $Apr(B) = (\underline{Apr}(B), \overline{Apr}(B))$ be any two rough sets in the approximation space (U, θ). Then

1. $Apr(A) \sqcup Apr(B) = (\underline{Apr}(A) \cup \underline{Apr}(B), \overline{Apr}(A) \cup \overline{Apr}(B))$,

2. $Apr(A) \sqcap Apr(B) = (\underline{Apr}(A) \cap \underline{Apr}(B), \overline{Apr}(A) \cap \overline{Apr}(B))$,

3. $Apr(A) \sqsubseteq Apr(B) \iff Apr(A) \sqcap Apr(B) = Apr(A)$.

When $Apr(A) \sqsubseteq Apr(B)$, we say that $Apr(A)$ is a rough subset of $Apr(B)$.
Thus in the case of rough sets $Apr(A)$ and $Apr(B)$,
$Apr(A) \sqsubseteq Apr(B)$ if and only if $\underline{Apr}(A) \subseteq \underline{Apr}(B)$ and $\overline{Apr}(A) \subseteq \overline{Apr}(B)$.
This property of rough inclusion has all the properties of set inclusion. The rough complement of $Apr(A)$ denoted by $Apr^c(A)$ is defined by

$$Apr^c(A) = (U \setminus \overline{Apr}(A), U \setminus \underline{Apr}(A)).$$

Also, we can define $Apr(A) \setminus Apr(B)$ as follows:

$$Apr(A) \setminus Apr(B) = Apr(A) \sqcap Apr^c(B) = (\underline{Apr}(A) \setminus \overline{Apr}(B), \overline{Apr}(A) \setminus \underline{Apr}(B)).$$

Definition 18. [8] Let (U, θ) be an approximation space and X a non-empty subset of U.

1. If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable.

2. If $\underline{Apr}(X) = \emptyset$, then X is called empty interior.

3. If $\overline{Apr}(X) = U$, then X is called empty exterior.

The lower approximation of X in (U, θ) is the greatest definable set in U contained in X. The upper approximation of X in (U, θ) is the least definable set in U containing X. Therefore we have:

$$\underline{Apr}(X) = \bigcup \{S | S \subseteq X, \text{ S is definable}\},$$

$$\overline{Apr}(X) = \bigcap \{S | X \subseteq S, \text{ S is definable}\}.$$

A rough set X is the family of all subsets of U having the same lower and the same upper approximations of X.

2 Rough A-ideals in MV-modules over PMV-algebras

Throughout this paper M is an MV-module over a PMV-algebra A. We recall that in an MV-algebra M, the Chang distance function is

$$d : M \times M \longrightarrow M, \quad d(a, b) := (a \odot b^*) \oplus (b \odot a^*).$$

Let I be an A-ideal of M. We recall that the relation ρ_I defined by:

$$(x, y) \in \rho_I \quad \text{if and only if} \quad d(x, y) \in I,$$

for any $x, y \in M$, is a congruence with respect to the MV-algebra operations and $(x, y) \in \rho_I$ implies $(\alpha x, \alpha y) \in \rho_I$, for any $\alpha \in A$. Thus, the quotient MV-algebra M/I has a canonical structure of A-module

$$\alpha[x]_I := [\alpha x]_I \quad or \quad \alpha(x/I) := (\alpha x)/I,$$

where $[x]_I = x/I$ is the congruence class of x. $x/I = y/I$ or $x\rho_I y$ if and only if $d(x, y) \in I$ and if $x, y \in M$, then $x/I \leq y/I$ if and only if $x \odot y^* \in I$. Also $a \in x/I$ if and only if $d(a, x) \in I$.

Let I be an A-ideal of an A-module M. Then the quotient group M/I is an A-module with the action of A on M/I given by the well-defined map

$$\alpha(a/I) = (\alpha a)/I, \text{ for all } \alpha \in A, \ a \in M.$$

Let I be an A-ideal of M and X a non-empty subset of M, then the sets

$$\underline{\rho}_I(X) = \underline{Apr}_I(X) = \{x \in M | x/I \subseteq X\} \text{ and } \overline{\rho}_I(X) = \overline{Apr}_I(X) = \{x \in M | x/I \cap X \neq \emptyset\}$$

are called lower and upper approximations of the set X with respect to the A-ideal I. In this case we use the pair (M, I) instead of the approximation space (U, θ).

Now, we give an example of the lower and upper approximations theory applied the MV-module theory.

Example 19. Let $M_2(\mathbb{R})$ be the ring of square matrices of order 2 with real elements and 0 be the matrix with all element 0. If we define the order relation on components $A = (a_{ij})_{i,j=1,2} \geq 0$ iff $a_{ij} \geq 0$ for any i, j, such that $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, then $A = \Gamma(M_2(\mathbb{R}), v)$ is a PMV-algebra. Let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ be the direct product with the order relation defined on components. If $M = \Gamma(\mathbb{R}^2, u)$ is an MV-algebra, where $u = (1, 1)$, $(x, y)^* = u - (x, y)$, $(x, y) \oplus (z, t) = \min\{u, (x + z, y + t)\}$, and $(x, y) \odot (z, t) = \max\{(0, 0), (x, y) + (z, t) - u\}$, then M is an A-module [10], where the external operation is the usual matrix multiplication

$$(A, (x, y)) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now, let $I = \{(0, 0)\}$. Then I is an A-ideal of $R \times R$. We consider the maps

$$f(x) = 1/2(\text{sinx} - 4) \text{ and } g(x) = 1/2(\text{sinx} + 4),$$

and suppose that

$$X = \{(x, y) | f(x) \leq y \leq g(x)\}.$$

Then we have $(x, y)/I = \{(a, b) \in M | d((x, y), (a, b)) = 0\} = \{(a, b) \in M | (x, y) = (a, b)\}$. Thus the lower and upper approximations of this set can be calculate in the following way:

$$\underline{Apr}_I(X) = X = \overline{Apr}_I(X)$$

In general, we can prove that:

Remark 20. Let $I = \{0\}$ be a trivial A-ideal of M. Then $\underline{Apr}_I(X) = X = \overline{Apr}_I(X)$, for every non-empty subset X of M. Hence every non-empty subset of M is definiable.

Proof. Note that

$$\begin{aligned}
[x]_I &= \{y \in M : d(y,x) \in I\}, \\
&= \{y \in M : x \odot y^* = 0 \text{ and } y \odot x^* = 0\}, \\
&= \{y \in M : x \leq y \text{ and } y \leq x\}, \\
&= \{y \in M : x = y\}, \\
&= \{x\}.
\end{aligned}$$

Hence $\underline{Apr}_I(X) = X = \overline{Apr}_I(X)$. Thus this completes proof. □

Theorem 21. *For every approximation space (M, I) and every subsets $X, Y \subseteq M$, we have:*

1. $\underline{Apr}_I(X) \subseteq X \subseteq \overline{Apr}_I(X)$,
2. $\underline{Apr}_I(\emptyset) = \emptyset = \overline{Apr}_I(\emptyset)$,
3. $\underline{Apr}_I(M) = M = \overline{Apr}_I(M)$,
4. *If* $X \subseteq Y$, *then* $\underline{Apr}_I(X) \subseteq \underline{Apr}_I(Y)$ *and* $\overline{Apr}_I(X) \subseteq \overline{Apr}_I(Y)$,
5. $\underline{Apr}_I(\underline{Apr}_I(X)) = \underline{Apr}_I(X)$,
6. $\overline{Apr}_I(\overline{Apr}_I(X)) = \overline{Apr}_I(X)$,
7. $\overline{Apr}_I(\underline{Apr}_I(X)) = \underline{Apr}_I(X)$,
8. $\underline{Apr}_I(\overline{Apr}_I(X)) = \overline{Apr}_I(X)$,
9. $\underline{Apr}_I(X \cap Y) = \underline{Apr}_I(X) \cap \underline{Apr}_I(Y)$,
10. $\overline{Apr}_I(X \cap Y) \subseteq \overline{Apr}_I(X)\overline{Apr}_I(Y)$,
11. $\underline{Apr}_I(X \cup Y) \supseteq \underline{Apr}_I(X) \cup \underline{Apr}_I(Y)$,
12. $\overline{Apr}_I(X \cup Y) = \overline{Apr}_I(X) \cup \overline{Apr}_I(Y)$.

Proof. The proof is similar to the proof of Theorem 2.1 in [17]. □

The following example shows that the converse of (10) and (11) in the Theorem 21 is not true.

Example 22. Let $M = \{0, a, b, c, d, 1\}$, where $0 < a < c < 1$, $0 < b < d < 1$ and elements of $\{a, b\}$ and $\{c, d\}$ are pairwise incomparable. Define $\oplus$, $\odot$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(M, \oplus, \odot, *, 0, 1)$ is an MV-algebra [15]. Consider $A = \Gamma(\mathbb{Z}, 1) = \{0, 1\}$, then M is A-module with natural product αx, for any $\alpha \in A$ and $x \in M$. It is clear $I = \{0, a\}$ is an A-ideal of M. Let $X = \{0, a, c\}$ and $Y = \{0, b, 1\}$ are subsets of M. Then the equivalence classes are $[a]_I = [0]_I = \{0, a\}$, $[1]_I = \{1, d\}$, $[b]_I = \{b, c\}$, $[c]_I = \{b, c\}$ and $[d]_I = \{d, 1\}$. Thus we have

$$\begin{aligned}
\underline{Apr}_I(X) &= \{0, a\},\\
\underline{Apr}_I(Y) &= \emptyset,\\
\underline{Apr}_I(X \cup Y) &= \{0, a, b, c\},\\
\overline{Apr}_I(X) &= \{0, a, b, c\},\\
\overline{Apr}_I(Y) &= \{0, a, b, c, d, 1\},\\
\overline{Apr}_I(X \cap Y) &= \{0, a\}.
\end{aligned}$$

It follows that $\underline{Apr}_I(X \cup Y) \neq \underline{Apr}_I(X) \cup \underline{Apr}_I(Y)$ and $\overline{Apr}_I(X) \cap \overline{Apr}_I(Y) \neq \overline{Apr}_I(X \cap Y)$.

Remark 23. For every approximation space (M, I) and for all $x \in M$, we have

$$\underline{Apr}_I(x/I) = \overline{Apr}_I(x/I)$$

Proof. It follows from Theorem 21 (1) that $\underline{Apr}_I(x/I) \subseteq \overline{Apr}_I(x/I)$. Conversely, let $a \in \overline{Apr}_I(x/I)$. Hence $[a]_I \cap x/I \neq \emptyset$. So there exists $t \in [a]$ and $t \in x/I$.

Now, we only show that $[a \subseteq x/I$. Let $y \in [a]$. Hence $(y, a) \in \rho_I$ and we have $(t, a) \in \rho_I$. We obtain $(y, t) \in \rho_I$ and also we have $(t, x) \in \rho_I$. It follows that $(y, x) \in \rho_I$, that is $y \in x/I$. Thus $[a] \subseteq x/I$. This results $a \in \underline{Apr}_I(x/I)$. □

Remark 24. [21] We recall that if X and Y are non-empty subsets of M, then we have

$$X \vee Y = \{a \in M | a \leq x \oplus y, x \in X, y \in Y\}.$$

Proposition 25. *Let I be an A-ideal of an A-module M and X, Y be non-empty subsets of M. Then*

(i) $\overline{Apr}_I(X \vee Y) \subseteq \overline{Apr}_I(X) \vee \overline{Apr}_I(Y)$. *In the particularly, if M is linearly ordered, then* $\overline{Apr}_I(X \vee Y) = \overline{Apr}_I(X) \vee \overline{Apr}_I(Y)$.

(ii) $\underline{Apr}_I(X) \vee \underline{Apr}_I(Y) \subseteq \underline{Apr}_I(X \vee Y)$.

Proof. The proof is similar to the proofs of Propositions 3.2.1 and 3.2.4 in [23]. □

The following example shows that we can not replace the inclusion symbol $\subseteq$ by an equal sign in Proposition 25 (ii).

Example 26. Let $\Omega = \{1, 2\}$ and $\mathcal{A} = \mathcal{P}(\Omega)$. $\mathcal{A}$ is a PMV-algebra with $\oplus = \cup$ and $\odot = \cdot = \cap$. If we consider $\mathcal{M} = \mathcal{A} = \mathcal{P}(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \phi\}$, then $\mathcal{M}$ becomes an MV-module over $\mathcal{A}$ with the external operation defined by $AX := A \cap X$ for any $A \in \mathcal{A}$ and $X \in \mathcal{M}$. Consider $X = \{\{1\}\}$ and $Y = \{\{2\}\}$. Obviously, $I = \{\emptyset, \{1\}\}$ is an $\mathcal{A}$-ideal of $\mathcal{M}$. We have $X \vee Y = \mathcal{M}$ and $[\{1\}]_I = \{\emptyset, \{1\}\}$, $[\emptyset]_I = \{\emptyset, \{1\}\}$, $[\{2\}]_I = \{\{2\}, \{1, 2\}\}$ and $[\{1, 2\}]_I = \{\{2\}, \{1, 2\}\}$. Hence we obtain $\underline{Apr}_I(X) = \emptyset$, $\underline{Apr}_I(Y) = \emptyset$. Thus $\underline{Apr}_I(X \vee Y) = \mathcal{M}$ is not a subset of $\underline{Apr}_I(X) \vee \underline{Apr}_I(Y) = \emptyset$.

Lemma 27. *Let I, J be two A-ideals of M such that $I \subset J$ and let X be a non-empty subset of M. Then*

(i) $\underline{Apr}_J(X) \subseteq \underline{Apr}_I(X)$,

(ii) $\overline{Apr}_I(X) \subseteq \overline{Apr}_J(X)$.

Proof. (i) Let $x \in \underline{Apr}_J(X)$. Then $x/I \subseteq x/J \subseteq X$, so $x/I \subseteq X$. Thus $x \in \underline{Apr}_I(X)$. Therefore $\underline{Apr}_J(X) \subseteq \underline{Apr}_I(X)$.

(ii) Let $x \in \overline{Apr}_I(X)$. Then $x/I \cap X \neq \emptyset$. We get $t \in x/I$ and $t \in X$. Hence $d(t,x) \in I \subseteq J$ and $t \in X$. It follows that $d(t,x) \in J$ and $t \in X$. This results $t \in x/J \cap X$. Thus $x/J \cap X \neq \emptyset$. Therefore $x \in \overline{Apr}_J(X)$. □

We recall that an element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. We denote the set of complemented of A by $B(A)$.

Proposition 28. *Let I, J be two A-ideals of an MV-module M and X be a non-empty subset of M.*

(i) *If $X \subseteq B(M)$ or M is linearly ordered, then $\overline{Apr}_{I\vee J}(X) \subseteq \overline{Apr}_I(X) \vee \overline{Apr}_J(X)$.*

(ii) $\underline{Apr}_{I\vee J}(X) \subseteq \underline{Apr}_I(X) \vee \underline{Apr}_J(X)$.

Proof. The proof is similar to the proofs of Propositions 3.2.7 and 3.2.9 in [23]. □

The following examples show that in Proposition 28 (i), (ii), the symbol inclusion can be proper.

Example 29. (i) Consider $\mathcal{M}$ as the MV-module in Example 26. Let $I = \{\emptyset, \{1\}\}$ and $J = \{\emptyset\}$ be two A-ideals of $\mathcal{M}$ and $X = \{\{2\}\} \subseteq B(M)$ be a subset of $\mathcal{M}$. It is easy to check that $[\emptyset]_J = \{\emptyset\}$, $[\{1\}]_J = \{\{1\}\}$, $[\{2\}]_J = \{\{2\}\}$ and $[\{1,2\}]_J = \{\{1,2\}\}$. Hence $\overline{Apr}_I(X) = \{\{2\}, \{1,2\}\}$ and $\overline{Apr}_J(X) = \{\{2\}, \{1,2\}\}$. Thus $\overline{Apr}_{I\vee J}(X) = \{\{2\}, \{1,2\}\}$ and by Remark 24, we have $\overline{Apr}_I(X) \vee \overline{Apr}_J(X) = \mathcal{M}$.

(ii) Consider $\mathcal{M}$ as the MV-module in Example 26. Let $I = \{\emptyset, \{1\}\}$ and $J = \{\emptyset\}$ be two A-ideals of $\mathcal{M}$ and $X = \{\emptyset, \{1\}, \{2\}\}$ be a subset of $\mathcal{M}$. We have $\underline{Apr}_{I\vee J}(X) = \{\emptyset, \{1\}\}$, $\underline{Apr}_I(X) = \{\emptyset, \{1\}\}$ and $\underline{Apr}_J(X) = \{\emptyset, \{1\}, \{2\}\}$. Thus $\underline{Apr}_{I\vee J}(X) \neq \underline{Apr}_I(X) \vee \underline{Apr}_J(X) = \mathcal{M}$.

Lemma 30. *Let I be an A-ideal of an MV-module M and X be a non-empty subset of M. Then X is definable if and only if $\underline{Apr}_I(X) = X$ or $\overline{Apr}_I(X) = X$.*

Proof. The proof is similar to the proof of Lemma 3.2.2 in [23]. □

We recall that X is an MV-subalgebra (for short, subalgebra) of M if and only if X is closed under the MV-operations defined in M.

Proposition 31. *Let M be an A-module and I be an A-ideal of M.*

(i) *If X is an A-ideal of M, then $\overline{Apr}_I(X)$ is a subalgebra too.*

(ii) *In particular, if M is a linearly ordered A-module and J is an A-ideal of M, then $\overline{Apr}_I(J)$ is an A-ideal of M.*

Proof. (i) By Proposition 3.3.3 (i) in [23], we obtain $\overline{Apr}_I(X)$ is a subalgebra of MV-algebra M. It sufficient to show that if $\alpha \in A$ and $x \in \overline{Apr}_I(X)$, then $\alpha x \in \overline{Apr}_I(X)$.

Since $x \in \overline{Apr}_I(X)$, so $x_1 \in [x]_I \cap X$, hence we have $d(x_1, x) \in I$ and $x_1 \in X$. Also by Lemma 12 (f) we deduce that $d(\alpha x_1, \alpha x) \leq \alpha d(x_1, x) \in I$ and since X is A-ideal M, $\alpha x_1 \in X$ and $\alpha x_1 \in [\alpha x]_I$. Thus $\alpha x_1 \in [\alpha x]_I \cap X$. Thus $\alpha x \in \overline{Apr}_I(X)$.

(ii) By Proposition 3.3.3 (ii) in [23] and similar to part (i), we can easily show $\alpha x \in \overline{Apr}_I(J)$, for $\alpha \in A$ and $x \in \overline{Apr}_I(J)$. It can be concluded that $\overline{Apr}_I(J)$ is an A-ideal of M. □

2.1 Rough sets in a quotient MV-module

Let I be an A-ideal of M. It is important to note that the equivalence class X/I containing x plays dual roles. It is a subset of M if considered in relation to the A-module M, and an element of M/I if considered in relation to the quotient MV-module. Therefore the lower and upper approximations can be presented in an equivalent form as shown below:

Let I be an A-ideal of M, and X a non-empty subset of M. Then

$$\underline{\underline{Apr}}_I(X) = \{x/I \in M : x/I \subseteq X\},$$

$$\overline{\overline{Apr}}_I(X) = \{x/I \in M : (x/I) \cap X \neq \emptyset\}.$$

Now, we discuss these sets as subsets of the quotient MV-module M/I.

Proposition 32. *Let I and J be two A-ideals of linearly ordered MV-module M. Then $\overline{\overline{Apr}}_I(J)$ is an A-ideal of M/I.*

Proof. Obviously, $\overline{\overline{Apr}}_I(J)$ is non-empty. Assume that $a/I, b/I \in \overline{\overline{Apr}}_I(J)$ and $\alpha \in A$. Then $a/I \cap J \neq \emptyset$ and $b/I \cap J \neq \emptyset$. So there exist $x \in a/I \cap J$ and $y \in b/I \cap J$. Since J is an A-ideal of M, we have $x \oplus y \in J$ and $\alpha x \in J$. Hence $d(x, a) \in I$ and $d(y, b) \in I$. It follows from Lemma 2 (11) that $d(x \oplus y, a \oplus b) \leq d(x, a) \oplus d(y, b) \in I$. Thus $x \oplus y \in (a \oplus b)/I \cap J$. So $(a/I \oplus b/I) \cap J = (a \oplus b)/I \cap J \neq \emptyset$. Therefore $a/I \oplus b/I \in \overline{\overline{Apr}}_I(J)$.

Now, if $x/I \leq y/I$ and $y/I \in \overline{\overline{Apr}}_I(J)$, then $y/I \cap J \neq \emptyset$. Hence there exists $t \in y/I \cap J$. Since M is linearly ordered MV-algebra, $x \leq y$ or $y \leq x$.

Case 1. If $x \leq y$, then by Lemma 6, for each $s \in [x]_I$, we have $s \leq t$ and since J is an A-ideal, we obtain $s \in J$. Hence $s \in J \cap x/I$, so $x/I \cap J \neq \emptyset$. Thus $x/I \in \overline{\overline{Apr}}_I(J)$.

Case 2. If $y \leq x$, then $y \odot x^* = 0 \in I$, hence $y/I \leq x/I$. So $x/I = y/I$. Thus the proof is complete.

Let $\alpha \in A$ and $x/I \in \overline{\overline{Apr}}_I(J)$. We show that $\alpha(x/I) \in \overline{\overline{Apr}}_I(J)$. Since $x/I \cap J \neq \emptyset$, $x_1 \in [x]_I \cap J$, so we have $d(x_1, x) \in I$, $x_1 \in J$. It follows from Lemma 12 (f) that $d(\alpha x_1, \alpha x) \leq \alpha d(x_1, x) \in I$ and since J is an A-ideal, $\alpha x_1 \in J$ and $\alpha x_1 \in [\alpha x]_I$. Thus $(\alpha x)/I \cap J \neq \emptyset$. Hence $\alpha(x/I) \in \overline{\overline{Apr}}_I(J)$. □

Theorem 33. *Let I, J be two A-ideals of M. Then $\underline{\underline{Apr}}_I(J) \neq \emptyset$ is an A-ideal, when $I \subseteq J$ and J is non-empty interior.*

Proof. Assume that $a/I, b/I \in \underline{\underline{Apr}}_I(J)$ and $\alpha \in A$, then $a \in [a]_I = a/I \subseteq J$ and $b \in [b]_I = b/I \subseteq J$. Let $z \in (a \oplus b)/I$, hence $d(z, a \oplus b) \in J$. Since $a \oplus b \in J$, we obtain $z \in (a \oplus b)/J = J$, thus $(a \oplus b)/I \subseteq J$. It proved that $a/I \oplus b/I \in \underline{\underline{Apr}}_I(J)$.

Now, let $x/I \in \underline{\underline{Apr}}_I(J)$ and $y/I \leq x/I$. We have $x \in [x]_I = x/I \subseteq J$ and since $y \odot x^* \in I \subseteq J$ and $x \in J$, we obtain $y \leq x \vee y = x \oplus (x^* \odot y) \in J$, hence $y \in J$. We must show that $y/I \subseteq J$. Let $z \in [y]_I$, then $d(z, y) \in I \subseteq J$. So $z \in y/J = J$. Hence $z \in J$. Thus $[y]_I = y/I \subseteq J$. Therefore $y/I \in \underline{\underline{Apr}}_I(J)$.

Finally, we show that $\alpha(a/I) \in \underline{\underline{Apr}}_I(J)$. Since $a/I \subseteq J$, we have $a \in J$. Since J is an A-ideal of M, then $\alpha a \in J$. It is sufficient to show that $(\alpha a)/I \subseteq J$. Let $z \in [\alpha a]_I$. Then $d(z, \alpha a) \in I \subseteq J$, this result $z \in [\alpha a]_J = J$. It follows $(\alpha a)/I \subseteq J$. So $\alpha(a/I) \in \underline{\underline{Apr}}_I(J)$. Therefore $\underline{\underline{Apr}}_I(J)$ is an A-ideal of M. $\square$

3 Lower and Upper Approximations with Respect to Fuzzy Congruences

[14] Let M be an A-module. A function θ from $M \times M$ to the unit interval $[0, 1]$ will be called a fuzzy congruence relation on M, if it satisfies the following for $x, y, z \in M$ and $\alpha \in A$:

(C1) $\theta(0, 0) = \theta(x, x)$,

(C2) $\theta(x, y) = \theta(y, x)$,

(C3) $\theta(x, z) \geq \theta(x, y) \wedge \theta(y, z)$,

(C4) $\theta(x \oplus z, y \oplus z) \geq \theta(x, y)$,

(C5) $\theta(x^*, y^*) = \theta(x, y)$,

(C6) $\theta(\alpha x, \alpha y) \geq \theta(x, y)$.

Lemma 34. *[14] If θ is a fuzzy congruence in M, then $\theta(0, 0) \geq \theta(x, y)$, for all $x, y \in M$.*

Let θ and ϕ be two fuzzy relations on M. Then the product $\theta \circ \phi$ is defined by

$$(\theta \circ \phi)(a, b) = \sup_{x \in M}[\min\{\theta(a, x), \phi(x, b)\}]$$

for all $a, b \in M$.

Let θ be a fuzzy congruence relation on M. For each $a \in M$, we define a fuzzy subset θ^a as follows:

$$\theta^a(x) = \theta(a, x)$$

for all $x \in M$. This fuzzy subset θ^a is called a fuzzy congruence class containing $a \in M$. We set

$$M/\theta = \{\theta^a : a \in M\}$$

is called a fuzzy quotient set by θ.

Lemma 35. *Let θ be a fuzzy congruence relation on an A-module M. Then*

$$\theta^{-1}(s) = \{(a, b) \in M \times M : \theta(a, b) = \theta(0, 0) = s\}$$

is a congruence relation on M.

Proof. It is clear that $\theta^{-1}(s)$ is reflexive and symmetric. To prove that $\theta^{-1}(s)$ is transitive, let $(a,b),(b,c) \in \theta^{-1}(s)$. Then $\theta(a,b) = \theta(b,c) = s$. Since θ is a fuzzy congruence relation on M, we have

$$\theta(a,c) \geq \theta(a,b) \wedge \theta(b,c) = s = \theta(0,0),$$

hence $\theta(a,c) = \theta(0,0) = s$, so $(a,c) \in \theta^{-1}(s)$, and $\theta^{-1}(s)$ is transitive. Thus $\theta^{-1}(s)$ is an equivalence relation on M.

Now, let $(a,b) \in \theta^{-1}(s)$ and $(c,d) \in \theta^{-1}(s)$. Hence $\theta(a,b) = \theta(c,d) = s$. Since θ is a fuzzy congruence relation on M, we have

$$\theta(c\oplus a, b\oplus d) \geq \theta(c\oplus a, d\oplus a)\wedge\theta(a\oplus d, b\oplus d) \geq \theta(c,d)\wedge\theta(a,b) = \theta(0,0)\wedge\theta(0,0) = \theta(0,0) = s.$$

Hence $\theta(c \oplus a, b \oplus d) = \theta(0,0)$, so $(a,b) \oplus (c,d) \in \theta^{-1}(s)$.

Let $(a,b) \in \theta^{-1}(s)$. Since θ is a fuzzy congruence relation on M, we have $s = \theta(a,b) = \theta(a^*,b^*)$, this results $(a^*,b^*) \in \theta^{-1}(s)$.

Let $(a,b) \in \theta^{-1}(s)$, and $\alpha \in A$. Then, since θ is a fuzzy congruence relation on M, we have

$$\theta(\alpha a, \alpha b) \geq \theta(a,b) = \theta(0,0) = s,$$

and so $\theta(\alpha a, \alpha b) = s$. Similarly, we have $(a\alpha, b\alpha) \in \theta^{-1}(s)$. Therefore we obtain $\theta^{-1}(s)$ is a congruence relation on M. □

Theorem 36. *Let θ and ϕ be fuzzy congruence relations on an A-module M. Then $\theta \cap \phi$ is a fuzzy congruence relation on M, and*

$$(\theta \cap \phi)^{-1}(s) = \theta^{-1}(s) \cap \phi^{-1}(s), \text{ where } s = \theta(0,0) = \phi(0,0).$$

Proof. It can be easily proved that $\theta \cap \phi$ is a fuzzy congruence relation on M. Let $(a,b) \in (\theta \cap \phi)^{-1}(s)$. Then we have

$$\min\{\theta(a,b), \phi(a,b)\} = (\theta \cap \phi)(a,b) = s,$$

and so

$$\theta(a,b) = \phi(a,b) = s.$$

Thus $(a,b) \in \theta^{-1}(s)$ and $(a,b) \in \phi^{-1}(s)$, and so

$$(a,b) \in \theta^{-1}(s) \cap \phi^{-1}(s).$$

Therefore we obtain that

$$(\theta \cap \phi)^{-1}(s) \subseteq \theta^{-1}(s) \cap \phi^{-1}(s).$$

Conversely, let $(a,b) \in \theta^{-1}(s) \cap \phi^{-1}(s)$. Then

$$(a,b) \in \theta^{-1}(s) \text{ and } (a,b) \in \phi^{-1}(s).$$

Thus we have

$$\theta(a,b) = \phi(a,b) = s.$$

Then we have

$$(\theta \cap \phi)(a,b) = \min\{\theta(a,b), \phi(a,b)\} = \min\{s,s\} = s,$$

and so

$$(a,b) \in (\theta \cap \phi)^{-1}(s).$$

Therefore we have

$$\theta^{-1}(s) \cap \phi^{-1}(s) \subseteq (\theta \cap \phi)^{-1}(s).$$

Thus we obtain that

$$(\theta \cap \phi)^{-1}(s) = \theta^{-1}(s) \cap \phi^{-1}(s).$$

□

Theorem 37. *Let ρ and λ be congruence relations on an A-module M. If X is a non-empty subset of M, then*

$$\overline{(\rho \cap \lambda)}_I(X) \subseteq \overline{\rho}_I(X) \cap \overline{\lambda}_I(X).$$

Proof. Note that $\rho \cap \lambda$ is also a congruence relation on M. Let $c \in \overline{(\rho \cap \lambda)}_I(X)$. Then

$$[c]_{\rho\cap\lambda} \cap X \neq \emptyset.$$

Then there exists an element $a \in [c]_{\rho\cap\lambda} \cap X$. Since $(a,c) \in \rho \cap \lambda$, we have $(a,c) \in \rho$ and $(a,c) \in \lambda$. Thus we have $a \in [c]_\rho$ and $a \in [c]_\lambda$. Since $a \in X$, we have

$$a \in [c]_\rho, \quad a \in X, \text{ and } a \in [c]_\lambda, \quad a \in X.$$

This implies that

$$c \in \overline{\rho}_I(X) \text{ and } c \in \overline{\lambda}_I(X),$$

and so

$$c \in \overline{\rho}_I(X) \cap \overline{\lambda}_I(X).$$

Thus we get

$$\overline{(\rho \cap \lambda)}_I(X) \subseteq \overline{\rho}_I(X) \cap \overline{\lambda}_I(X).$$

□

Theorem 38. *Let ρ and λ be congruence relations on an A-module M. If M is a non-empty subset of M, then*

$$\underline{(\rho \cap \lambda)}_I(X) = \underline{\rho}_I(X) \cap \underline{\lambda}_I(X).$$

Proof.

$$\begin{aligned} c \in \underline{(\rho \cap \lambda)}_I(X) \quad &\Leftrightarrow \quad [c]_{\rho\cap\lambda} \subseteq X, \\ &\Leftrightarrow \quad [c]_\rho \subseteq X \text{ and } [c]_\lambda \subseteq X, \\ &\Leftrightarrow \quad c \in \underline{\rho}_I(X) \text{ and } c \in \underline{\lambda}_I(X), \\ &\Leftrightarrow \quad c \in \underline{\rho}_I(X) \cap \underline{\lambda}_I(X). \end{aligned}$$

Thus we obtain that

$$\underline{(\rho \cap \lambda)}_I(X) = \underline{\rho}_I(X) \cap \underline{\lambda}_I(X).$$

□

Theorem 39. *Let θ and ϕ be fuzzy congruence relations on an A-module M and X a non-empty subset of M, where $s = \theta(0,0) = \phi(0,0)$. Then*

(1) $\underline{(\theta \cap \phi)^{-1}(s)}_I(X) = \underline{(\theta^{-1}(s) \cap \phi^{-1}(s))}_I(X) = \underline{\theta^{-1}(s)}_I(X) \cap \underline{\phi^{-1}(s)}_I(X).$

(2) $\overline{(\theta \cap \phi)^{-1}(s)}_I(X) = \overline{(\theta^{-1}(s) \cap \phi^{-1}(s))}_I(X) \subseteq \overline{\theta^{-1}(s)}_I(X) \cap \overline{\phi^{-1}(s)}_I(X).$

Proof. Those follow from Theorem 36, Theorem 37 and Theorem 38. □

Let α and β be binary relations on an A-module M. Then the product $\alpha\cdot\beta$ of α and β is defined as follows:

$$\alpha\cdot\beta=\{(a,b)\in M\times M:(a,c)\in\alpha \text{ and } (c,d)\in\beta \text{ for some } c\in M\}.$$

Assume α and β are congruence relations on an A-module M. Then, we can easily prove that $\alpha\cdot\beta$ is a congruence if and only if $\alpha\cdot\beta=\beta\cdot\alpha$.

Theorem 40. *Let ρ and λ be congruence relations on a linearly ordered A-module M such that $\rho\cdot\lambda=\lambda\cdot\rho$. If M is an A-module of M and X is an A-ideal of M, then*

$$\overline{\rho}_I(X)\vee\overline{\lambda}_I(X)\subseteq\overline{(\rho\cdot\lambda)}_I(X).$$

Proof. Let c be any element of $\overline{\rho}_I(X)\vee\overline{\lambda}_I(X)$. Then $c\le a\oplus b$ with $a\in\overline{\rho}_I(X)$ and $b\in\overline{\lambda}_I(X)$. Then there exist elements $x,y\in M$ such that

$$x\in[a]_\rho\cap X \text{ and } y\in[b]_\lambda\cap X.$$

Thus $x\in[a]_\rho$, $y\in[b]_\lambda$, and $x,y\in X$. Since X is an A-ideal of M, we have $x\oplus y\in X$. Then $(x,a)\in\rho$ and $(y,b)\in\lambda$, and since ρ and λ are congruence relations, we have

$$(x\oplus y,a\oplus y)\in\rho \text{ and } (a\oplus y,a\oplus b)\in\lambda.$$

Thus we have $(x\oplus y,a\oplus b)\in\rho\cdot\lambda$, and so $x\oplus y\in[a\oplus b]_{\rho\cdot\lambda}$. Therefore we have

$$x\oplus y\in[a\oplus b]_{\rho\cdot\lambda}\cap X,$$

which yields

$$c\le a\oplus b\in(\overline{\rho\cdot\lambda})_I(X).$$

Since by Proposition 31 (ii), $(\overline{\rho\cdot\lambda})_I(X)$ is an A-ideal, we obtain that $c\in(\overline{\rho\cdot\lambda})_I(X)$. Hence we have

$$\overline{\rho}_I(X)\vee\overline{\lambda}_I(X)\subseteq\overline{(\rho\cdot\lambda)}_I(X).$$ □

We note that if θ and ϕ are fuzzy congruence relations, then $\theta\circ\phi$ is a fuzzy congruence relation on M if and only if $\theta\circ\phi=\phi\circ\theta$ (see [14]).

Theorem 41. *Let θ and ϕ be fuzzy congruence relations on an A-module M such that $\theta\circ\phi=\phi\circ\theta$, where $\theta(0,0)=\phi(0,0)$. Then*

$$\theta^{-1}(s)\cdot\phi^{-1}(s)\subseteq(\theta\circ\phi)^{-1}(s).$$

Proof. Let $(a,b)\in\theta^{-1}(s)\cdot\phi^{-1}(s)$. Then there exists an element $c\in M$ such that $(a,c)\in\theta^{-1}(s)$ and $(c,b)\in\phi^{-1}(s)$. Then, since

$$\theta(a,c)=\phi(c,b)=s,$$

we have

$$\begin{aligned}(\theta\circ\phi)(a,b)&=\sup_{x\in M}[\min\{\theta(a,x),\phi(x,b)]\\&\ge\min\{\theta(a,c),\phi(c,b)\}\\&=\min\{s,s\}\\&=s.\end{aligned}$$

and so $(\theta\circ\phi)(a,b)=s$. This implies that $(a,b)\in(\theta\circ\phi)^{-1}(s)$. Thus we obtain that

$$\theta^{-1}(s)\cdot\phi^{-1}(s)\subseteq(\theta\circ\phi)^{-1}(s).$$ □

Remark 42. Let ρ and λ be congruence relations on A-module M. If X and Y are nonempty subsets of M, then the following hold:

(i) $\rho \subseteq \lambda$ implies $\underline{\rho}_I(X) \supseteq \underline{\lambda}_I(Y)$,

(ii) $\rho \subseteq \lambda$ implies $\overline{\rho}_I(X) \subseteq \overline{\lambda}_I(Y)$.

Theorem 43. *Let θ and ϕ be fuzzy congruence relations on an A-module M such that $\theta \circ \phi = \phi \circ \theta$. If X is a nonempty subset of M, then*

(1) $\underline{(\theta^{-1}(s) \cdot \phi^{-1}(s))}_I(X) \supseteq \underline{(\theta \circ \phi)^{-1}(s)}_I(X)$.

(2) $\overline{\theta^{-1}(s) \cdot \phi^{-1}(s))}_I(X) \subseteq \overline{(\theta \circ \phi)^{-1}(s)}_I(X)$.

Proof. Those follow from Theorem 41 and Remark 42 (i), (ii). □

Theorem 44. *Let θ and ϕ be fuzzy congruence relations on a linearly ordered A-module M such that $\theta \circ \phi = \phi \circ \theta$, where $\theta(0,0) = \phi(0,0) = s$. If X is an A-ideal of M, then*

$$(\overline{\theta^{-1}(s)}_I(X)) \vee (\overline{\phi^{-1}(s)}_I(X)) \subseteq \overline{(\theta \circ \phi)^{-1}(s)}_I(X).$$

Proof. Let c be any element of $\overline{\theta^{-1}(s)}_I(X) \vee (\overline{\phi^{-1}(s)}_I(X)$. Then $c \leq a \oplus b$ with $a \in \overline{\theta^{-1}(s)}_I(X)$ and $b \in (\overline{\phi^{-1}(s)}_I(X)$. Then there exist elements $x, y \in M$ such that

$$x \in \theta^a \cap X \text{ and } y \in \phi^b \cap X.$$

This implies that

$$(a, x) \in \theta^{-1}(s) \text{ and } (b, y) \in \phi^{-1}(s),$$

and $x, y \in X$. Then we have

$$\theta(a, x) = \phi(b, y) = s.$$

Since θ and ϕ are fuzzy congruence relations on M, we have and so

$$(\theta \circ \phi)(a \oplus b, x \oplus y) = s.$$

Note that, since X is an A-ideal of M, thus we have

$$x \oplus y \in (\theta \circ \phi)^{a \oplus b} \cap X.$$

This implies that

$$c \leq a \oplus b \in \overline{(\theta \circ \phi)^{-1}(s)}_I(X).$$

Since by Proposition 31(ii), $\overline{(\theta \circ \phi)^{-1}(s)}_I(X)$ is an A-ideal, we obtain that $c \in \overline{(\theta \circ \phi)^{-1}(s)}_I(X)$. We get $(\overline{\theta^{-1}(s)}_I(X)) \vee (\overline{\phi^{-1}(s)}_I(X)) \subseteq \overline{(\theta \circ \phi)^{-1}(s)}_I(X)$. □

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Uniform bounds on locations of zeros of partial theta function

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Abstract
We consider the partial theta function $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, where $(q,z) \in \mathbb{C}^2$, $|q| < 1$. We show that for any $0 < \delta_0 < \delta < 1$, there exists $n_0 \in \mathbb{N}$ such that for any q with $\delta_0 \leq |q| \leq \delta$ and for any $n \geq n_0$ the function θ has exactly n zeros with modulus $< |q|^{-n-1/2}$ counted with multiplicity.

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1 Introduction

We consider the bivariate series $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, where $(q,z) \in \mathbb{C}^2$, $|q| < 1$. This series defines a *partial theta function*. The terminology is explained by the fact that the Jacobi theta function is defined by the series $\sum_{j=-\infty}^{\infty} q^{j^2} z^j$ and the following equality holds true: $\theta(q^2, z/q) = \sum_{j=0}^{\infty} q^{j^2} z^j$. The word "partial" is justified by the summation in θ ranging from 0 to ∞ and not from $-\infty$ to ∞. In what follows we consider z as a variable and q as a parameter. For each fixed value of the parameter q the function θ is an entire function in the variable z.

The function θ finds applications in various domains, such as statistical physics and combinatorics (see [17]), Ramanujan type q-series (see [18]), the theory of (mock) modular forms (see [3]), asymptotic analysis (see [2]), and also in problems concerning real polynomials in one variable with all roots real (such polynomials are called *hyperbolic*, see [4], [5], [15], [14], [6], [13] and [7]). Other facts about θ can be found in [1].

The zeros of θ depend on the parameter q. For some values of q (called *spectral*) confluence of zeros occurs, so it would be correct to regard the zeros as multivalued functions of q; about the spectrum of θ see [13], [11] and [12].

We denote by $\mathbb{D}_\rho$ the open disk in the q-space centered at 0 and of radius ρ, by $\mathcal{C}_\rho$ the corresponding circumference, and by $A_{\delta_0,\delta}$ the closed annulus $\{q \in \mathbb{C} \,|\, \delta_0 \leq |q| \leq \delta\}$.

In the present paper we prove the following theorem:

Theorem 1. *For any couple of numbers (δ_0, δ) such that $0 < \delta_0 < \delta < 1$, there exists $n_0 \in \mathbb{N}$ such that for any $q \in A_{\delta_0,\delta}$ and for any $n \geq n_0$ the function θ has exactly n zeros in $\mathbb{D}_{|q|^{-n-1/2}}$ counted with multiplicity.*

Remark 2. 1. The proof of the theorem is based on a comparison between θ and the function

$$u(q,z) := \prod_{\nu=1}^{\infty}(1+q^{\nu}z) \tag{1.1}$$

We use the equality

$$u = \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j / (q;q)_j \ , \tag{1.2}$$

where $(q;q)_j := (1-q)(1-q^2)\cdots(1-q^j)$ is the q-Pochhammer symbol; it follows directly from Problem I-50 of [16] (see pages 9 and 186 of [16]). The analog of the above theorem for the *deformed exponential function* $\sum_{j=0}^{\infty} q^{j(j+1)/2} z^j/j!$ is proved in a non-published text by A. E. Eremenko using a different method.

2. For q close to 0 the zeros of θ are of the form $-q^{-\ell}(1+o(1))$, $\ell \in \mathbb{N}$, see more details about this in [8], [9] and [10].

2 Proofs

Proof of Theorem 1. It is shown in [8] that for $0 < |q| \leq 0.108$ the zeros of θ can be expanded in convergent Laurent series. Recall that the function u (defined by (1.1)) satisfies equality (1.2), i.e. the zeros of u are the numbers $-q^{-\ell}$, $\ell \in \mathbb{N}$. We show that for $n \in \mathbb{N}$ sufficiently large the functions u and θ have one and the same number of zeros in the open disk $\mathbb{D}_{|q|^{-n-1/2}}$. To this end we show that for the restrictions u^0 and θ^0 of u and θ to the circumference $\mathcal{C}_{|q|^{-n-1/2}}$ one has $|u^0 - \theta^0/(q;q)_n| < |u^0|$ after which we apply the Rouché theorem.

For $0 < |q| \leq 0.108$ one can establish a bijection between the zeros of θ and u, because their ℓth zeros are of the form $-q^{-\ell}(1+o(1))$ and the moduli of the zeros increase with ℓ, see part 2 of Remark 2.

Set $P_k(|q|) := \prod_{\ell=0}^{k}(1-|q|^{\ell+1/2})$, $k \in \mathbb{N} \cup \infty$. For $|u^0|$ one obtains the estimation

$$|u^0| \geq |q|^{-n^2/2} P_{n-1}(|q|) P_{\infty}(|q|) > |q|^{-n^2/2}(P_{\infty}(|q|))^2 \geq |q|^{-n^2/2}(P_{\infty}(\delta))^2 \ . \tag{2.1}$$

Indeed, for $|z| = |q|^{-n-1/2}$ one can set $z := |q|^{-n-1/2}\omega$, $|\omega| = 1$. For $1 \leq \nu \leq n$ (resp. for $\nu > n$), the factor $(1+q^{\nu}z)$ in (1.1) is of the form $(1-|q|^{-\ell-1/2}\omega_{\ell})$, where $\ell = n-\nu$ and $|\omega_{\ell}| = 1$ (resp. of the form $(1-|q|^{\ell+1/2}\omega_{\ell}^*)$, where $\ell = \nu - n - 1$ and $|\omega_{\ell}^*| = 1$). Thus

$$u(q, |q|^{-n-1/2}\omega^{-n-1/2}) = \prod_{\ell=0}^{n-1}(1-|q|^{-\ell-1/2}\omega_{\ell}) \prod_{\ell=0}^{\infty}(1-|q|^{\ell+1/2}\omega_{\ell}^*) \ .$$

The first of the factors in the right-hand side can be represented in the form $|q|^{-n^2/2}\tilde{\omega}\prod_{\ell=0}^{n-1}(1-|q|^{\ell+1/2}\omega_{\ell}^{**})$ with $|\tilde{\omega}| = |\omega_{\ell}^{**}| = 1$. Therefore

$$u(q, |q|^{-n-1/2}\omega^{-n-1/2}) = |q|^{-n^2/2}\tilde{\omega}\prod_{\ell=0}^{n-1}(1-|q|^{\ell+1/2}\omega_{\ell}^{**}) \prod_{\ell=0}^{\infty}(1-|q|^{\ell+1/2}\omega_{\ell}^*) \ .$$

The modulus of the right-hand side is minimal for $\omega_{\ell}^* = \omega_{\ell}^{**} = 1$ in which case one obtains the leftmost inequality in (2.1).

Consider the monomial $\beta_j := \alpha_j z^j$ in the series $u - \theta/(q;q)_n$. Hence for $j = n$ it vanishes and for $j > n$ one has

$$\begin{array}{rcl} \alpha_j & = & q^{j(j+1)/2}(1/(q;q)_j - 1/(q;q)_n) = q^{j(j+1)/2}U_{j,n} \quad , \quad \text{where} \\ U_{j,n} & := & (1 - \prod_{\ell=n+1}^{j}(1-q^\ell))/(q;q)_j \ , \end{array}$$

so for $|z| = |q|^{-n-1/2}$ one has $|\beta_j| = |q|^{-n^2/2+(j-n)^2/2}|U_{j,n}|$. One can observe that $U_{j,n} = q^{n+1} + O(q^{n+2})$. Set

$$U_{j,n} := \sum_{\nu \geq n+1} u_{j,n;\nu}q^\nu \quad \text{and} \quad U := ((\prod_{\ell=1}^{\infty}(1+q^\ell)) - 1)/(q;q)_\infty = \sum_{\nu=1}^{\infty} u_\nu q^\nu \ .$$

The Taylor series of U converges for $|q| < 1$ because the infinite products defining U converge. Clearly $u_{j,n;\nu} \in \mathbb{Z}$, $u_\nu \in \mathbb{N}$ (because all coefficients of the series $1/(q;q)_j$ and $1/(q;q)_\infty$ are positive integers) and $u_{j,n;n+1} = u_1 = 1$.

The following lemma explains in what sense the series U majorizes the series $U_{j,n}$.

Lemma 3. *One has* $|u_{j,n;n+\nu}| \leq u_\nu$, $\nu \in \mathbb{N}$.

Before proving Lemma 3 (the proof is given at the end of the paper) we continue the proof of Theorem 1.

Set $R(|q|) := \sum_{j>n} |q|^{(j-n)^2/2}$. The following inequality results immediately from the lemma:

$$Z_1 := \sum_{j>n} |\beta_j| \leq |q|^{-n^2/2}|q|^n U(|q|)R(|q|) \leq |q|^{-n^2/2}\delta^n U(\delta)R(\delta) \ . \tag{2.2}$$

The first condition which we impose on the choice of n is the following inequality to be fulfilled:

$$\delta^n U(\delta)R(\delta) < (P_\infty(\delta))^2/4 \ . \tag{2.3}$$

For $j < n$ and $|z| = |q|^{-n-1/2}$ one has $|\beta_j| = |q|^{-n^2/2+(j-n)^2/2}|\tilde{U}_{j,n}|$, where

$$\tilde{U}_{j,n} := (\prod_{\ell=j+1}^{n}(1-q^\ell) - 1)/(q;q)_n \ . \tag{2.4}$$

Hence $|\tilde{U}_{j,n}| \leq T(|q|) := (\prod_{\ell=1}^{\infty}(1+|q|^\ell)+1)/(|q|;|q|)_\infty$ and

$$|\beta_j| \leq |q|^{-n^2/2}|q|^{(j-n)^2/2}T(\delta) \tag{2.5}$$

Choose $m \in \mathbb{N}$ such that $T(\delta)\sum_{s=m}^{\infty}\delta^{s^2/2} \leq (P_\infty(\delta))^2/4$. Inequality (2.5) implies that

$$Z_2 := \sum_{j=0}^{n-m} |\beta_j| \leq |q|^{-n^2/2}(P_\infty(\delta))^2/4 \tag{2.6}$$

Notice that for $n < m$ the above sum is empty and the inequality trivially holds true.

The finite sum

$$Z_3 := \sum_{j=n-m+1}^{n-1} |\beta_j| \tag{2.7}$$

is of the form $|q|^{-n^2/2}O(|q|^n)$. Indeed, consider formula (2.4). There exists $M > 0$ depending only on δ_0 and δ such that

$$0 < |1/(q;q)_n| \leq 1/(|q|;|q|)_n < 1/(|q|;|q|)_\infty \leq M \quad \text{for} \quad \delta_0 \leq |q| \leq \delta \ .$$

Thus

$$|\tilde{U}_{j,n}| \leq M(\prod_{\ell=j+1}^{n}(1+|q|^\ell) - 1) \ .$$

The index j can take only the values $n-m+1, \ldots, n-1$. In the last product each monomial $|q|^\ell$ can be represented in the form $|q|^n|q|^{\ell-n}$, where $\ell - n = 2-m, \ldots, 0$. The modulus of each factor $|q|^{\ell-n}$ is not larger than $1/\delta_0^{\max(0,m-2)}$. Therefore

$$|\tilde{U}_{j,n}| \leq M((1+|q|^n/\delta_0^{\max(0,m-2)})^{m-1} - 1) = O(|q|^n) \ .$$

The sum Z_3 (see (2.7)) can be made less than $|q|^{-n^2/2}(P_\infty(\delta))^2/4$ by choosing n large enough. Thus inequalities (2.1), (2.2) and (2.6) yield

$$|u^0 - \theta^0/(q;q)_n| \leq Z_1 + Z_2 + Z_3 \leq (3/4)|q|^{-n^2/2}(P_\infty(\delta))^2 < |q|^{-n^2/2}(P_\infty(\delta))^2 \leq |u^0|$$

which proves the theorem. □

Proof of Lemma 3. We first compare the coefficients of the series

$$\prod_{\ell=p}^{r}(1+q^\ell) - 1 = \sum_{\nu \geq p}\gamma_\nu^1 q^\nu \quad \text{and} \quad \prod_{\ell=p}^{r}(1-q^\ell) - 1 = \sum_{\nu \geq p}\gamma_\nu^2 q^\nu \quad , \quad p \leq r \ .$$

They are obtained respectively as a sum of the non-negative coefficients of monomials and as a linear combination of the same coefficients some of which are taken with the + and the rest with the − sign. Therefore $\gamma_\nu^1 \geq |\gamma_\nu^2|$, $\nu \geq p$. This means that $|u_{j,n;\nu}| \leq v_{j,n;\nu} \leq v_{\infty,n;\nu}$, where

$$V_{j,n} := (\prod_{\ell=n+1}^{j}(1+q^\ell) - 1)/(q;q)_j = \sum_{\nu \geq n+1} v_{j,n;\nu}q^\nu \quad , \quad V_{\infty,0} = U \quad \text{and} \quad v_{\infty,0;\nu} = u_\nu \ .$$

To prove the lemma it suffices to show that

$$v_{\infty,n;n+\nu} \leq v_{\infty,0;\nu} \ . \tag{2.8}$$

Consider the series $S_r := \prod_{\ell=r+1}^{\infty}(1+q^\ell) - 1 = \sum_{\nu \geq r+1} s_{r;\nu}q^\nu$ for $r = 0$ and $r = n$. Compare the coefficients $s_{0;\nu}$ and $s_{n;n+\nu}$. The coefficient $s_{0;\nu}$ is equal to the number of ways in which ν can be represented as a sum of distinct natural numbers forming an increasing sequence whereas $s_{n;n+\nu}$ is the number of ways in which $n+\nu$ can be represented as a sum of distinct natural numbers $\geq n+1$ forming an increasing sequence. Clearly $s_{n;n+\nu} \leq s_{0;\nu}$. This implies inequality (2.8) and the lemma, because one has $V_{\infty,r} = S_r/(q;q)_\infty$ and the coefficients of the series $1/(q;q)_\infty$ are all positive. □

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A note on non-unital homomorphisms on C^*-convex sets in $*$-rings

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Abstract
Recently, Ebrahimi et al. [C^*-convexity and C^*-faces in $*$-rings, *Turk. J. Math.* **36** (2012), 131–145] identified the optimal points of continuous unital homomorphisms on some C^*-convex sets of a topological $*$-ring. In this short note, we generalize their results for continuous (non-unital) homomorphisms in a topological $*$-ring. Moreover, for continuous unital homomorphisms, we point out the same conclusion does not hold everywhere that a Krein-Milman type theorem exists. An important issue is so that we do not assume that homomorphisms are unital.

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1 Introduction

The term non-commutative convexity refers to any one of the various forms of convexity in which convex coefficients need not commute among themselves. Formal study of C^*-convexity as a form of non-commutative convexity, was initiated by Loebl and Paulsen in [3], where the notion of C^*-extreme point, as a non-commutative analog of extreme point was also studied.

Recently, Ebrahimi et al. [2] motivated the following general question. Operator algebras are equipped with rich algebraic, geometric and topological structures such that one naturally asks: which of these structures have made a particular theorem work. In the algebraic direction this question has led to evolution of the algebraic theory of operator algebras. Indeed, they defined the notions of C^*-convexity and C^*-extreme point and discussed some illustrative examples of C^*-convex subsets of $*$-rings. For example, they showed that the set $\{x\}$ is C^*-convex, when $x \in Z(R)$, $Z(R)$ is the center of R, and R is a unital $*$-ring (see Example 2.2 of [2]). In this situation, when $x \notin Z(R)$ the set $\{x\}$ is not C^*-convex and the C^*-convex hull of $\{x\}$ is the smallest C^*-convex set containing $\{x\}$ and is denoted by C^*-Co({x}). They investigated some properties of C^*-convex sets and C^*-extreme points and identified some C^*-convex subsets of $*$-rings by applying C^*-convex maps. Moreover, they identified optimal points of some unital homomorphisms on some C^*-convex sets.

In this note, we generalize the theorem appearing as Theorem 4.8 in [2] for continuous (non-unital) homomorphism in a topological $*$-ring. We know that many algebras have no characters, for instance M_n for $n \geq 1$, $\mathcal{B}(H)$, etc. (see for example Exercise IV.1 of [1]) and we also know that the set of real–valued unital homomorphisms on an algebra is a subset of the set of its characters. So, we notice that Corollary 4.9 of [2] discusses on the maximum and minimum of an empty set of functions on M_n .

2 Main results

Throughout this note $\mathcal{R}$ is a unital $*$-ring, that is, a ring with an involution which has an identity element.

Definition 1. A subset K of $\mathcal{R}$ is called C^*-convex if $\sum_{i=1}^n a_i^* x_i a_i \in K$, whenever $x_i \in K$, $a_i \in \mathcal{R}$ and $\sum_{i=1}^n a_i^* a_i = 1_{\mathcal{R}}$.

For example, the positive cone in $\mathcal{R}$ is C^*-convex and in general the segment $[0, a]$ for $a \in \mathcal{R}$ is not C^*-convex, see Example 2.11 of [2].

Definition 2. If K is a C^*-convex subset of $\mathcal{R}$, then $x \in K$ is called a C^*-extreme point for K if the condition

$$x = \sum_{i=1}^n a_i^* x_i a_i, \quad \sum_{i=1}^n a_i^* a_i = 1_{\mathcal{R}}, \; x_i \in K, n \in \mathbb{N},$$

where a_i is invertible in $\mathcal{R}$ implies that all x_i are unitarily equivalent to x in $\mathcal{R}$, that is, there exist unitaries $u_i \in \mathcal{R}$ such that $x_i = u_i^* x u_i$ for all i. The set of all C^*-extreme point of K is denoted by C^*-ext(K).

Recall that a homomorphism $f : \mathcal{R} \to \mathbb{R}$ is unital if $f(1_{\mathcal{R}}) = 1$. The following theorem appears as Theorem 4.8 in [2] for a C^*-convex subset K of $\mathcal{R}$.

Theorem 3. *Suppose $\mathcal{R}$ is a topological $*$-ring, C^*-ext(K) is closed and S is a compact subset of $\overline{C^*}$-Co$(C^*$-ext$(K))$ containing C^*-ext(K). Then every continuous unital homomorphism $f : \mathcal{R} \to \mathbb{R}$ attains its maximum and minimum on S at C^*-extreme points of K. Moreover, maximum and minimum of f on S is equal with its maximum and minimum on C^*-ext(K), respectively.*

As a consequence of this theorem together with the generalized Krein-Milman theorem (Theorem 4.5 of [4]) the following corollary presented as Corollary 4.9 in [2].

Corollary 4. *If $S \subseteq M_n$ is compact, C^*-convex, and the set of all C^*-extreme points of S is closed, then every continuous unital homomorphism $f : M_n \to \mathbb{R}$, attains its maximum and minimum on S at C^*-extreme points of S.*

Remark 5. In the above mentioned theorem, if $\mathcal{R}$ is a simple $*$-ring and $f : \mathcal{R} \to \mathbb{R}$ is a continuous unital homomorphism, then the null space of f is an ideal of $\mathcal{R}$. On the other hand, $\mathcal{R}$ is simple and so it has no nontrivial ideals. Thus, either f is zero or $\mathcal{R}$ is isomorphic to the real numbers. Recall that f is unital, so f is not zero and then it remains that $\mathcal{R}$ is isomorphic to the real numbers, which is of course a trivial case. Since in this case all continuous homomorphisms $f : \mathbb{R} \to \mathbb{R}$ are of the form $f(x) = cx$, where $c \in \mathbb{R}$ is a constant. Moreover, C^*-convex hull and C^*-extreme points of a set in $\mathbb{R}$ are identical with its convex hull and extreme points in the usual sense, respectively. We can observe that the closed convex hull of a subset of $\mathbb{R}$ is a closed interval and the extreme points of a closed interval are its initial and end points. So, for instance, if $c > 0$, then $f(x) = cx$ is increasing and obviously attains its minimum and maximum at the initial and end points of the interval, respectively.

Remark 6. Note that the set of all real–valued unital homomorphisms on M_n are a subset of characters of M_n and note that M_n has no characters (cf. Exercise IV.1 of [1]) and so the above mentioned corollary discusses on the maximum and minimum of an empty set of characters on M_n .

We would remark that for compact C^*-convex subsets of M_n a Krein–Milman type theorem was established by Morenz (Theorem 4.5 of [4]). The authors in [2] claim that the same conclusion (Corollary 4) holds everywhere that a Krein-Milman type theorem exists. For example in the generalized state space of a C^*-algebra with bounded–weak topology such a conclusion holds. However, Remark 6 ensures we can not claim that the same conclusion holds everywhere that a Krein-Milman type theorem exists.

Taking ideas from Remarks 5 and 6, we are going to consider continuous (non-unital) homomorphism on C^*-convex sets. We now state and prove an extended version of Theorem 1.1 to non-unital maps and we show the assumption that f is unital would be dropped. That is, we prove the following theorem.

Theorem 7. *Suppose $\mathcal{R}$ is a topological $*$-ring, C^*-ext(K) is closed and S is a compact subset of $\overline{C^*}$-Co$(C^*$-ext$(K))$ containing C^*-ext(K). Then every continuous homomorphism $f : \mathcal{R} \to \mathbb{R}$ attains its maximum and minimum on S at C^*-extreme points of K. Moreover, maximum and minimum of f on S is equal with its maximum and minimum on C^*-ext(K), respectively.*

Proof. Suppose that f attains its maximum on S at a point $x \in S$. Then, there exists a net $\{x_\lambda\}$ in C^*-Co$(C^*$-ext$(K))$ such that x_λ converges to x. For every λ, x_λ is C^*-convex combination of points of C^*-ext(K), i.e., $x_\lambda = \sum_{i=1}^{n(\lambda)} a^*_{\lambda,i} x_{\lambda,i} a_{\lambda,i}$, where $n(\lambda)$ is a positive integer, $x_{\lambda,i} \in C^*$-ext(K), and $\sum_{i=1}^{n(\lambda)} a^*_{\lambda,i} a_{\lambda,i} = 1_{\mathcal{R}}$. Define $f(x_{\lambda,i_\lambda}) := \max_{1\le i\le n(\lambda)} f(x_{\lambda,i})$. Then,

$$\begin{aligned} f(x_\lambda) &= f(\sum_{i=1}^{n(\lambda)} a^*_{\lambda,i} x_{\lambda,i} a_{\lambda,i}) = \sum_{i=1}^{n(\lambda)} f(a^*_{\lambda,i} x_{\lambda,i} a_{\lambda,i}) \\ &= \sum_{i=1}^{n(\lambda)} f(a^*_{\lambda,i}) f(x_{\lambda,i}) f(a_{\lambda,i}) \le \max_{1\le i\le n(\lambda)} f(x_{\lambda,i}) \sum_{i=1}^{n(\lambda)} f(a^*_{\lambda,i}) f(a_{\lambda,i}) \\ &= f(x_{\lambda,i_\lambda}) f(\sum_{i=1}^{n(\lambda)} a^*_{\lambda,i} a_{\lambda,i}) \\ &= f(x_{\lambda,i_\lambda}) f(1_{\mathcal{R}}) = f(x_{\lambda,i_\lambda}.1_{\mathcal{R}}) \\ &= f(x_{\lambda,i_\lambda}). \end{aligned} \tag{2.1}$$

Since f is multiplicative, in the forth line of (2.1) we use the equality $f(x_{\lambda,i_\lambda}) f(1_{\mathcal{R}}) = f(x_{\lambda,i_\lambda}.1_{\mathcal{R}})$. This shows the assumption that f is unital can be dropped in Theorem 4.8 of [2]. The rest of the proof is similar to that of Theorem 4.8 of [2]. □

However, by removing the assumption that f is unital, we can state a correct version of Corollary 4.9 in [2] as follows:

Corollary 8. *If $S \subseteq M_n$ is compact, C^*-convex, and the set of all C^*-extreme points of S is closed, then every continuous homomorphism $f : M_n \to \mathbb{R}$ attains its maximum and minimum on S at C^*-extreme points of S.*

We recall that every real–valued homomorphism on M_n is zero and it is clear that its maximum and minimum value are zero. Indeed, in this case the above corollary is an obvious corollary. However, unlike Corollary 4, this corollary does not discuss on an empty set of functions on M_n and the discussed set contains at least the zero function.

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On Ultimate Boundedness and Existence of Periodic Solutions of Kind of Third Order Delay Differential Equations

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Abstract
In this paper we first study the problem of uniform ultimate boundedness of a certain third order nonlinear differential equation with delay. Further the existence of periodic solutions for the considered equation are also given, as a consequence of uniform ultimate boundedness results. Finally, some criteria to guarantee the uniform asymptotic stability are derived via the Lyapunov's second method. We also give an example to illustrate our results.

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1 Introduction

Nonlinear differential equations of higher order have been extensively studied with high degree of generality. In particular, boundedness, uniform boundedness, ultimate boundedness, uniform ultimate boundedness and asymptotic behavior of solutions have in the past and also recently been discussed. See for instance Reissig et al. [13], Rouche et al. [19], Yoshizawa [22] and [23]. It is well known that the ultimate boundedness is a very important problem in the theory and applications of differential equations. An effective method for studying the ultimate boundedness of nonlinear differential equations is still the Lyapunov's direct method.

Because of their applications, the existence of periodic solutions of third order differential equations has been also investigated by many researchers in recent years. Besides it is worth-mentioning that there are a few results on the same topic for third order delay differential equations, for example, Chukwu [6], Gui[10], Tunç [21] and Zhu[24].

In 1992, Zhu[24], established some sufficient conditions to ensure the stability, boundedness, ultimate boundedness of the solutions of the following third order non-linear delay differential equation

$$x''' + ax'' + bx' + f(x(t-r)) = e(t). \tag{1.1}$$

*corresponding author

The existence of periodic solutions was also discussed in the case where e(t) is a periodic function.
Recently, in [8], the authors extend results obtained in [24] to the following third order non autonomous differential equation with delay

$$[g(x(t))x'(t)]'' + (h(x(t))x'(t))' + \varphi(x(t))x'(t) + f(x(t-r)) = e(t), \tag{1.2}$$

In this paper, we are concerned with the third order delay differential equation

$$\Big(q(t)\big(g(x(t))x'(t)\big)'\Big)' + a(t)\big(h(x(t))x'(t)\big)' + b(t)\varphi(x(t))x'(t) + c(t)f(x(t-r)) = e(t), \tag{1.3}$$

where $r > 0$ is a fixed delay and $a,\ b,\ c,\ e,\ f,\ g,\ h,$ and φ are continuous functions and depend only on the arguments shown explicitly; $f(0) = 0;\ f'(x), g'(x),\ h'(x),$ and $\varphi'(x)$ exist and are continuous for all x. Our objective here is to extend results obtained in [8] to (1.3). The paper is organized as follows. In section 3 we study the problems of the boundedness and ultimate boundedness of solutions when $e(t) \neq 0$. The assumptions will also give us an opportunity to discuss the existence of periodic solutions of the same equation when $a,\ b,\ c,\ e,\ q,$ are periodic functions. Finally we investigate the asymptotic stability of the zero solution of the delay differential equation (1.3) with $e(t) = 0$. We give an example to illustrate the effectiveness of main results obtained in Section 3.
Clearly the equation discussed by Zhu in [24] is a special case of equation (1.3) when $g(x) = h(x) = \varphi(x) = 1$, $a(x) = a$, and $b(t) = b$, also (1.2) is a special case of (1.3) with $q(t) = 1$.

2 Preliminaries

To describe the main result of this paper, we include some preliminary knowledge on the stability and ultimate boundedness for a general class of nonlinear delay differential system

$$x' = f(t, x_t),\quad x_t(\theta) = x(t+\theta)\ ,\quad -r \le \theta \le 0,\quad t \ge 0, \tag{2.1}$$

where $f : C_H \to \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$, $C_H := \{\phi \in (C[-r,0],\ \mathbb{R}^n) : \|\phi\| \le H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Lemma 1. *[12] If there is a continuous functional* $V(t,\phi) : [0,+\infty[\times C_H \to [0,+\infty[$ *locally Lipschitz in ϕ and wedges W_i such that:*
(i) *If $W_1(\|\phi\|) \le V(t,\phi)$, $V(t,0) = 0$ and $V'_{(2,1)}(t,\phi) \le 0$.*

Then, the zero solution of (2.1) is stable. If in addition $V(t,\phi) \le W_2(\|\phi\|)$. Then, the zero solution of (2.1) is uniformly stable.
(ii) *If $W_1(\|\phi\|) \le V(t,\phi) \le W_2(\|\phi\|)$ and $V'_{(2,1)}(t,\phi) \le -W_3(\|\phi\|)$.*

Then, the zero solution of (2.1) is uniformly asymptotically stable.

Definition 2. [4] Solutions of (2.1) are uniform ultimate bounded for bound B at $t = 0$ if for each $A > 0$ there is a $K > 0$ such that $\phi \in C_H,\ ||\phi|| < A,\ t \ge K$ imply that $x(t, 0, \phi) < A$.

Lemma 3. *[4] Let $V(t,\varphi) : \mathbb{R} \times C \to \mathbb{R}$ be continous and locally Lipschitz in φ. If*

$$i)\qquad W_0(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2\Big(\int_{t-r}^{t} W_3(|x(t)|)ds\Big),$$

$$ii)\qquad V'_{(2.1)} \le -W_3(|x(t)|) + M,$$

for some $M > 0$, where $W_i (i = 0, 1, 2, 3)$ are wedges, then the solutions of (2.1) are uniformly bounded and uniformly ultimately bounded for bound B.

If (2.1) is periodic system with period T, we have the following result:

Lemma 4. *[20] Suppose that, for $\alpha > 0$, there exists $L(\alpha) > 0$ such that $|f(t, x_t)| \leq L(\alpha)$, for $t \in [-T, 0]$ and $\|x_t\| \leq \alpha$, and suppose that the solutions of (2.1) are equi-bounded and equi-ultimately bounded for bound B, then there exists a periodic solution of (2.1) of period T.*

3 Main Results

We shall give here some assumptions which will be used on the functions that appeared in equation (1.3). Suppose that there are positive constants $a_0, a_1, b_0, b_1, c_0, c_1, g_0, g_1, h_0, h_1, \varphi_0, \varphi_1, \delta_0, \delta_1, \mu_1$ and μ_2 such that the following conditions are satisfied:

i) $0 < a_0 \leq q(t) \leq a(t) \leq a_1,\ 0 < b_0 \leq b(t) \leq b_1,\ 0 < c_0 \leq c(t) \leq c_1.$

ii) $\int_{-\infty}^{+\infty} (|q'(u)| + |a'(u)| + |b'(u)| + |c'(u)|) du < \infty.$

iii) $0 < g_0 \leq g(x) \leq g_1,\quad 0 < h_0 \leq h(x) \leq h_1,\quad 0 < \varphi_0 \leq \varphi(x) \leq \varphi_1.$

iv) $\int_{-\infty}^{+\infty} (|g'(u)| + |h'(u)| + |\varphi'(u)|) du < \infty.$

v) $f(0) = 0, \dfrac{f(x)}{x} \geq \delta_0 > 0\ (x \neq 0)$, and $|f'(x)| \leq \delta_1$ for all x.

vi) $\dfrac{c_1 g_1 \delta_1}{b_0 \varphi_0} < \mu_1 < \dfrac{a_0 h_0}{a_1}$ and

$$\mu_2 = \min\left\{1,\ 2D_2,\ \frac{2g_0^2 D_1}{g_1^2\left(2a_1 h_1 + 2 + b_1\varphi_1\right)}\right\}, \text{ where}$$

$$D_1 = \frac{a_0\left(\mu_1 b_0 \varphi_0 - c_1 \delta_1 g_1\right)}{g_1^2} > 0,\ D_2 = \frac{a_0 h_0 - \mu_1 a_1}{a_1 g_1} > 0,$$

Before stating theorems, let us introduce the following notations:

$$\begin{aligned}
\Theta_1(t) &= \frac{1}{\beta_1}\Big(\frac{1}{g(x(t))}\Big)', \Theta_2(t) = \frac{1}{\beta_1}\Big(\frac{h(x(t))}{g(x(t))}\Big)',\ \Theta_3(t) = \frac{1}{\beta_1}\Big(\frac{\varphi(x(t))}{g(x(t))}\Big)', \\
\Theta_4(t) &= \frac{1}{\beta_1}\Big(\frac{q(t)}{g(x(t))}\Big)',\ \Theta_5(t) = \Big(|q'(t)| + |a'(t)| + |b'(t)| + |c'(t)|\Big), \\
\Theta_6(t) &= \frac{1}{\beta_3}\Big(h(x(t))\Big)',
\end{aligned} \tag{3.1}$$

and

$$\Omega(t) = \int_0^t \Big[|\Theta_1(s)| + |\Theta_2(s)| + |\Theta_3(s)| + |\Theta_4(s)| + (\frac{1}{\beta_1} + \frac{1}{\beta_2})\Theta_5(s) + |\Theta_6(s)|\Big] ds.$$

Also,

$$\begin{aligned}
\gamma_1 &= \max\left\{\frac{a_1\varphi_1}{2g_0}, \frac{a_1}{2\mu_1}, \frac{\mu_1 a_1 h_1}{g_0^2}+\frac{b_1\varphi_1}{2g_0}+\frac{c_1}{2\mu_1}\right\},\\
\gamma_2 &= \max\left\{\frac{h_1}{g_0}(\frac{\mu_2+1}{2}+\frac{a_1h_1}{g_0}), \frac{a_1\varphi_1}{2g_0}, \frac{a_1\alpha}{2}, \frac{\varphi_1 b_1}{2g_0}+\frac{c_1\alpha}{2}\right\}, \text{ such that } \alpha=\frac{a_0b_0\varphi_0}{c_1g_1a_1},\\
D_3 &= c_0\delta_0-\frac{(1+b_1\varphi_1)}{2}, D_4=\frac{a_0b_0h_0\varphi_0-a_1c_1\delta_1g_1}{g_0^2}>0.
\end{aligned}$$

Now, our main result on the boundedness and ultimate boundedness of (1.3) with $e(t)\neq 0$.

Theorem 5. *If hypotheses (i)-(vi) hold true, and in addition the following conditions are satisfied*

vii) $|e(t)|\leq m$,

viii) $D_3>0$.

Then all solutions of (1.3) are uniformly bounded and uniformly ultimately bounded provided r *satisfies*

$$r<\min\left\{\frac{2D_2-\mu_2}{\delta_1c_1}, \frac{2D_3}{\delta_1c_1}, \frac{2g_0^3D_4}{\delta_1c_1[g_0(2+\mu_2)+a_1(\mu_1+h_1)(1+g_0^2)]}\right\}. \tag{3.2}$$

Proof. We write the equation (1.3) as the following equivalent system

$$\begin{cases}
x'=\dfrac{1}{g(x)}y,\\
y'=\dfrac{1}{q(t)}z,\\
z'=-\dfrac{a(t)h(x)}{q(t)g(x)}z-a(t)\Theta_2(t)y-\dfrac{b(t)\varphi(x)}{g(x)}y-c(t)f(x)+e(t)\\
\qquad +c(t)\displaystyle\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds.
\end{cases} \tag{3.3}$$

Note that the continuity of the functions a, b, c, e, f, g, h, φ, f', g', h', and φ' guarantees the existence of the solutions of (1.3) (see [7], pp.15). It is assumed that the right hand side of the system (3.3) satisfies a Lipschitz condition in $x(t), y(t), z(t)$, and $x(t-r)$. This assumption guarantees the uniqueness of solutions of (1.3) (see [7], pp.15). We shall use as a tool to prove our main results a Lyapunov function $W=W(t,x_t,y_t,z_t)$ defined by

$$W(t,x_t,y_t,z_t)=e^{-\Omega(t)}V(t,x_t,y_t,z_t)=e^{-\Omega(t)}V, \tag{3.4}$$

where

$$V=V_1(t,x_t,y_t,z_t)+V_2(t,x_t,y_t,z_t)+\lambda\int_{-r}^{0}\int_{t+s}^{t}y^2(\xi)d\xi ds,$$

$$\begin{aligned}
V_1 &= \mu_1q(t)c(t)G(x,y)+\frac{\mu_1q(t)}{2}\big(\frac{a(t)h(t)-\mu_1q(t)}{g^2(x)}\big)y^2+\frac{1}{2}\big(z+\frac{\mu_1q(t)}{g(x)}y\big)^2\\
&\quad+\frac{q(t)}{2}\big(\frac{\varphi(t)b(t)}{g(x)}-\frac{c(t)\delta_1}{\mu_1}\big)y^2,
\end{aligned} \tag{3.5}$$

$$\begin{aligned} V_2 &= a(t)c(t)h(x)F(x) - \frac{q(t)c^2(t)g(x)}{2b(t)\varphi(x)}f^2(x) + \frac{1}{2}\left(z + \frac{a(t)h(x)}{g(x)}y + \mu_2 x\right)^2 \\ &\quad + \frac{q(t)b(t)\varphi(x)}{2g(x)}\left(y + \frac{c(t)f(x)g(x)}{b(t)\varphi(x)}\right)^2 + \frac{1}{2}\mu_2(1-\mu_2)x^2, \end{aligned} \tag{3.6}$$

such that $F(x) = \int_0^x f(u)du$ and $G(x,y) = F(x) + \frac{1}{\mu_1}f(x)y + \frac{\delta_1}{2\mu_1^2}y^2$. λ is positive constant which will be specified later in the proof. We easily rearrange the above functional V_1 as follows

$$\begin{aligned} V_1 &= \mu_1 q(t)c(t)F(x) + \frac{q(t)b(t)\varphi(x)}{2g(x)}\left(y + \frac{c(t)f(x)g(x)}{b(t)\varphi(x)}\right)^2 - \frac{q(t)c^2(t)g(x)f^2(x)}{2b(t)\varphi(x)} \\ &\quad + \frac{1}{2}(z + \frac{\mu_1 q(t)}{g(x)}y)^2 + \frac{\mu_1 q(t)(a(t)h(x) - \mu_1 q(t))}{2g^2(x)}y^2. \end{aligned} \tag{3.7}$$

Using (i), (iii) and (vi) we have

$$\frac{\mu_1 q(t)(a(t)h(x) - \mu_1 q(t))}{2g^2(x)} \geq \frac{\mu_1 a_0(a_0 h_0 - \mu_1 a_1)}{2g^2(x)} > 0.$$

Thus there exists a positive constant δ_2 such that

$$\frac{1}{2}(z + \frac{\mu_1 q(t)}{g(x)}y)^2 + \frac{\mu_1 q(t)(a(t)h(x) - \mu_1 q(t))}{2g^2(x)}y^2 \geq \delta_2 y^2 + \delta_2 z^2. \tag{3.8}$$

On the other hand, using the assumptions (i), (iii), (v) and (vi) we obtain

$$\begin{aligned} \mu_1 q(t)c(t)F(x) - \frac{q(t)c^2(t)g(x)f^2(x)}{2b(t)\varphi(x)} &\geq \mu_1 q(t)c(t)\int_0^x (1 - c(t)\frac{g(x)f'(u)}{\mu_1 b(t)\varphi(x)})f(u)du \\ &\geq \mu_1 a_1 c_1 \int_0^x (1 - \frac{g_1 c_1 \delta_1}{\mu_1 b_0 \varphi_0})f(u)du \\ &\geq \delta_3 F(x), \end{aligned}$$

where $\delta_3 = \mu_1 a_1 c_1 (1 - \frac{g_1 c_1 \delta_1}{\mu_1 b_0 \varphi_0}) > 0$. Hence, from the last inequality, (3.8) and (3.7),

$$V_1 \geq \delta_3 F(x) + \delta_2 y^2 + \delta_2 z^2. \tag{3.9}$$

Clearly, using hypothesis (v) we have the following estimate

$$V_1 \geq \frac{\delta_3 \delta_0}{2}x^2 + \delta_2 y^2 + \delta_2 z^2. \tag{3.10}$$

By adding and subtracting some terms together with condition (i) we can estimate the functional V_2 above thus

$$\begin{aligned} V_2 &\geq q(t)c(t)H(x,y) + \frac{1}{2}\left(z + \frac{h(x)}{g(x)}y + \mu_2 x\right)^2 + \frac{1}{2}\mu_2(1-\mu_2)x^2 \\ &\quad + \frac{q(t)}{2}\left(\frac{b(t)\varphi(x)}{g(x)} - \alpha c(t)\right)y^2, \end{aligned}$$

where

$$H(x,y) = h(x)F(x) + f(x)y + \frac{\alpha}{2}y^2.$$

From condition (vi) we have $\dfrac{b(t)\varphi(x)}{g(x)} - \alpha c(t) \geq 0$, and $1-\mu_2 \geq 0$, it follows that

$$V_2 \geq q(t)c(t)H(x,y).$$

But

$$\begin{aligned} H(x,y) &= h(x)F(x) + \frac{\alpha}{2}\big(y + \frac{1}{\alpha}f(x)\big)^2 - \frac{1}{2\alpha}f^2(x) \\ &\geq h(x)F(x) - \frac{1}{2\alpha}f^2(x) \\ &\geq \int_0^x \big(h_0 - \frac{\delta_1}{\alpha}\big)f(u)du. \end{aligned}$$

From condition (vi) $H(x,y) \geq 0$. Hence, by (i) we get

$$V_2 \geq a_0 c_0 H(x,y). \tag{3.11}$$

It is easily seen from (3.6) that

$$\begin{aligned} V_2 &\geq a(t)c(t)h(x)F(x) - \frac{q(t)c^2(t)g(x)}{2b(t)\varphi(x)}f^2(x) \\ &\geq c(t)\big(a_0h_0F(x) - \frac{a_1c(t)g_1}{2b_0\varphi_0}f^2(x)\big) \\ &\geq c_1\int_0^x (a_0h_0 - \frac{a_1g_1c_1\delta_1}{b_0\varphi_0})f(u)du \\ &\geq \delta_4 F(x), \end{aligned}$$

where $\delta_4 = c_1\big(a_0h_0 - \dfrac{a_1c_1g_1\delta_1}{b_0\varphi_0}\big) > 0$. Thus from (v) we obtain,

$$V_2 \geq \frac{\delta_4\delta_0}{2}x^2. \tag{3.12}$$

Clearly, from (3.12), (3.10) and the fact that the integral $\int_{-r}^0\int_{t+s}^t y^2(\xi)d\xi ds$ is positive, we deduce that

$$V \geq \delta_2 y^2 + \delta_2 z^2 + \frac{\delta_5\delta_0}{2}x^2,$$

where $\delta_5 = \delta_3 + \delta_4$. Further simplification of the last inequality gives

$$V \geq k(x^2+y^2+z^2), \tag{3.13}$$

where $k = min\{\delta_2; \frac{\delta_5\delta_0}{2}\}$. In view of the hypotheses (i)-(iv) we have

$$\begin{aligned}
\Omega(t) &= \int_0^t \Big[|\Theta_1(s)| + |\Theta_2(s)| + |\Theta_3(s)| + |\Theta_4(s)| + \big(\frac{1}{\beta_1} + \frac{1}{\beta_2}\big)\Theta_5 + |\Theta_6(s)| \Big] ds \\
&\leq \frac{(1+\varphi_1+h_1+a_1)}{\beta_1} \int_{\sigma_1(t)}^{\sigma_2(t)} \frac{|g'(u)|}{g^2(u)} du + \frac{1}{\beta_1} \int_{\sigma_1(t)}^{\sigma_2(t)} \frac{|\varphi'(u)| + |h'(u)|}{g(u)} du \\
&\quad + \frac{1}{\beta_3} \int_{\sigma_1(t)}^{\sigma_2(t)} |h'(u)|\, du + \frac{1}{\beta_1 g_0} \int_0^t |q'(u)| du \\
&\quad + \big(\frac{1}{\beta_1} + \frac{1}{\beta_2}\big) \int_0^t (|q'(u)| + |a'(u)| + |b'(u)| + |c'(u)|) du \\
&\leq \frac{(1+\varphi_1+h_1+a_1)}{\beta_1 g_0^2} \int_{-\infty}^{+\infty} |g'(u)|\, du + \frac{1}{\beta_1 g_0} \int_{-\infty}^{+\infty} \big(|\varphi'(u)| + |h'(u)|\big) du \\
&\quad + \frac{1}{\beta_3} \int_{-\infty}^{+\infty} |h'(u)|\, du + \frac{1}{\beta_1 g_0} \int_{-\infty}^{+\infty} |q'(u)| du \\
&\quad + \big(\frac{1}{\beta_1} + \frac{1}{\beta_2}\big) \int_{-\infty}^{+\infty} (|q'(u)| + |a'(u)| + |b'(u)| + |c'(u)|) du \leq N < \infty,
\end{aligned}$$

where $\sigma_1(t) = \min\{x(0), x(t)\}$, and $\sigma_2(t) = \max\{x(0), x(t)\}$. Therefore we can find a continuous function $W_1(|\Phi(0)|)$ with

$$W_1(|\Phi(0)|) \geq 0 \quad \text{and} \quad W_1(|\Phi(0)|) \leq W(t, \Phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $W(t, \phi) \leq W_2(\|\phi\|)$, is easily verified.

For the time derivative of the Lyapunov functional $V(t, x_t, y_t, z_t)$, along the trajectories of the system (3.3), we have

$$V'_{(3.3)} = V'_{1_{(3.3)}} + V'_{2_{(3.3)}} + \lambda r y^2 - \lambda \int_{t-r}^t y^2(\xi) d\xi,$$

where

$$\begin{aligned}
V'_{1_{(3.3)}} &= \mu_1 \big(q(t)c(t)\big)' G(x,y) + \Big[\frac{q(t)c(t)g(x)f'(x) - \mu_1 q(t)b(t)\varphi(x)}{g^2(x)} \Big] y^2 \\
&\quad + \Big[\frac{\mu_1 q(t) - a(t)h(x)}{q(t)g(x)} \Big] z^2 - \frac{\mu_1 a(t)q(t)}{2} \frac{h(x)}{g(x)} \Theta_1(t) y^2 \\
&\quad + a(t)\Big(yz + \mu_1 q(t)\big(1 - \frac{1}{g(x)}\big) y^2 \Big) \Theta_2(t) + \frac{q(t)b(t)}{2} \Theta_3(t) y^2 + \mu_1 \Theta_4(t) yz \\
&\quad + \mu_1 \big(a(t)q(t)\big)' \frac{h(x)}{g^2(x)} y^2 + \frac{1}{2} \big(b(t)q(t)\big)' \frac{\varphi(x)}{g(x)} y^2 - \frac{1}{2\mu_1} \big(c(t)q(t)\big)' y^2 \\
&\quad + \mu_1 \frac{q(t)}{g(x)} e(t) y + e(t) z + c(t) \Big(z + \frac{\mu_1 q(t)}{g(x)} y \Big) \int_{t-r}^t \frac{y(s)}{g(x(s))} f'(x(s)) ds.
\end{aligned}$$

In view of conditions (i), (iii) and (v) we get

$$\begin{aligned}
V'_{1_{(3.3)}} \leq\ & \mu_1 A\Theta_5(t)G(x,y) - D_1y^2 - D_2z^2 + \gamma_1\Theta_5(t)y^2 \\
& + \frac{\mu_1 a_1^2}{2}\frac{h_1}{g_0}|\Theta_1(t)|y^2 + \frac{a_1b_1}{2}|\Theta_3(t)|y^2 + \Big(a_1|yz| + \mu_1a_1^2(1+\frac{1}{g_0})y^2\Big)|\Theta_2(t)| \\
& + \mu_1|\Theta_4(t)||yz| + \mu_1\frac{a_1}{g_0}|y|m + |z|m \\
& + c(t)\Big(z + \frac{\mu_1 q(t)}{g(x)}y\Big)\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds,
\end{aligned}$$

where $A = \max\{a_1, c_1\}$. Using the Schwartz inequality $|uv| \leq \frac{1}{2}(u^2+v^2)$, we obtain

$$\begin{aligned}
& \frac{\mu_1 a_1^2}{2}\frac{h_1}{g_0}|\Theta_1(t)|y^2 + \frac{a_1b_1}{2}|\Theta_3(t)|y^2 + \Big(a_1|yz| + \mu_1a_1^2(1+\frac{1}{g_0})y^2\Big)|\Theta_2(t)| \\
& \qquad + \mu_1|\Theta_4(t)||yz| \leq k_1\Big[|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)|\Big](y^2+z^2),
\end{aligned}$$

where $k_1 = max\{\frac{\mu_1}{2}(1+\frac{h_1}{g_0}), \frac{1}{2}(1+\frac{\mu_1}{g_0}), \frac{1}{2}\}$.

From condition (v) and the Schwartz inequality, we obtain the following

$$c(t)\frac{\mu_1 q(t)}{g(x)}y\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds \leq \frac{\delta_1\mu_1a_1c_1r}{2g_0}y^2 + \frac{\mu_1a_1c_1\delta_1}{2g_0^3}\int_{t-r}^{t}y^2(\xi)d\xi, \tag{3.14}$$

and

$$c(t)z\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds \leq \frac{\delta_1c_1r}{2}z^2 + \frac{\delta_1c_1}{2g_0^2}\int_{t-r}^{t}y^2(\xi)d\xi.$$

After some rearrangements we get

$$\begin{aligned}
V'_{1_{(3.3)}} \leq\ & \mu_1 A\Theta_5(t)G(x,y) - \Big[D_1 - \frac{\mu_1a_1\delta_1c_1}{2g_0}r\Big]y^2 - \Big[D_2 - \frac{\delta_1c_1r}{2}\Big]z^2 \\
& + \gamma_1\Theta_5(t)y^2 + k_1\Big[|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)|\Big](y^2+z^2) \\
& + \mu_1\frac{a_1}{g_0}|y|m + |z| + \frac{\delta_1c_1}{2g_0^2}(1+\frac{\mu_1a_1}{g_0})\int_{t-r}^{t}y^2(\xi)d\xi.
\end{aligned} \tag{3.15}$$

In addition,

$$\begin{aligned}
V'_{2_{(3.3)}} =\ & \big(a(t)c(t)\big)'h(x)F(x) + \big(q(t)c(t)\big)'f(x)y + a(t)c(t)\Theta_6(t)F(x) + \mu_2\frac{a(t)h(x)}{g^2(x)}y^2 \\
& + \frac{q(t)c(t)f'(x)g(x) - a(t)b(t)h(x)\varphi(x)}{g^2(x)}y^2 + \frac{\mu_2}{g(x)}\Big(1 - b(t)\varphi(x)\Big)xy + \frac{\mu_2}{g(x)}yz \\
& + \frac{b(t)q(t)}{2}\Theta_3(t)y^2 - \mu_2c(t)xf(x) + \mu_2xe(t) + \frac{a(t)h(x)}{g(x)}ye(t) + ze(t) \\
& + a'(t)\frac{h(x)}{g(x)}\big(\mu_2xy + yz + a(t)\frac{h(x)}{g(x)}y^2\big) + \frac{1}{2}\big(b(t)q(t)\big)'\frac{\varphi(x)}{g(x)}y^2 \\
& + c(t)(\mu_2x + z + \frac{a(t)h(x)}{g(x)}y)\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds.
\end{aligned}$$

We can now proceed analogously to (3.14)

$$\mu_2 c(t)x\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds \le \frac{\mu_2\delta_1 c_1 r}{2}x^2+\frac{\mu_2\delta_1 c_1}{2g_0^2}\int_{t-r}^{t}y^2(\xi)d\xi,$$

$$\frac{a(t)c(t)h(x)}{g(x)}y\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds \le \frac{\delta_1 a_1 c_1 h_1 r}{2g_0}y^2+\frac{a_1 h_1\delta_1 c_1}{2g_0^3}\int_{t-r}^{t}y^2(\xi)d\xi,$$

and

$$c(t)z\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds \le \frac{\delta_1 c_1 r}{2}z^2+\frac{\delta_1 c_1}{2g_0^2}\int_{t-r}^{t}y^2(\xi)d\xi.$$

These estimates and Schwartz inequality imply the following

$$\begin{aligned}
V'_{2_{(3.3)}} \le\ & \Big[\big(a(t)c(t)\big)'-\big(q(t)c(t)\big)'\Big]h(x)F(x)+\big(q(t)c(t)\big)'H(x,y)\\
&+a(t)c(t)\Theta_6(t)F(x)-\mu_2\Big[c_0\delta_0-\frac{(1+b(t)\varphi(t))}{2}\Big]x^2\\
&+\frac{q(t)c(t)f'(x)g(x)-a(t)b(t)h(x)\varphi(x)}{g^2(x)}y^2\\
&+\frac{\mu_2}{2g^2(x)}\Big(2+b(t)\varphi(x)+2a(t)h(x)\Big)y^2\\
&+\frac{\mu_2}{2}z^2+\frac{b(t)q(t)}{2}\Theta_3(t)y^2+\big(\mu_2|x|+\frac{a(t)h(x)}{g(x)}|y|+|z|\big)m\\
&+a'(t)\frac{h(x)}{g(x)}\big(\mu_2 xy+yz+a(t)\frac{h(x)}{g(x)}y^2\big)+\frac{1}{2}\big(b(t)q(t)\big)'\frac{\varphi(x)}{g(x)}y^2\\
&-\frac{\alpha}{2}\big(q(t)c(t)\big)'y^2+\frac{\mu_2\delta_1 c_1 r}{2}x^2+\frac{\delta_1 a_1 c_1 h_1 r}{2g_0}y^2+\frac{\delta_1 c_1 r}{2}z^2\\
&+\frac{\delta_1 c_1}{2g_0^2}\big(\mu_2+\frac{a_1 h_1}{g_0}+1\big)\int_{t-r}^{t}y^2(\xi)d\xi.
\end{aligned}$$

It is easy to check that by (i), (iii) and (v) we have

$$\begin{aligned}
\Big[\big(a(t)c(t)\big)'-\big(q(t)c(t)\big)'\Big]h(x)F(x) &\le \Big[|\big(a(t)c(t)\big)'|+|\big(q(t)c(t)\big)'|\Big]\frac{h_1\delta_1}{2}x^2\\
&\le \frac{h_1\delta_1 B}{2}\Theta_5(t)x^2,
\end{aligned}$$

such that $B=\max\{2a_1,c_1\}$. By conditions (i), (iii) and (v) we have

$$\frac{q(t)c(t)f'(x)g(x)-a(t)b(t)h(x)\varphi(x)}{g^2(x)} \le \frac{a_1 c_1\delta_1 g_1-a_0 b_0 h_0\varphi_0}{g_0^2}<0.$$

Using condition (i) and (iii) again we get

$$
\begin{aligned}
V'_{2_{(3.3)}} \leq\ & \frac{h_1\delta_1 B}{2}\Theta_5(t)x^2 + A\Theta_5(t)H(x,y) + a_1c_1|\Theta_6(t)|F(x) - \mu_2 D_3 x^2 \\
& -\Big[D_4 - \frac{\mu_2}{2g_0^2}\Big(2 + b_1\varphi_1 + 2a_1h_1\Big)\Big]y^2 + \frac{\mu_2}{2}z^2 + \frac{b_1a_1}{2}|\Theta_3(t)|y^2 \\
& +\big(\mu_2|x| + \frac{a_1h_1}{g_0}|y| + |z|\big)m + \gamma_2\Theta_5(t)(x^2+y^2+z^2) \qquad (3.16)\\
& +\frac{\mu_2\delta_1c_1r}{2}x^2 + \frac{\delta_1a_1c_1h_1r}{2g_0}y^2 + \frac{\delta_1c_1r}{2}z^2 + \frac{\delta_1c_1}{2g_0^2}\big(\mu_2 + \frac{a_1h_1}{g_0} + 1\big)\int_{t-r}^{t} y^2(\xi)d\xi.
\end{aligned}
$$

Combining (3.16), (3.15) and condition (vi) we get

$$
\begin{aligned}
V'_{(3.3)} \leq\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x) \\
& -\mu_2\Big[D_3 - \frac{\delta_1c_1r}{2}\Big]x^2 - \Big[D_4 - r(\lambda + \frac{\delta_1a_1c_1\mu_1}{2g_0} + \frac{\delta_1a_1c_1h_1}{2g_0})\Big]y^2 \\
& -\Big[D_2 - \frac{\mu_2}{2} - \frac{\delta_1c_1r}{2}\Big]z^2 + \mu_2|x|m + \frac{h_1+\mu_1}{g_0}|y|m + 2|z|m \\
& +k_2\Big[\,|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + \Theta_5(t)\Big](x^2+y^2+z^2) \\
& +\Big(\frac{\delta_1c_1[g_0(2+\mu_2) + a_1(\mu_1+h_1)]}{2g_0^3} - \lambda\Big)\int_{t-r}^{t} y^2(\xi)d\xi,
\end{aligned}
$$

where $k_2 = \max\left\{\gamma_1+\gamma_2,\ \frac{h_1\delta_1B}{2},\ k_1 + \frac{1}{2}\right\}$.

Choosing $\frac{\delta_1c_1[g_0(2+\mu_2) + a_1(\mu_1+h_1)]}{2g_0^3} = \lambda$, since r and D_3 satisfy (3.2) and condition (viii) respectively, there is $\eta > 0$ such that

$$
\begin{aligned}
V'_{(3.3)} \leq\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x) \\
& -\eta(x^2+y^2+z^2) + \eta M(|x| + |z| + |y|) \\
& +k_2\big(\,|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + \Theta_5(t)\big)(x^2+y^2+z^2), \qquad (3.17)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta &= \min\left\{D_4 - r(\lambda + \frac{\delta_1a_1c_1\mu_1}{2g_0} + \frac{\delta_1a_1c_1h_1}{2g_0}),\ D_3 - \frac{\delta_1c_1r}{2},\ D_2 - \frac{\mu_2}{2} - \frac{\delta_1c_1r}{2}\right\}. \\
M &= \frac{m}{\eta}\max\left\{2,\ \frac{h_1+\mu_1}{g_0},\ \mu_2\right\}.
\end{aligned}
$$

The above inequality may be written as

$$
\begin{aligned}
V'_{(3.3)} \leq\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x) \\
& -\frac{\eta}{2}(x^2+y^2+z^2) + k_2\big(\,|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)|\big)(y^2+z^2) \\
& -\frac{\eta}{2}[(x-M)^2 + (y-M)^2 + (z-M)^2] + \frac{3\eta}{2}M^2 \\
\leq\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x) - \frac{\eta}{2}(x^2+y^2+z^2) \\
& +k_2\big(\,|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + \Theta_5(t)\big)(x^2+y^2+z^2) + \frac{3\eta}{2}M^2.
\end{aligned}
$$

It is easily verified that

$$\begin{aligned} W'_{(3.3)} &= e^{-\Omega(t)}\Bigg[V'_{(3.3)} - \Big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| \\ &\qquad + \big(\frac{1}{\beta_1} + \frac{1}{\beta_2}\big)\Theta_5(t) + |\Theta_6(t)|\Big)V\Bigg]. \end{aligned}$$

Using the fact that

$$\begin{aligned} G(x,y) &= F(x) + \frac{\delta_1}{2\mu_1^2}\big(y + \frac{\mu_1}{\delta_1}f(x)\big)^2 - \frac{1}{2\delta_1}f^2(x) \\ &\geq F(x) - \frac{1}{2\delta_1}f^2(x) \\ &= \int_0^x \big(1 - \frac{f'(u)}{\delta_1}\big)f(u)du \geq 0, \end{aligned}$$

since $1 - \dfrac{f'(u)}{\delta_1} \geq 0$. It can be followed from (3.5) and (iii) that there exist $\delta_6 > 0$ such that

$$V_1 \geq \mu_1 a_0 c_0 G(x,y) + \delta_6 y^2 + \delta_6 z^2. \tag{3.18}$$

Combining (3.9) and (3.12) we have

$$V_1 \geq \delta_3 F(x) + \delta_2 y^2 + \delta_2 z^2 \text{ and } V_2 \geq \frac{\delta_4\delta_0}{2}x^2.$$

From (3.11) and (3.18) we get

$$V \geq \mu_1 a_0 c_0 G(x,y) + \delta_6 y^2 + \delta_6 z^2 + a_0 c_0 H(x,y).$$

Hence, by (3.13) and the last inequalities we have the following estimate

$$\begin{aligned} W'_{(3.3)} \leq e^{-\Omega(t)}\Bigg[V'_{(3.3)} - \Bigg(& k\big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| \\ & + \frac{1}{\beta_1}\Theta_5(t)\big)\big(x^2 + y^2 + z^2\big) \\ & + \frac{1}{\beta_2}\Theta_5(t)\big(\mu_1 a_0 c_0 G(x,y)\delta_6 y^2 + \delta_6 z^2 + a_0 c_0 H(x,y)\big) \\ & + |\Theta_6(t)|\big(\delta_3 F(x) + \delta_2 y^2 + \delta_2 z^2 + \frac{\delta_4\delta_0}{2}x^2\big)\Bigg)\Bigg]. \end{aligned}$$

So choosing $\beta_1 = \dfrac{k}{k_2}$, $\beta_2 = \dfrac{a_0 c_0}{A}$ and $\beta_3 = \dfrac{\delta_3}{a_1 c_1}$ this reduces to

$$W'_{(3.3)}(t, x_t, y_t, z_t) \leq L\left[-\frac{\eta}{2}(x^2 + y^2 + z^2) + \frac{3\eta}{2}^2\right], \quad \text{for some } \ L > 0.$$

Hence the conclusions of Theorem 5 can be followed from Lemma 3, this completes the proof of Theorem 5 □

The following Theorem being a consequence of Theorem 5 and Lemma 4

Theorem 6. *If hypotheses of Theorem 5 be satisfied and a, b, c, e, q are periodic functions of period T, then there exists at the east periodic solution of system (1.3) with the period T.*

Proof. It only remains to verify using the assumptions of Theorem 5 that the conditions of Lemma 4 follow easily. □

Example 7. We consider the following third order delay differential equation

$$\begin{aligned}
&\Big[ln(3+\cos t)\big[(\frac{\cos(x)}{1+x^2}+4)x'(t)\big]'\Big]' + (2ln(5+2\cos t)\big(\frac{\sin x+3e^x+3e^{-x}}{e^x+e^{-x}}x'(t)\big)' \\
&+(3ln(2+\cos t)+1)(\frac{\sin(x)}{1+x^2}+11)x'(t) \\
&+(\frac{1}{2}ln(4+\cos t))\left[x(t-r)+\frac{x(t-r)}{1+x^2(t-r)}\right] = 3\sin t+5. \qquad (3.19)
\end{aligned}$$

It can be seen that

$$\begin{aligned}
&2ln3 = a_0 \le a(t) = 2ln(5+2\cos t) \le 2ln7,\ a'(t) = -\frac{4\sin t}{5+2\cos t}, \\
&1 = b_0 \le b(t) = 3ln(2+\cos t)+1 \le 1+3ln3, b'(t) = -3\frac{\sin t}{2+\cos t}, \\
&\frac{ln3}{2} = c_0 \le c(t) = \frac{1}{2}ln(4+\cos t) \le \frac{ln5}{2},\ \le c'(t) = \frac{1}{2}\frac{\sin t}{5+\cos t}, \\
&ln2 \le q(t) = ln(3+\cos t) \le 2ln2, \le q'(t) = -\frac{\sin t}{3+\cos t}, \\
&50 \le \frac{f(x)}{x} = 50+\frac{1}{1+x^2} \text{ with } x \neq 0,\ |f'(x)| \le \delta_1 = 2,\ t \ge 0,
\end{aligned}$$

Moreover,

$$\begin{aligned}
&2 \le e(t) = 3\sin t+5 \le 8,\ 3 \le g(x) = \frac{\cos(x)}{1+x^2}+4 \le 5, \\
&10 \le \varphi(x) = \frac{\sin(x)}{1+x^2}+11 \le 12,\ \frac{5}{2} \le h(x) = \frac{\sin x+3e^x+3e^{-x}}{e^x+e^{-x}} \le \frac{7}{2}.
\end{aligned}$$

Also, $0.80 = \dfrac{c_1g_1\delta_1}{b_0\varphi_0} < \mu_1 < \dfrac{a_0h_0}{a_1} = 1.41$ and $50 = \delta_0 > \dfrac{1+\varphi_1b_1}{2c_0} = 47.83.$
It is straightforward to verify that

$$\begin{aligned}
\int_{-\infty}^{+\infty} |g'(u)|\,du &\le \int_{-\infty}^{+\infty} \left[\left|\frac{\sin u}{1+u^2}\right| + \left|\frac{2u\cos u}{(1+u^2)^2}\right|\right] du \\
&\le \pi+2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\varphi'(u)|\,du &\le \int_{-\infty}^{+\infty} \left[\left|\frac{\cos u}{1+u^2}\right| + \left|\frac{2u\sin u}{(1+u^2)^2}\right|\right] du \\
&\le \pi+2.
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} |h'(u)|\,du &= \int_{-\infty}^{+\infty} \left|\frac{(e^u+e^{-u})\cos u-(e^u-e^{-u})\sin u}{(e^u+e^{-u})^2}\right| du \\
&\le \int_{-\infty}^{+\infty} \left(\frac{1}{e^u+e^{-u}}+\frac{u}{(e^u+e^{-u})^2}\left(e^u-e^{-u}\right)\right) du = \pi.
\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|q'(u)|\,du &= \int_{-\infty}^{+\infty}|\frac{\sin u}{3+\cos u}|du \leq \int_{-\infty}^{+\infty}\frac{1}{3+\cos u}du\\ &= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{1}{2+u^2}du = \frac{2}{\sqrt{2}}tan^{-1}(\frac{\pi}{2\sqrt{2}}).\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|a'(u)|\,du &= \int_{-\infty}^{+\infty}|\frac{4\sin u}{5+2\cos u}|du \leq \int_{-\infty}^{+\infty}\frac{4}{5+2\cos u}du\\ &= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{8}{7+3u^2}du = \frac{16}{\sqrt{21}}tan^{-1}(\frac{\pi\sqrt{3}}{2\sqrt{7}}).\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|b'(u)|\,du &= 3\int_{-\infty}^{+\infty}|\frac{\sin u}{2+\cos u}|du \leq 3\int_{-\infty}^{+\infty}\frac{1}{3+\cos u}du\\ &= 3\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{1}{2+u^2}du = \frac{4}{\sqrt{3}}tan^{-1}(\frac{\pi}{2\sqrt{3}}).\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|c'(u)|\,du &= \frac{1}{2}\int_{-\infty}^{+\infty}|\frac{\sin u}{4+\cos u}|du \leq \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{4+\cos u}du\\ &= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{1}{5+3u^2}du = \frac{2}{\sqrt{15}}tan^{-1}(\frac{\pi\sqrt{3}}{2\sqrt{5}}).\end{aligned}$$

Thus all the assumptions of Theorem 5. hold, this shows that every solution of (3.19) is uniformly bounded and uniformly ultimately bounded. Since a, b, c, e, q are periodic functions of period 2π, then there exists a periodic solution of (3.19) of period 2π.

For the case $e(t) = 0$, the equation (1.3) is equivalent to the system

$$\begin{cases} x' = \dfrac{1}{g(x)}y,\\ y' = \dfrac{1}{q(t)}z,\\ z' = -\dfrac{a(t)h(x)}{q(t)g(x)}z - a(t)\Theta_2(t)y - \dfrac{b(t)\varphi(x)}{g(x)}y - c(t)f(x) + c(t)\displaystyle\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds.\end{cases} \tag{3.20}$$

The following result is introduced.

Corollary 8. *One assumes that all the assumptions (i)-(vi) and (vii) hold. Then the zero solution of equation (1.3) is uniformly asymptotically stable.*

Proof. If $e(t) = 0$, similarly to above proof, the inequality (3.17) becomes

$$\begin{aligned}V'_{(3.20)} \leq\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x) - \eta(x^2+y^2+z^2)\\ &+k_2\big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + \Theta_5(t)\big)(x^2+y^2+z^2),\end{aligned}$$

Hence

$$W'_{(3.20)}(t, x_t, y_t, z_t) \leq L\left[-\eta(x^2+y^2+z^2)\right], \quad \text{for some } \ L > 0.$$

Thus, all the conditions of Lemma 1 are satisfied. This shows that the zero solution of equation (1.3) is uniformly asymptotically stable. □

4 Conclusions

Liapunov's method has proved to be a popular and useful technique in the study of the stability and boundedness of solutions of higher order non-linear differential equations. In this paper we examine the boundedness and ultimate boundedness of solutions for certain third order non-linear non-autonomous differential equations with delay. Sufficient conditions were obtained for the existence of at least one periodic solution of the equation. Finally, we investigate the asymptotic stability of the zero solution of the same equation for the case $e(t) = 0$.

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A note on Euclid's Theorem concerning the infinitude of the primes

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Abstract

We present another elementary proof of Euclid's Theorem concerning the infinitude of the prime numbers. This proof is "geometric" in nature and it employs very little beyond the concept of "proportion."

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Euclid's Theorem ([2], Book IX, Proposition 20) establishes the existence of infinitely many prime numbers. It has been one of the cornerstones of mathematical thought. More than a dozen different proofs of this result, with many clever simplifications and variants, have been published over the past two millennia (for lists of proofs and good discussions of their historical relevance, see [1], [3], [4] and [6]). A decade ago, in [5], we gave a short direct proof of Euclid's Theorem that has received a surprising amount of attention. Here we would like to present another idea, not quite as simple as the first one, but perhaps equally fundamental. It makes use of the ancient concept of proportion, the theory of which was perfected by Pythagoras, Eudoxus and finally Euclid himself (a fact demonstrated by the results summarized in Book V of his *Elements* [2]).

We rephrase the problem slightly. The question we ask is: Why cannot products of powers of a finite number of primes cover the entire set $\mathbb{N}$?

We investigate the factorization geometrically and consider the canonical representation as an operation (on exponents) in two dimensions, with single prime powers representing what we will call the "vertical" and their products the "horizontal" dimensions.

Vertical Dimension. For a fixed prime number p, and $0 \le i \le m$, there are $m+1$ positive integers that can be written in the form p^i, the largest of which is p^m. Since, clearly, $m+1 \le (1+1)^m = 2^m \le p^m$, many integers are not of this form; so for the proportion $\nabla(p^m)$ of these powers (up to p^m) we not only have $\nabla(p^m) < 1$, for all $m > 1$ (as well as $\nabla(p^m) \to 0$, as $m \to \infty$), but also $\nabla(p^m) > \nabla(p^{m+1})$, because

$$\frac{m+1}{p^m} > \frac{m+2}{p^{m+1}} \iff 1 - \frac{1}{m+2} > \frac{1}{p}. \tag{1}$$

Thus, considered vertically, the proportions are monotonically decreasing.

Horizontal Dimension. Recall that a function $f : \mathbb{N} \to \mathbb{C}$ is called multiplicative, if $f(1) = 1$ and $f(ab) = f(a)f(b)$, for all $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. A critically important property of the proportions ∇ is their multiplicativity. For all $k \geq 2$, let us define

$$\nabla(p_1^{m_1} \cdots p_k^{m_k}) \ := \ \frac{\#\{n = p_1^{a_1} \cdots p_k^{a_k} \ : \ 0 \leq a_j \leq m_j, \text{ for } 1 \leq j \leq k\}}{p_1^{m_1} \cdots p_k^{m_k}},$$

then, for all permutations of exponents m_i, we have

$$\nabla(p_1^{m_1} \cdots p_k^{m_k}) = \nabla(p_1^{m_1}) \cdots \nabla(p_k^{m_k}) < 1. \tag{2}$$

In other words, the multiplicativity of ∇ implies the horizontal monotonicity.

Combining these two monotonic orthogonal trends is enough to prove the infinitude of the prime numbers. This is because the vertical dimension is (trivially) infinite, and (1) implies an ever-increasing sparseness of integers represented by a given prime power; while from the monotonicity property of (2) it follows that the same will remain true upon any finite composition of such powers, and therefore only an infinite horizontal dimension could possibly compensate for the growing deficit and create a complete cover of $\mathbb{N}$, guaranteed by the unique factorization theorem.

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