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Twenty-five years of Acta UMB Math

Editorial

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Our journal Acta Universitatis Matthiae Belii, series Mathematics (shortly, Acta UMB Math), was founded in 1993 by Alfonz Haviar, a father of the author of this editorial (for an editorial dedicated to the seventies of the founder see [1]). The journal publishes original research articles and survey papers in selected areas of mathematics and theoretical computer science which are, said in a pragmatic way, defined as the union of the areas of interest of the members of its Editorial Board. The printed edition of Acta UMB Math has from its beginning been covered by the world renowned reviewing mathematical journals Zentralblatt MATH of European Mathematical Society, founded in 1931, and Mathematical Rewiews, founded in 1940, of American Mathematical Society with its famous mathematical electronic database MathSciNet since 1996.

Alfonz Haviar, the founder of Acta UMB Math.

During the twenty-five years of its existence the journal has been led by three editors-in-chief: Alfonz Haviar for the first eight volumes in 1993–2000, Roman Nedela for the next seven volumes in 2001–2009 (no volumes appeared in the years 2002 and 2008) and Miroslav Haviar for the subsequent ten volumes in 2010–2017 (two volumes appeared in the years 2010 and 2011). As the managing editors of the journal served Gabriela Monoszová in 1993–2000, Vladimír Janiš in 1994, Dana Smutná in 2001, Petr Hliněný in 2002–2011 excepting 2009 and Volume 19, Marián Grendár in 2005–2010 excepting 2009, Miroslav Haviar in 2009 and Ján Karabáš since Volume 19 in 2011 (however, Ján Karabáš has served the journal as its technical editor already since 2004).

Online edition of Acta UMB Math was founded in 2013 to make the publication process more flexible and to shorten it: every article accepted for publication into Acta

UMB Math almost immediately appears in the online edition. So far all articles from the online version have been selected for publishing in the printed edition. However, the idea since the birth of the online version has been that in case of possibly many articles in the online edition only some would be selected for the printed edition which always appears towards the end of the calendar year.

The journal started with publishing papers of members of Banská Bystrica mathematical community (the first contributors from outside the city were Pavol Híc from Trnava and Judita Lihová from Košice in Volume 3 and the first contributor from abroad was Andrew Bucki from Oklahoma City, USA, in Volume 4). Until Volume 14 in 2007, roughly half of the contributors were from Banská Bystrica mathematical community. We would still be very pleased with having members of Banská Bystrica mathematical community and more generally, of Slovak mathematical community, as prevailing contributors for Acta UMB Math. The "publishing culture" in Slovakia that has so dramatically changed over the last decade has unfortunately led to less and less Slovak authors appearing in the journal. The journal has also been recently offered an Open Access platform within one of the global scientific publishing houses De Gruyter (headquartered in Berlin, with offices in Basel, Beijing, Boston and Munich), but for various reasons the move to De Gruyter Open has not yet happened.

The journal would welcome high quality long papers in the spirit of Transactions of American Mathematical Society – the journal was privileged to publish two such papers by Brian A. Davey (Melbourne) as well as by Yan-Quan Feng (Beijing) with Roman Nedela (Banská Bystrica) in Volume 13 in 2006. Recent invitations to renowned authors with connections to Banská Bystrica to publish high quality survey papers in Acta UMB Math have led to two such great contributions: by Alex Rosa from Canada in Volume 23 (2015) and by Mikhail H. Klin from Israel and Andrew J. Woldar from the USA in the present Volume 25. We believe that not only the 25th anniversary, but mainly the wonderful 58 pp. survey article by Misha and Andy, with 229 references, make the present volume so special.

On behalf of the present Editorial Board of Acta UMB Math I wish to thank all contributors of the present jubilee issue, in particular Misha and Andy. Let me also thank all people who over the past 25 years contributed by their articles or by their work (editorial, reviewing, managerial or technical) to the journal. I wish the journal to celebrate its other jubilees in good shape and with having its name being wider recognized on the Slovak and worldwide publishing scenes.

References

[1] M. Haviar and Ľ. Snoha, Fonzo Haviar is seventy this year, Special issue dedicated to 70th birthday of Alfonz Haviar, *Acta Universitatis Matthiae Belii, series Mathematics* **15** (2009), pp. 5–10.

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The strongly regular graph with parameters (100,22,0,6): Hidden history and beyond

Dedicated to the memory of Dale Marsh Mesner (1923–2009)

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Abstract

Discovery of the strongly regular graph Γ with parameters $(100, 22, 0, 6)$ is almost universally attributed to D. G. Higman and C. C. Sims, stemming from their innovative 1968 paper. While such attribution is surely appropriate, this graph has a most intriguing history that for decades has remained hidden to the vast majority of mathematicians. In this paper, we reveal that Γ was in fact constructed as early as 1956 by Dale M. Mesner, who later established its uniqueness in 1964. We provide a detailed account of both independent discoveries, paying special attention to differing perspectives, styles, motivation and methodologies of these accomplished mathematicians, and discuss how their contrasting presentations influenced future generations of researchers. It is also hoped that the new analysis of Γ we arrange in Section 11 will stimulate a renewed interest in the problem of classifying all primitive strongly regular graphs with no triangles.

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Keywords Algebraic graph theory, association scheme, strongly regular graph (SRG), Higman-Sims graph, negative Latin square-type graph, triangle-free graph, balanced incomplete block design (BIBD), quasi-symmetric design (QSD), 3-design, negative Latin square design, Witt design, Steiner system, biplane, generalized quadrangle, spread, Higman-Sims group, Mathieu group.
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1 Introduction

The discovery of a rank 3 graph on 100 vertices by Higman and Sims [101] was a definite breakthrough in group theory and combinatorics. Aside from its extraordinary significance on the dawn of the era of the classification of finite simple groups, this discovery served also as a strong impetus for further development of the theory of rank 3 groups.

It turns out that the same graph Γ on 100 vertices was discovered 12 years earlier and described in much detail in the Ph.D. thesis [159] of Dale Marsh Mesner, see also [160]. While the motivating factors and employed techniques of Mesner and Higman & Sims are essentially different, it is quite surprising to observe that the final form in which this graph was independently described is nearly identical.

Our objectives in this paper are multifold:

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- to provide an historic account of the origins of the graph Γ, in particular to recreate the drama of competing ideas from diverse scientific traditions, backgrounds and experiences;
- to describe how these independent discoveries influenced future development in group theory and combinatorics;
- to pay tribute to Dale M. Mesner by promoting awareness of his results to a wide mathematical audience, in particular to convince the reader that Mesner's ideas, hidden from view for so many years, still have fresh and promising potential.

A further hope is that our text will help to promote future investigations of such extremely rare objects as primitive strongly regular graphs with no triangles.

We do not aim to provide the reader with all necessary preliminaries, however there is an abundant supply of references throughout. For the purpose of general background information, the text [42] is extraordinary in its scope and accessibility. The survey [73] should also be helpful (see, as well, the references cited in Section 2). For a good initial exposure to the life and mathematics of Dale Mesner we recommend [118], as well as [10] which fulfills a similar role for Donald G. Higman.

The following conventions will be used freely throughout the text: The abbreviation (P)BIBD stands for "(partially) balanced incomplete block design". Likewise, SRG stands for "strongly regular graph". Occasionally we write SRG(v, k, λ, μ) to denote an SRG with the indicated parameters, although these parameters will sometimes stand alone. An SRG Γ is said to be *primitive* if both Γ and its complementary graph $\overline{\Gamma}$ are connected. The standard parameters for a BIBD are denoted by (v, b, r, k, λ); an appended caret is used (e.g., $\widehat{v}$ replacing v) when it is necessary to distinguish these parameters from those of an SRG. Intersection numbers of an association scheme are denoted by p_{ij}^k. Often, we shall abuse notation by freely identifying a (symmetric) 2-class association scheme with its corresponding SRG (or pair of SRGs). We use the abbreviation AGT for "algebraic graph theory". Likewise, due to its sheer frequency alone, we abbreviate the name "Dale Mesner" by DM. In the same spirit we freely append DM to various nouns (e.g., DM-theory, DM-approach, DM-series, etc.).

The balance of our paper is organized as follows. Sec. 2 is devoted to the most significant preliminaries, exposing the reader to the core essentials of our exposition. Sec. 3 is a microcosm of the entire story: a brief summary with minimum detail. A rather thorough account of the texts [159] and [160] appears in Secs. 4 and 5, respectively. Secs. 6-8 introduce the reader to the main techniques, ideas and results of DM as they relate to the graph SRG$(100, 22, 0, 6)$. These sections are meticulously detailed, as are Secs. 9 and 10 which deal with the Witt design [224] and the Higman-Sims group [101], respectively. The authors' own personal vision of DM's construction of SRG$(100, 22, 0, 6)$ is presented in Sec. 11. Here, our own (re)construction relies on the use of computer algebra packages, with the helpful assistance of Matan Ziv-Av. In Sec. 12 we survey many important developments that cascaded from the earlier seminal ideas of DM and others. In Sec. 13 we consolidate all extra material that we feel may be of compelling interest to the reader, but which we intentionally omitted from the main expository thread so as not to disrupt the flow of our presentation. Finally, in Sec. 14 we briefly explain how the current text evolved from its earlier incarnations over the course of a dozen or so years.

In broad terms the genre of our paper is dynamic survey; thus we hope to create its future updates. Any new and illuminating information relevant to our presentation would be greatly appreciated.

2 Preliminaries

Our goals in this section are quite modest: to recall central definitions of AGT, to establish notation and terminology, and to warm up the reader to a few simple examples treated in a deliberately naive fashion.

Let Ω be a finite set of cardinality $|\Omega| = n$. By $S(\Omega)$ we mean the group of all permutations of Ω with respect to composition of functions, hence we may identify $S(\Omega)$ with the symmetric group S_n of degree n and order $n!$. We denote by α^g the image of $\alpha \in \Omega$ under the action of $g \in S_n$. Clearly $(\alpha^g)^h = \alpha^{gh}$ for all $\alpha \in \Omega$ and $g, h \in S_n$, and $\alpha^e = \alpha$ for all $\alpha \in \Omega$ where e is the identity element of S_n.

We call (G, Ω) a *permutation group* of degree n provided G is a subgroup of $S(\Omega)$, $|\Omega| = n$. In this case, each $g \in G$ may be identified with an $n \times n$ permutation matrix $X_g \in M^{n\times n}(\mathbb{C})$. We next consider the algebra $\mathcal{V}(G, \Omega)$ defined as follows:

$$\mathcal{V}(G, \Omega) = \{A \in M^{n\times n}(\mathbb{C}) \mid AX_g = X_gA \text{ for all } g \in G\}.$$

It is immediate that $\mathcal{V}(G, \Omega)$ is a matrix algebra with standard basis consisting of $(0, 1)$-matrices. Observe that this algebra contains the identity matrix I, the all-ones matrix J, and is closed with respect to both complex conjugation and Schur-Hadamard (entry-wise) multiplication. We call $\mathcal{V}(G, \Omega)$ the *centralizer algebra* of (G, Ω).

The centralizer algebra has a very nice formulation in terms of 2-orbits of (G, Ω). Here, by 2-*orbit* we mean an orbit of G on $\Omega \times \Omega$ induced from the action (G, Ω) in the most natural sense: $(\alpha, \beta)^g = (\alpha^g, \beta^g)$. For each directed graph (digraph) (Ω, R) which is invariant with respect to a prescribed permutation group (G, Ω), the arc set R will be a union of suitable relations from the set 2-$orb(G, \Omega)$ of all 2-orbits of (G, Ω). In fact, many central concepts in modern AGT are based axiomatically on the salient properties of this set 2-$orb(G, \Omega)$.

Again, let Ω be a finite set, $|\Omega| = n$, and let $\mathcal{R} = \{R_1, R_2, \dots, R_r\}$ be a partition of the Cartesian square Ω^2. A pair $\mathfrak{X} = (\Omega, \mathcal{R})$ is called a *coherent configuration* (briefly, *CC*) *of rank* r provided the following conditions hold:

(CC1) $R_i \cap R_j = \emptyset$ for all $1 \le i \ne j \le r$;

(CC2) $\bigcup_{i=1}^{r} = \Omega^2$;

(CC3) For each $i \in \{1, 2, \dots, r\}$ there exists $i' \in \{1, 2, \dots, r\}$ such that $R_{i'} = R_i^T$ where $R_i^T = \{(\beta, \alpha) \mid (\alpha, \beta) \in R_i\}$;

(CC4) There exists a subset $Y \subset \{1, 2, \dots, r\}$ such that $\bigcup_{i\in Y} R_i = \Delta \equiv \{(\alpha, \alpha) \mid \alpha \in \Omega\}$;

(CC5) For each $i, j, k \in \{1, 2, \dots, r\}$ the number p_{ij}^k of elements $z \in \Omega$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is constant for all pairs $(x, y) \in R_k$.

There is a lot to be said about axioms (CC1)-(CC5). First observe that axioms (CC1) and (CC2) merely reassert that $\mathcal{R}$ is a partition of Ω^2. Each nonempty subset R_i is thus a binary relation on Ω, called a *basis relation.* In axiom (CC3) we refer to R_i^T as the *transpose* relation of R_i. In axiom (CC4) we refer to Δ as the *diagonal* (or *reflexive*) relation on Ω. Finally, in axiom (CC5) we refer to the numbers p_{ij}^k as the *intersection numbers* of $\mathfrak{X}$.

To each basis relation R_i of $\mathfrak{X}$ we associate a digraph $\Gamma_i = (\Omega, R_i)$, which we call a *basis* (*di*)*graph* of $\mathfrak{X}$. Let us denote its adjacency matrix by $A_i = A(\Gamma_i)$. It is then easy to verify that the set $\{A_1, A_2, \ldots, A_r\}$ forms a basis for the vector subspace $\mathfrak{S} = \mathfrak{S}(\mathfrak{X})$ of $M^{n\times n}(\mathbb{C})$, in particular $A_iA_j = \sum_{k=1}^{r} p_{ij}^k A_k$ for all i, j.

The reader will now observe that axiom (CC5) asserts that $\mathfrak{S}$ is actually a subalgebra of $M^{n\times n}(\mathbb{C})$. In fact, it may be seen from axioms (CC1)-(CC5) that $\mathfrak{S}$ contains I, J, and is closed with respect to complex conjugation and Schur-Hadamard multiplication. Indeed, $\mathfrak{S}$ specializes to the centralizer algebra $\mathcal{V}(G, \Omega)$ in the case where $\mathfrak{X} = (\Omega, 2\text{-}orb(G, \Omega))$. We call $\mathfrak{S}$ a *coherent algebra* of order n and rank r.

A hallmark of the theory is the apparent ease with which one is able to pass between the languages of relations, graphs and matrices. We cannot stress strongly enough that this mode of passage is anything but superficial. As one striking example, the intersection numbers of $\mathfrak{X}$ (see axiom (CC5)) correspond to the structure constants of the coherent algebra $\mathfrak{S}(\mathfrak{X})$. From this observation alone, one sees that the links between the combinatorial and algebraic perspectives of the theory are not only deep but inescapable.

Observe that axioms (CC1) and (CC4) together imply the existence of the unique partition $\{F_1, F_2, \ldots, F_m\}$ of Ω, namely via $\Delta = \bigcup_{i=1}^{m}\{(\alpha, \alpha) \mid \alpha \in F_i\}$. We refer to $F_1, F_2, \ldots, F_m$ as the *fibers* of $\mathfrak{X}$. For each basis relation R_i of $\mathfrak{X}$ there exists $s, t \in \{1, 2, \ldots, m\}$ for which $R_i \subseteq F_s \times F_t$. Based on this, if we consider an arbitrary subset $\Omega' \subset \Omega$ formed by the union of certain fibers of $\mathfrak{X}$ we can then consider the corresponding subset $\mathcal{R}' \subset \mathcal{R}$ defined by $\mathcal{R}' = \{R \in \mathcal{R} \mid R \subseteq \Omega' \times \Omega'\}$. This gives rise to another CC, namely $\mathfrak{X}' = (\Omega', \mathcal{R}')$, which we call the *CC induced on* Ω'.

A CC with only one fiber is called *homogeneous*. An alternate name for a homogeneous CC is an *association scheme* (briefly, *AS*). We remark that in the case of an AS it is customary to indicate the diagonal relation Δ by R_0.

An association scheme $\mathfrak{X}$ is called *symmetric* if each of its basis relations is symmetric (i.e., equal to its transpose). We call $\mathfrak{X}$ *commutative* if its corresponding coherent algebra $\mathfrak{S}(\mathfrak{X})$ is commutative. It is easy to check that a symmetric AS is commutative but not vice versa.

It is customary to refer to the coherent algebra $\mathfrak{S}(\mathfrak{X})$ of an AS $\mathfrak{X}$ as the *adjacency algebra* of $\mathfrak{X}$. In the case where $\mathfrak{X}$ is a commutative AS, the term *Bose-Mesner algebra* (or *BM algebra*) is generally applied, stemming from the seminal work [19] of Bose and Mesner.

As previously mentioned, the axioms for a coherent configuration are modeled after the special class of CC's of the form $\mathfrak{X} = (\Omega, 2\text{-}orb(G, \Omega))$, where (G, Ω) is a permutation group. Such CC's are said to be *Schurian*. In particular, $\mathfrak{X} = (\Omega, 2\text{-}orb(G, \Omega))$ is a Schurian association scheme precisely when (G, Ω) is transitive.

Not surprisingly, non-Schurian CC's comprise a special focus of modern AGT. Smallest examples exist of orders 14, 15, 16, see [142]. In Example 3 below, we exhibit a non-Schurian CC of order 16 in rather full detail.

Let $\mathfrak{S}'$ be a coherent subalgebra of the coherent algebra $\mathfrak{S}(\mathfrak{X})$. There corresponds to $\mathfrak{S}'$ a CC $\mathfrak{X}' = (\Omega, \mathcal{R}')$ in which each basis relation of $\mathfrak{X}'$ is a suitable union of basis relations of $\mathcal{X}$. Following [226] we shall refer to $\mathfrak{X}'$ as a *fusion CC* of $\mathfrak{X}$, although we shall sometimes use the term *merging* in this precise context, see [24].

Mergings play a significant role in the construction of association schemes; both Schurian and non-Schurian CC's arise in this way. In this text, we mostly consider AS's as fusions of non-homogeneous CC's, in fact in most cases the resulting fusions will be symmetric AS's.

In addition to induced CC's and fusion CC's, there is one additional general construction. Let $\mathcal{M}$ be a subset of $M^{n\times n}(\mathbb{C})$. We denote by $\langle\langle \mathcal{M} \rangle\rangle$ the smallest CC that contains $\mathcal{M}$. Clearly, such a configuration exists because the intersection of any number of CC's is again a CC. We call $\langle\langle\mathcal{M}\rangle\rangle$ the *coherent closure* of $\mathcal{M}$.

Let $\Delta = (\Omega, R)$ be an undirected graph of diameter d with adjacency matrix $A(\Delta)$. We call Δ a *distance regular graph* (briefly, DRG) provided $\langle\langle A(\Delta)\rangle\rangle$ is an AS of rank $d+1$. The case $d = 2$ is very special, as we now discuss.

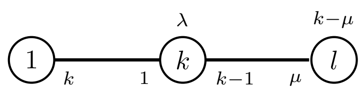
Let Δ be a regular graph of valency k and order v such that each pair of adjacent vertices has λ common neighbors, and each pair of non-adjacent vertices has μ common neighbors. Then we call Δ a *strongly regular graph* (briefly, SRG). We refer to the sequence (v, k, l, λ, μ) as the *main parameters* of Δ. (Frequently, the redundant parameter $l = v - k - 1$ is omitted.) The reader can easily verify that a connected SRG is nothing more than a DRG of diameter $d = 2$.

The complement $\overline{\Delta}$ of an SRG Δ is also an SRG, in fact its parameters $(\bar{v}, \bar{k}, \bar{l}, \bar{\lambda}, \bar{\mu})$ are readily described in terms of the parameters of Δ, namely

$$\bar{v} = v, \quad \bar{k} = l, \quad \bar{l} = v - l - 1, \quad \bar{\lambda} = v - 2 - 2k + \mu, \quad \bar{\mu} = v - 2k + \lambda.$$

We call Δ *primitive* if both Δ and $\overline{\Delta}$ are connected.

The parameters of a DRG are frequently depicted with the aid of an *intersection diagram*, e.g., see [24]. We indicate this diagram for a connected SRG as follows:



There are certain conditions that the parameters of an SRG must satisfy. We cite $k(k-1) = l\mu$ as just one example, although many others are far more sophisticated in nature and depend on spectral techniques. These conditions were investigated by contemporaries of DM, who was already employing them in his 1956 thesis [159]. We call a sequence *feasible* if it satisfies these necessary conditions.

Example 1. Consider the permutation group $(F_5^4, [0,4])$ where F_5^4 is the Frobenius group of degree 5 and order 20, and $[0,4] = \{0,1,2,3,4\}$. Using the fact that $(F_5^4, [0,4])$ is generated by the two permutations $(0,1,2,3,4)$ and $(1,2,4,3)$, we see at once that $(F_5^4, [0,4])$ may be realized as the automorphism group of the pair $\{C_5, \overline{C}_5\}$, where C_5 is the suitably labeled pentagon in Fig. 1. (Note that here automorphisms may interchange the two graphs in the unordered pair, cf. our definition of $CAut(X)$ below.)

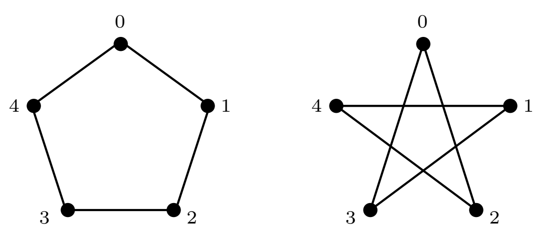


Figure 1. The pentagon and its complement

Let Ω denote the set of 2-element subsets of $[0,4]$. It then follows from 2-transitivity of $(F_5^4, [0,4])$ that (F_5^4, Ω) is a transitive permutation group. With the aid of a computer we determined the centralizer algebra $\mathcal{V} = \mathcal{V}(F_5^4, \Omega)$, from which it followed that $\mathfrak{X} = (\Omega, 2\text{-}orb(F_5^4, \Omega))$ is a rank 6 AS with "color" adjacency matrix $A = A(\mathfrak{X})$ given as follows.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 1 & 3 & 2 & 5 \\ 1 & 0 & 2 & 1 & 5 & 2 & 3 & 3 & 5 & 4 \\ 5 & 5 & 0 & 2 & 1 & 3 & 2 & 4 & 3 & 1 \\ 3 & 1 & 5 & 0 & 2 & 2 & 3 & 1 & 4 & 5 \\ 4 & 2 & 1 & 5 & 0 & 3 & 2 & 5 & 1 & 3 \\ 2 & 5 & 3 & 5 & 3 & 0 & 4 & 2 & 1 & 1 \\ 1 & 3 & 5 & 3 & 5 & 4 & 0 & 1 & 2 & 2 \\ 3 & 3 & 4 & 1 & 2 & 5 & 1 & 0 & 5 & 2 \\ 5 & 2 & 3 & 4 & 1 & 1 & 5 & 2 & 0 & 3 \\ 2 & 4 & 1 & 2 & 3 & 1 & 5 & 5 & 3 & 0 \end{bmatrix}$$

Here, matrix A delineates an arc-coloring of the complete digraph on 10 vertices with color set $\{0, 1, 2, 3, 4, 5\}$. The reader will further observe that $A(\mathfrak{X}) = \sum_{i=0}^{5} iA_i$ where A_i is the usual adjacency matrix of the basis graph (Ω, R_i), $0 \leq i \leq 5$.

In fact, $\mathfrak{X}$ is one of the smallest nontrivial examples of a non-commutative AS. To see this, simply observe that there is a 2-path from vertex 0 to vertex 2 along arcs colored 1 then 2, but no such path exists along arcs colored 2 then 1.

There are two primitive fusions of $\mathfrak{X}$ corresponding to the following mergings of basis relations: $\mathcal{R}' = \{R_0, R_1 \cup R_4, R_2 \cup R_3 \cup R_5\}$ and $\mathcal{R}'' = \{R_0, R_3 \cup R_4, R_1 \cup R_2 \cup R_5\}$. The resulting schemes are isomorphic; graphs $(\Omega, R_1 \cup R_4)$ and $(\Omega, R_3 \cup R_4)$ both yield copies of the Petersen graph, as shown in Fig. 2.

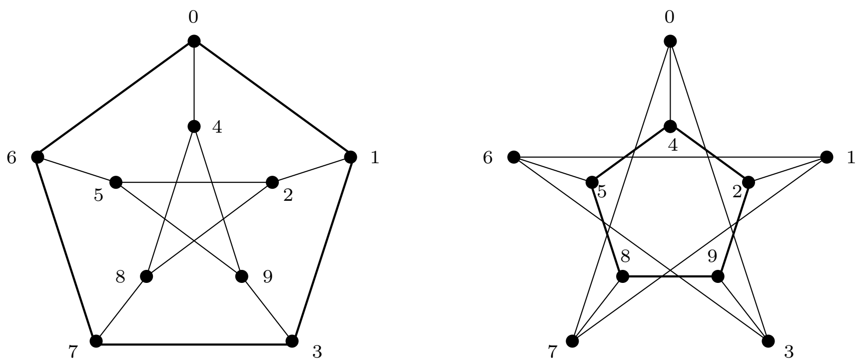


Figure 2. Two copies of the Petersen graph via merged relations

The point of Example 1 is to convince the reader that even for a relatively simple AS on 10 points, manipulation of computer data is a far from trivial task. Indeed, human ingenuity and intermediation are key to the process.

To each CC $\mathfrak{X} = (\Omega, \mathcal{R})$ we may associate three groups.

1. The (usual) *automorphism group* $Aut(\mathfrak{X}) = \bigcap_{i=1}^{r} Aut(\Gamma_i)$ is that subgroup of $S(\Omega)$ which preserves each color graph $\Gamma_i = (\Omega, R_i)$, $1 \leq i \leq r$.
2. The *color automorphism group* $CAut(\mathfrak{X})$ is a less restrictive subgroup of $S(\Omega)$ in the sense that it allows colors to be permuted in a uniform manner:

$$CAut(\mathfrak{X}) = \{g \in S(\Omega) \mid R_i^g \in \mathcal{R} \text{ for all } R_i \in \mathcal{R}\}.$$

3. The *algebraic automorphism group* $AAut(\mathfrak{X})$ preserves the tensor of structure constants of $\mathfrak{X}$:
$$AAut(\mathfrak{X}) = \{\sigma \in S(\{1,2,\ldots,r\}) \mid p_{i^\sigma j^\sigma}^{k^\sigma} = p_{ij}^k\}.$$
It is the only type of automorphism of $\mathfrak{X}$ that need not arise from a permutation of the set Ω.

It is routine to verify that $Aut(\mathfrak{X})$ is a normal subgroup of $CAut(\mathfrak{X})$, and that the quotient group $CAut(\mathfrak{X})/Aut(\mathfrak{X})$ embeds in $AAut(\mathfrak{X})$. Groups $Aut(\mathfrak{X})$ and $CAut(\mathfrak{X})$ are very helpful in cases in which $\mathfrak{X}$ has a limited number of fusion schemes.

Let $\Delta = (\Omega, E)$ be an (undirected) graph, and let $V = \{\Omega_1, \Omega_2, \ldots, \Omega_s\}$ be a partition of its vertex set Ω. For every $i, j \in \{1,2,\ldots,s\}$ let $e_{ij} = e_{ij}(x)$ denote the number of neighbors in Ω_j of a fixed $x \in \Omega_i$. We call V an *equitable partition* (briefly, *EP*) of Δ if e_{ij} does not depend on the particular vertex $x \in \Omega_i$ chosen.

Observe that for each subgroup $H \leq Aut(\Delta)$, the orbits of (H, Ω) form an EP. We refer to such an EP as *automorphic*. Clearly, non-automorphic EP's are a special source of interest.

To each EP there corresponds a collapsed $s \times s$ matrix $E = (e_{ij})$. Note that if Δ is regular of valency k, then each row sum of E is equal to k.

Example 2. It is well known that the Petersen graph is hypohamiltonian, see Fig. 3(a). Consider the partition of its vertices into two parts (the 9 black vertices comprising the outer cycle, and single white vertex at its center). This partition is not an EP simply because the subgraph induced on the vertices of the 9-cycle is not regular.

In Fig. 3(b), we depict a partition $\pi = \{\text{black}, \text{white}\}$ of the vertex set V of the octahedron. Here π is an EP with collapsed matrix E given by
$$E = \begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}.$$
In fact π is automorphic, as its cells are orbits of the permutation group $(\mathbb{Z}_2 \times D_4, V)$.

Finally, in Fig. 3(c) we depict the partition $\pi = \{\text{black}, \text{white}\}$ of the cuneane graph (i.e., graph of the cuneane molecule, see [213] for details). The partition π is again an EP with collapsed matrix
$$E = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$
However in this case π is not automorphic. Indeed, there are two orbits of white vertices under the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is the full automorphism group of the cuneane graph.

Example 3. Here we provide a classical example of a non-Schurian AS. It is generated by the Shrikhande graph Sh of order 16, one of two SRG's with parameters $(16, 6, 2, 2)$.

There are a few well known constructions of this graph, for example as the complement of a Latin square graph over $\mathbb{Z}_4$, e.g., see [96]. However, we shall deliberately approach its construction in a more sophisticated way, starting from a CC with two fibers of size 4 and 12. A candidate for the collapsed matrix here is
$$E = \begin{bmatrix} 0 & 6 \\ 2 & 4 \end{bmatrix},$$
as it satisfies some simple necessary combinatorial conditions as well as one strong spectral condition, see [83, Theorem 9.3.3]. The $(1,1)$-entry of E indicates the existence of a coclique of size 4 in Sh. Hence if the resulting EP is to be automorphic one would

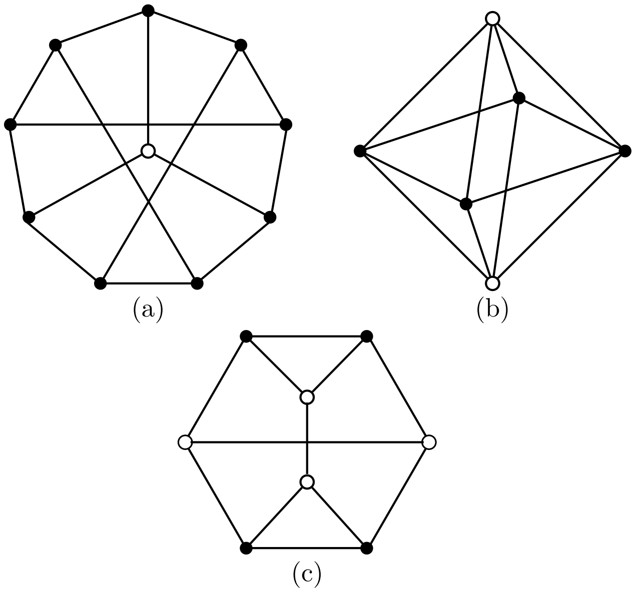


Figure 3. Vertex partitions (black and white vertices) for three graphs

expect to encounter a natural action of the group S_4 on the cell of size 4. Of the two faithful transitive degree 12 actions of S_4, we chose the more natural one: the set $\binom{X}{2}$ of all ordered pairs of distinct elements of $X = \{0, 1, 2, 3\}$.

Using the computer package COCO we constructed the centralizer algebra $\mathcal{V} = \mathcal{V}(S_4, \Omega)$ where $\Omega = X \cup \binom{X}{2}$. The CC $\mathfrak{X}$ corresponding to $\mathcal{V}$ turned out to be of rank 15, comprised of two reflexive, five symmetric, and four pair of antisymmetric 2-orbits of (S_4, Ω). COCO subsequently returned 18 nontrivial fusion schemes of $\mathfrak{X}$, the most interesting of which were five mergings corresponding to SRG's with parameters $(16, 6, 2, 2)$. Of these, three had automorphism groups of order 1152, while the remaining two had automorphism groups of order 192. One of these latter two, "`Merging #17`" (the name so assigned by COCO), is explained in detail below. As usual, different letters are meant to indicate different elements of X.

One basis graph $\Delta = (\Omega, R)$ comes by way of merging $R = R_3 \cup R_7 \cup R_{12} \cup R_{13}$, where $R_3 = \{a, (b, c)\}$, $R_7 = \{(b, c), a\}$, $R_{12} = \{(a, b), (a, c)\}$, $R_{13} = \{(b, c), (c, a)\}$. Clearly R_{12} and R_{13} are symmetric relations, while $R_3 = R_7^T$. It is fairly straightforward to see that Δ is a regular graph of valency 6 with collapsed matrix E. In fact, Δ is the Shrikhande graph Sh, see Fig. 4.

Our next step is to understand the structure of $Aut(Sh)$. At the moment, all that we know for certain is that $S_4 \leq Aut(Sh)$. However one can show that the stabilizer $Aut(Sh)_{(0,1)}$ of vertex $(0, 1)$ is the dihedral group D_6 of order 12. Indeed, using Fig. 5 as a partial aid, the reader is encouraged to verify this directly by showing that $Aut(Sh)_{(0,1)}$ is generated by the following two involutions s, t, whose product st has order 6:

$$s = \big(2,3\big)\big((0,2),(0,3)\big)\big((2,1),(3,1)\big)\big((2,0),(3,0)\big)\big((1,2),(1,3)\big)\big((2,3),(3,2)\big),$$

$$t = \big(3,(0,3)\big)\big(2,(2,1)\big)\big(0,(3,0)\big)\big(1,(1,2)\big)\big((2,3),(1,0)\big)\big((2,0),(1,3)\big).$$

From this one concludes that $Aut(Sh)$ is a transitive group of degree 16 and order $16 \cdot 12 = 192$.

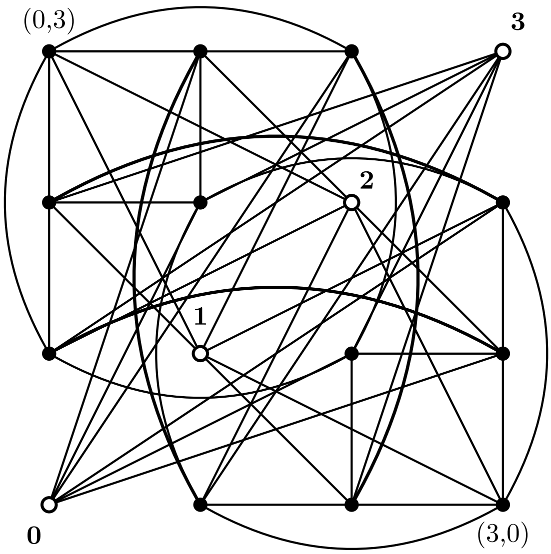


Figure 4. The Shrikhande graph Sh. Points along the main diagonal (white dots) correspond to elements of $X = \{0, 1, 2, 3\}$ while all remaining points correspond to elements of $\binom{X}{2}$.

While $Aut(Sh)$ acts transitively on edges, its action on non-edges is intransitive. This can be seen by considering the 7-vertex subgraph Δ' of Sh induced on $(0, 1)$ and its neighbor set, as depicted in Fig. 5. Transitivity on non-edges would require that $Aut(\Delta') = Aut(Sh)_{(0,1)}$ act transitively on the non-neighbors of $(0, 1)$ which is clearly not the case. In fact, $Aut(Sh)$ is a rank 4 group with subdegrees $1, 6, 6, 3$. This may be verified by establishing that $(0, 1)$, 2, 0, $(2, 3)$ are distinct orbit representatives under the action of $Aut(\Delta') \cong D_6$.

At the next stage we wish to show, without the aid of a computer, that the graph Sh is indeed an SRG. For this purpose it suffices to count the number of 2-paths from $(0, 1)$ to 2 (since $\{(0, 1), 2\}$ represents the single orbit of edges in Sh), as well as the number of 2-paths from $(0, 1)$ to each of 0 and $(2, 3)$ (since $\{(0, 1), 0\}$ and $\{(0, 1), (2, 3)\}$ represent the two orbits of non-edges in Sh). In all cases this number is 2, whence it is confirmed that Sh is an SRG with parameters $(16, 6, 2, 2)$. We conclude that `Merging #17` is a non-Schurian AS with two classes, Sh and $\overline{Sh}$, of valencies 6 and 9.

We end our example with an observation on the automorphism groups of the initial rank 15 CC $\mathfrak{X}$ with two fibers of size 4 and 12. Using COCO in conjunction with GAP, we were able to confirm that $AAut(\mathfrak{X}) \cong \mathbb{Z}_2 \cong CAut(\mathfrak{X})/Aut(\mathfrak{X})$, and that a generating involution of $AAut(\mathfrak{X})$ interchanges four pairs of the 18 fusion schemes of $\mathfrak{X}$, one pair of which consists of two rank 3 mergings with group of order 1152. (Later we shall compare this information with the situation arising in DM's construction of $NL_2(10)$.)

It is interesting to compare the fusion scheme `Merging #17` of this example with another fusion scheme of $\mathfrak{X}$, assigned the name `Merging #14` by COCO. This latter merging is a rank 3 AS generated by the lattice square graph $L_2(4)$. Like Sh, this graph also has a natural "grid-like" construction, namely here the neighbors of a vertex (a, b) in $L_2(4)$ are precisely those vertices that lie either in the same row or column as (a, b) (hence of the form (a, y) where $y \neq b$, or (x, b) where $x \neq a$). The key observation here is that Sh may be obtained from $L_2(4)$ by the process of "switching" with respect to a

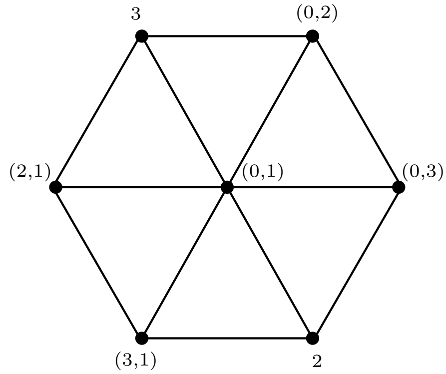


Figure 5. Subgraph Δ' of the Shrikhande graph, induced on $(0,1)$ and its neighbors

4-vertex coclique in $L_2(4)$. This procedure goes back to J. J. Seidel, e.g., see [26].

Finally, we wish to pay credit to S. S. Shrikhande [196], the discoverer of this remarkable graph.

For more details regarding the methodology of CC's and AS's, we refer the reader to the texts [24, 41, 83] and to our papers [73, 134, 135, 136, 142]. The latter two papers provide information about helpful computer tools for research in AGT, particularly the packages COCO, GAP, GRAPE, and nauty. These tools were exploited in the current paper in both visible and hidden form.

Details on the coherent closure of a set of matrices, including a very efficient algorithm for computing it, may be found in the seminal paper [219] of B. Ju. Weisfeiler and A. A. Leman. A reasonably elementary treatment of these ideas, plus an historical review, appears in [138].

Lastly, we mention that in various places in the text we employ a number of different kinds of incidence structures without further explanation. Among these are (partially) balanced incomplete block designs, biplanes, Steiner systems, etc. The texts [11, 109] should provide the reader with sufficient background material regarding these structures.

We also wish to remark that the ingredients of the language of AGT introduced in this section will not be fully exploited in our forthcoming exposition. Nevertheless, the reader who is familiar with these concepts will definitely benefit from his/her wider proficiency, being able to better comprehend many of the discussed links between modern state-of-the-art AGT and its initial seeds from the latter half of the XX-th Century.

3 Overview

An SRG Γ with the parameters $(100, 22, 0, 6)$ was first described in the 1956 thesis [159] of Dale M. Mesner in terms of its adjacency matrix A_1. This matrix was presented in block form with two of its blocks C_1 and C_1^T interpreted as incidence matrices of an auxiliary BIBD and its dual design. From here, properties of such a BIBD were postulated. DM proved that there was at least one such design, thereby establishing existence of Γ. Uniqueness of Γ would have followed from an examination of the three remaining putative designs mentioned by DM, however this was not persued.

This graph was later considered by DM in [160], where the initial arguments from [159] were presented in a more transparent and rigorous form. The notion of a negative Latin square association scheme with two classes, already coined in [159], was developed here to its full extent. The considered schemes were denoted $NL_g(n)$ with g, n free parameters. Special attention was paid to the case $p^1_{11} = 0$ (whereby $n = g^2 + 3g$), which led DM to feasible parameters of a putative infinite family of such schemes, denoted $NL_g(g^2 + 3g)$. For each given $g \geq 1$, existence of said scheme on n^2 vertices (and so, of the corresponding SRG) was reduced to the existence of a BIBD $\mathcal{C}$ on $\widehat{v} = g(g^2 + 3g + 1)$ points satisfying the properties earlier postulated by DM. For $g = 1$ uniqueness of $NL_1(4)$ was proved as an immediate illustration of DM's methods.

In contrast to [159], DM's investigation of the case $g = 2$ in [160] was more complete. Here design $\mathcal{C}$ was shown to be unique by analyzing the block decomposition of its putative incidence matrix C_1 (recall, a submatrix of A_1) and investigating certain auxiliary structures naturally appearing in this context.

In comparison to [160], which took the form of mimeographed notes, DM's later paper [162] enjoyed a quite wider distribution. Covering just a portion of the material from [160], it contains only a brief mention of the existence of $\Gamma = NL_2(10)$, and it does so without any supporting evidence or explanation.

Over the next few years, the results from [160, 162] would become known to only a narrow community of experts in design of experiments. Although this knowledge would stimulate further clarification and development of DM's approach, it is unfortunate that for many decades DM's results would remain unknown to virtually all experts in group theory and finite geometries. Indeed, during this period the community of experts specializing in design of statistical experiments was relatively isolated from the main body of mathematics, and the links that are fairly commonplace nowadays simply did not exist at that time.

The second appearance of the graph Γ is well known. It was discovered independently in 1967 by D. G. Higman and C. C. Sims. The result became known almost immediately, quite before its formal appearance in [101], and strongly influenced further investigation of Γ and $Aut(\Gamma)$. No doubt, interest in $Aut(\Gamma)$ stemmed from the fact that it contained a new sporadic simple group, nowadays denoted HS in honor of its discoverers. In particular, new proofs of the uniqueness of Γ and diverse characterizations of HS would be accomplished in the next few years by several authors.

The results in [101] were obtained by a clever combination of combinatorial arguments with elementary group theoretic observations. A construction of Γ was there given, based on the existence of an auxiliary design previously considered by E. Witt in [224]. (In subsequent papers one finds a proof of the uniqueness of Γ.) Although one now recognizes this design to be isomorphic to the design $\mathcal{C}$ of DM, its more common realization is the Steiner system $S(3, 6, 22)$ (also denoted W_{22}).

The graph Γ contains another SRG with parameters $(77, 16, 0, 4)$, which was also presented by DM. It follows from his construction that such an SRG is unique up to isomorphism. Again, this discovery was an immediate consequence of [101]. The pioneering paper here was due to by A. Gewirtz [80], establishing uniqueness not only of Γ but of several of its substructures.

More careful analysis soon revealed the 3-design $\mathcal{C}$ to be quasi-symmetric (i.e., having two allowable cardinalities for the intersection of distinct blocks). Moreover, it became clear that a quasi-symmetric design with suitable parameters yields an SRG by natural extension [86].

Very quickly, results about the Higman-Sims graph and its automorphism group inspired an explosion of fruitful activity in group theory and the newly developing AGT. Thus the main ideas of DM's discovery became known and influential to two generations of mathematicians, though without any attribution to his early pioneering work.

This concludes a summary of the main content of our paper. A comprehensive treatment begins in the next section.

4 Ph.D. thesis of Dale Mesner: a brief outline

The thesis of DM consists of 291+ix pages, is signed by adviser Leo Katz, with acknowledgments to the adviser and W. S. Connor, Jr.

Chapter I discusses general properties of partially balanced incomplete block designs (PBIBDs) and association schemes.

Chapter II provides a nice comprehensive discussion of the properties of 2-class association schemes and corresponding PBIBDs. In particular, DM derives results (with credits to [51]) on feasible spectral conditions for the existence of such schemes. (Later on a similar result appears independently as Lemma 6 in [98].)

In Chapter III, DM introduces the new notion of a "negative Latin square type" scheme. He establishes elements for a general theory of such schemes, and provides infinite series of examples with the aid of finite fields. This includes implicit consideration of the concept of a dual strongly regular graph (cf. Section 2.6.3 in [62]).[1] The content of Section 3.3 is discussed in more detail below.

Chapter IV is devoted to the investigation of Latin square type schemes, with special attention to the uniqueness of such schemes with two classes. The results are surprisingly strong, a real precursor to the more general approach developed later in [196] and [30]. Note that DM is not aware at this time of [29].

Chapter V is a summary of obtained results.

A comprehensive appendix consists of five parts, occupies more than 50 pages and provides a lot of interesting numerical data, in particular tables of parameter values for association schemes of small size.

The bibliography consists of 40 items.

Section 3.3 of the thesis is central to our presentation. In Section 3.1, DM suggests to consider graphs of negative Latin square type, denoted by him as $L_g^*(n)$. These are SRGs with the parameters $v = n^2$, $k = g(n+1)$, $l = (n+1-g)(n+1)$, $\lambda = (g+1)(g+2)-n-2$, $\mu = g(g+1)$. For n a prime-power, some families of such graphs are constructed in Section 3.2 using the notion of a Singer cycle [201].

In Section 3.3, DM faces the question of constructing an SRG of $L_2^*(10)$-type, designated #94 in his Table II ('Supplement' to thesis). He considers the adjacency matrix A_1 of a putative such graph, which by virtue of his Theorem 2.6 may be presented in the form

$$A_1 = \begin{pmatrix} 0 & 1\cdots 1 & 0\cdots 0 \\ 1 & & \\ \vdots & 0 & C_1 \\ 1 & & \\ 0 & & \\ \vdots & C_1^T & T \\ 0 & & \end{pmatrix}.$$

Here C_1 is the incidence matrix of a BIBD with the parameters $v = 22$, $r = 21$, $k = 6$, $b = 77$, $\lambda = 5$. Let us denote this BIBD as $\mathcal{C}$. Using simple combinatorial arguments in conjunction with variance counting, DM immediately concludes that each block of $\mathcal{C}$ is disjoint from 16 other blocks, and has exactly 2 common elements with any of the

[1] Of course, the term *strongly regular graph* does not appear explicitly in DM's thesis. It would later be coined in [18].

remaining 60 blocks. (In modern terminology, DM has proved that $\mathcal{C}$ is a quasi-symmetric BIBD with the intersection numbers $x = 0$, $y = 2$.)

Using his Lemmas 2.1 and 2.2, DM concludes that matrix T is the adjacency matrix of an SRG Γ_1, designated #64 in his Table II, which has the parameters $v = 77$, $k = 16$, $l = 60$, $\lambda = 0$, $\mu = 4$. It is implicit from DM's arguments that Γ_1 may be regarded as the complement of the block graph of the design $\mathcal{C}$. Together with the existence of $\mathcal{C}$, existence of graph #64 is another necessary condition for the existence of graph #94. For evident reasons, we denote this latter graph by Γ.

At the next stage the goal is to prove that the existence of $\mathcal{C}$ is sufficient, as well as necessary, to establish existence of the graph Γ. A series of brilliant *ad hoc* arguments and computations on pp. 134-137 allow DM to arrive at this conclusion.

The problem is now reduced to the construction of the design $\mathcal{C}$. (Of course, DM at this time is not aware of the existence of the Witt design on 22 points.) Without the aid of a computer, DM bravely attacks this problem. He claims in advance that he will show the existence of at most four possible solutions for $\mathcal{C}$, and that he will use the first solution encountered, neglecting consideration of other possibilities. Thus he starts his "systematic trials of possible solutions" (a very clever rigorous backtracking procedure in modern terms). The protocol of this computational procedure occupies pp. 137-145, while on p. 146 the result is presented: an explicit listing of the 77 blocks that make up his BIBD $\mathcal{C}$. Concluding his construction on p. 147, DM stresses the fact that he has arrived to at least one solution for the graph Γ, not concerning himself with other solutions. In DM's words: "It is not known whether any of the four solutions are equivalent under some permutation of treatments."

In actuality, a proof of the uniqueness of Γ is nearly achieved in the course of DM's presentation in [159]. He further pays consideration to the graph Γ_1, and to some other interesting combinatorial byproducts which still today are awaiting a proper interpretation.

5 Mimeographed notes of Dale Mesner: 1964 and beyond

Our main interest in this section focuses upon a comprehensive set of notes [160] published in the Mimeo Series of the Institute of Statistics of UNC-Chapel Hill. Many hundreds of texts published in this series form a scientific treasure, however they were not available to a wide audience for a very long while. All our attempts to gain access to [160] proved futile until we learned from Earl Kramer in February 2010 of the possibility of online access to most texts from this series.

Surprisingly, we were not able to secure a copy of [160] from Dale himself. He mentioned that he had lost his own personal copy, and that regardless, in Dale's own words, "it doesn't contain anything new." According to Dale, everything in [160] could be recovered from [159] in conjunction with [162].

To the modern reader the picture is quite the opposite: DM's mimeographed notes immediately became a personal source of great inspiration to us. We liken our impressions upon examining it to the emotions of an archeologist who has just uncovered a rare and priceless artifact.

The entire text [160] consists of 11 sections (100 pages + cover page) and is dated November 1964. Acknowledgment is paid to the NSF for support, as well as to Purdue University for the use of their computer facilities. No doubt, the latter figured prominently in the construction of DM's many tables.

Section 1 of [160] contains a brief general outline of designs and association schemes. Already by Section 2, we find ourselves in new territory, being exposed to the notion of an

NL-square design. The presented graphs[2] $NL_2(8)$, $NL_2(9)$, $NL_3(9)$ and $NL_2(10)$ are all new. Sections 3-4 introduce elements of a general theory of NL-graphs, while Section 5 gives some finite field constructions. Some more advanced geometric constructions form the content of Sections 6-7. All of these sections merit close examination, especially as a means of comparison to what are, nowadays, known results.

The real surprise, however, occurs in Section 8. Specifically, this is an introduction to the new theory of NL_g-graphs with $\lambda = 0$. In particular, Theorems 8.6 and 8.7 establish the existence and uniqueness of $\Gamma = NL_2(10)$.

Section 9 provides a proper treatment of PBIB designs, and is replete with many tables of feasible parameters. More or less, Section 10 coincides with Sec. 4 of [162]. Section 11 contains acknowledgments, followed by a list of 25 references including one to the 1963 notes of R. C. Bose from the same Chapel Hill Mimeo Series.[3]

DM's mimeographed notes comprise just one of three publications that grew out of his Ph.D. thesis, the other two being [161, 162] each of which was also submitted in 1964.

The short note [161] was received by the editors on March 16, 1964 and revised on September 10, 1964. Although DM is still not using the term graph, the subject of [161] is the investigation of necessary conditions for the existence of an SRG with given parameters in conjunction with conditions for the existence of a PBIB design. Here DM follows closely the spirit of [20, 21]. In particular, he investigates a few concrete putative families of parameters (citing [51] as an initial source of information) and coins the term "pseudo-cyclic" for one such family (with credit for its suggested usage to R. H. Bruck). Ultimately, DM proves that any SRG with a prime number of vertices must be pseudo-cyclic.

A number of diverse numerical conditions (some are new, one is attributed to J. S. Frame) are presented and exploited in [161] for consideration of particular parameter sets. As a simple exercise, one may readily obtain the result in [105] on putative parameters of Moore SRGs.

The more comprehensive article [162] (received July 20, 1964, revised August 12, 1966) is the most known and frequently cited of DM's three 1964 submissions. This is the text in which L^* is officially replaced by NL as a designation for schemes of negative Latin square type.

The main body of [162] deals with concrete methods for describing SRGs and BIBDs of NL-type based on the use of geometries over finite fields (in some cases just the field itself suffices). In this text, DM introduces the notation $NL_g(n)$ to refer to an SRG with the parameters $v = n^2$, $k = g(n+1)$, $\lambda = (g+1)(g+2) - n - 2$, $\mu = g(g+1)$.

At the close of Section 1 in [162] the author writes: "Methods to be presented in later papers give solutions for some of the foregoing as well as for $NL_2(9)$ and $NL_2(10)$. The schemes $NL_2(6)$, $NL_2(7)$, $NL_3(10)$ and $NL_4(10)$ are still unknown."

There is but one way to decode DM's message about $NL_2(10)$ above:

An SRG with the parameters $(100, 22, 0, 6)$ *does exist!*

It is bewildering that generations of experts (including the authors) were unaware of this accomplishment of DM until its formal disclosure in [117].

Remark 4. (a) We provide the current status of the schemes mentioned in DM's quote: Nonexistence of $NL_2(7)$ was proved in 1989 [34], existence of $NL_4(10)$ was proved in

[2] Here we are taking the liberty of extending DM's original notation and terminology for schemes (also used by him for designs) to apply as well to the corresponding SRGs. We shall uphold this convention in what follows.

[3] Despite much concerted effort we were unable to locate these notes of Bose. At present we have some doubts as to their actual existence.

2003 [123], existence of $NL_3(10)$ remains an open problem, and there are many graphs of type $NL_2(6)$, e.g., see [206, 133].

(b) In Section 4 of [162], DM gives feasible parameters for two families of association schemes. In modern terminology, such schemes are referred to as "amorphic", cf. [60].

6 The Clebsch graph: The DM-approach in miniature

By the *Clebsch graph* we refer to the graph $NL_1(4)$ in DM's notation. The name was coined by J. J. Seidel in [191], though more accurately he applied it to the complementary graph. Seidel in turn refers to Coxeter [56], who points out the relation of the corresponding polytope to the 16 lines on the Clebsch quartic surface [47].

An alternate notation for this graph is $\square_5$, reflecting its membership to an infinite series $\square_n$ of folded n-cubes, e.g., see [24]. In fact, $\square_5$ is an SRG with the parameters $(16, 5, 0, 2)$. It was rediscovered a few times in diverse contexts, see [45, 88, 124]. Our own vision of this graph is reflected in [137, 135]; in particular Fig. 6 is basically borrowed from [137]. Also see [83] for a proof of existence and uniqueness.

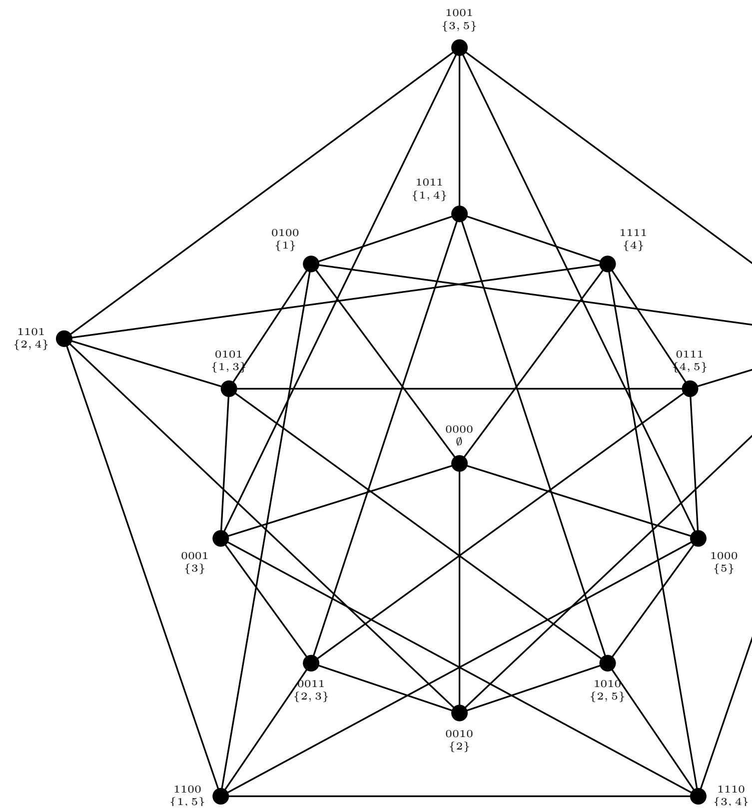


Figure 6. Clebsch graph

The graph $NL_1(4)$ was already a striking example in the thesis of DM [159]. On pp. 102–105 one finds a detailed construction which uses the finite field $GF(16)$ in the spirit of [201]. At this stage DM believed the example to be new, however in the text [160] he attributes it to Clatworthy [45]. In fact the graph $NL_1(4)$ plays a crucial role in [160], where it is constructed no fewer than four times, each time elaborating a different method for procuring a negative Latin square graph. Below we describe these four constructions in brief (Models 1–4), although we make no attempt to dogmatically preserve the original notation and terminology.

Model 1 ([160, Sec. 4]). Theorems 4.1 and 4.8 justify how to construct a 2-class association scheme from a suitable abelian group H and its partition into three sets: $S_0 = \{0\}$, S_1, S_2. DM introduces $NL_1(4)$ as an example on pp. 25-26; specifically, he works with the vector space $V = GF(2)^4$ (binary strings of length 4) and defines $S_1 = \{0001, 0010, 0100, 1000, 1111\}$. The graph $\square_5$ appears as the Cayley graph over V with connection set S_1 (see the binary labels in Fig. 6). In modern terminology, what DM has accomplished is a "merging of classes" in the Hamming scheme $H(4,2)$. His work in this section also touches upon elementary concepts in S-ring theory.

Model 2 ([160, Sec. 5]). Here $NL_1(4)$ appears in the guise of a cyclotomic association scheme (see [24, Sec. 2.10]). The presentation is similar to the one in [159], though more compact. On pp. 31-32 the field $GF(16)$ is described in such a way that the set S_1 of Model 1 appears as the subgroup of order 5 in the multiplicative group of $GF(16)$.

Model 3 ([160, Sec. 7]). In his previous Section 6, DM works with the k-dimensional Euclidean geometry over the field $GF(n)$ (so here n is a prime-power), in particular with the Desarguesian affine plane $EG(2,n)$ and the projective plane $PG(2,n)$. He explains how to derive association schemes from non-degenerate conics in $PG(2,n)$. In DM's presentation, he gives credit to B. Segre, R. C. Bose and R. H. Bruck. These techniques are more deeply exploited in Sec. 7, where DM starts by discussing a construction by D. K. Ray-Chaudhuri on $v = 2^{3t}$ points [177]. He asserts that this construction can never produce NL_g-graphs, and subsequently generalizes it to one that leads to infinitely many such graphs, as well as other association schemes. In his typical modest style, DM writes, "This generalization seems to have gone unnoticed until now."

Finally, $NL_1(4)$ appears in this context as one of a few illustrations of DM's developed techniques, together with such new objects as $NL_3(8)$ and $NL_2(9)$. Not aiming to provide precise formulations, we refer to [24, Sec. 9.5.C], where $NL_1(4)$ appears as the Hermitian forms graph over $GF(16)$. This latter family of graphs, along with many other classical families, was systematically considered in the early 1980s in the works of E. Bannai, A. M. Cohen and D. Stanton, see [9] for further details and references.

A detailed account of DM's new method refers to [178, 18] for necessary background. In fact, it is exactly this portion of [160] that is presented in [162] in a more rigorous manner. The latter article, which is highly readable and accessible to a wide audience, greatly influenced further development in AGT. Clearly, such implicit and explicit traces of DM's influence deserve a renewed attention.

Model 4 ([160, Sec. 8]). This model appears on p. 72 of [160] as a degenerate case of the introduced family $NL_g(g^2+3g)$, see our Section 7 below. Here DM writes, "Design $\mathcal{C}$ is trivial in the case $g = 1$, giving a fourth method of construction of the $NL_1(4)$ scheme. This construction gives an easy proof of the uniqueness of the scheme."

In our presentation we intentionally start with this simplest case, hoping to create a useful visual image for the reader.

Consider the vertex set $V = S_0 \cup S_1 \cup S_2$, where $S_0 = \emptyset$, $S_1 = [1,5]$, and $S_2 = \{{[1,5] \atop 2}\}$. (Here we adopt the notations $[i,j] = \{i, i+1, \ldots, j\}$ and $\{{S \atop k}\} = \{T \subseteq S \mid |T| = k\}$). Clearly, the pair $\mathcal{C} = (S_1, S_2)$ defines the trivial 2-$(5,2,1)$ design. We now define the graph $NL_1(4)$ with vertex set V, however we use the terminology of designs referring to vertices in S_1 as points and to those in S_2 as blocks. The vertex $\emptyset$ (called "initial vertex" by DM) is adjacent to all points of $\mathcal{C}$, each point is adjacent to those blocks of $\mathcal{C}$ which contain it, and two blocks are adjacent if and only if they have empty intersection.

Model 4 is also depicted in Fig. 6 if one focuses this time on the vertex labels given by k-element subsets of $[1, 5]$ with $k \leq 2$. This of course gives a visual isomorphism between Models 1 and 4. However, there is another isomorphism lurking about which we feel is more natural and esthetic. Namely, Let T be a k-element subset of $[1, 5]$, $k \leq 2$, and let $T' = T \cap [1, 4]$. Let $\xi_{T'}$ be the characteristic vector of T', and $\overline{\xi}_{T'}$ its bit complement. We define $\theta : T \mapsto \xi_{T'}$ if $5 \notin T$ and $\theta : T \mapsto \overline{\xi}_{T'}$ if $5 \in T$. It is straightforward to check that θ is indeed an isomorphism between the two models.

The uniqueness of $NL_1(4)$ as an SRG with the parameters $(16, 5, 0, 2)$ follows immediately from elements of DM-theory, see Section 8.

Although DM didn't at all explore the structure of the automorphism group $G = Aut(NL_1(4))$, such information may be readily obtained from his constructions. In particular, three subgroups of G are quite visible from Models 1, 2 and 4. We denote by H_i the subgroup arising naturally from consideration of Model i.

From Model 1, we observe that G contains the automorphism group $H_1 = Aut(Q_4)$ of the 4-dimensional cube Q_4. Group H_1 is the exponentiation $S_2 \uparrow S_4$ (see [131] for details). In particular, it is easy to see that G acts transitively on V.

From Model 2, we easily identify $H_2 \cong E_{16} \rtimes \mathbb{Z}_5$ as a subgroup of G.

Model 4 clearly depicts $H_4 \cong S_5$ as the stabilizer of the vertex $\emptyset$. Indeed, the graph $\Gamma_2(\emptyset)$ (that is, the subgraph of Γ induced on the vertices at distance 2 from $\emptyset$) is the famous Petersen graph which has automorphism group S_5.

Having already established that Models 1 and 4 are isomorphic, we may now deduce $|G| = |V| \cdot |H_4| = 1920$, which gives our desired structure $G \cong E_{16} \rtimes S_5$. (Alternatively, one may deduce $|G| = \frac{1}{2}|Aut(Q_5)| = 1920$ directly from $Aut(Q_5) \cong E_{32} \rtimes S_5$ and the fact that $\square_5$ is the folded 5-cube.)

Last but not least, we wish to elaborate one more model which, although not explicitly appearing in [160], still has visible traces to DM's work.

Model 5 ("Non-edge model"). We wish to start from a non-edge, so let a, b be two nonadjacent vertices. Now let S_{ij} denote the set of vertices simultaneously at distance i from vertex a and distance j from vertex b, $1 \leq i, j \leq 2$. We now define our vertex set as $V = \{a, b\} \cup S_{11} \cup (S_{12} \cup S_{21}) \cup S_{22}$. Taking into account that our objective is to construct a $(16, 5, 0, 2)$-SRG, the cardinalities $|S_{ij}|$ are uniquely determined: $|S_{11}| = \mu = 2$, $|S_{12}| = |S_{21}| = k - \mu = 3$, $|S_{22}| = v - 2k + \mu - 2 = 6$. (Note that in our model S_{12} and S_{21} are merged together.) Moreover, one may deduce the compact intersection diagram of our graph Γ which is depicted in Fig. 7. We again refer to [24] for a precise discussion of such types of diagrams.

One easily identifies subgraphs of Γ induced on S_{22} and $S_{12} \cup S_{21}$ as 6-cycles; the remaining edges of Γ are also quite evident. Model 5 thus makes visible a subgroup H_5 of G, namely the stabilizer of our non-edge $\{a, b\}$. Clearly H_5 can only preserve the mentioned 6-cycles, and its action on one determines its action on both. Likewise, the involution interchanging a with b (the "ends" of our non-edge) preserves each of these two 6-cycles. Thus $H_5 \cong S_2 \times D_6$, where D_6 is the automorphism group of a 6-cycle (i.e., the dihedral group of order 12).

Finally, we obtain that $G = \langle H_4, H_5 \rangle$ is an amalgam of groups of order 120 and 24 intersecting in a group $S_2 \times S_3 \cong D_6$ of order 12. This stresses the significance of the extra involution in G which interchanges a and b. Thus Model 5, together with borrowed group theoretic information from Model 4, gives an independent proof that G acts transitively on V.

There is one more "amalgam" we wish to discuss but it is not one of mathematical formalism. It is the amalgam of two ingenious approaches which greatly shaped the

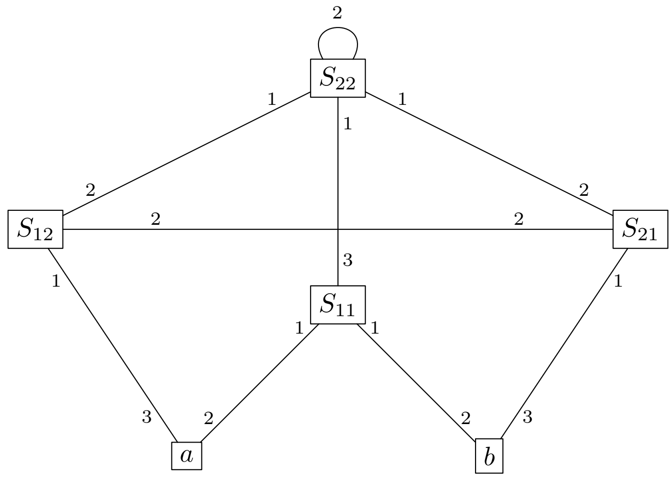


Figure 7. Intersection diagram of the Clebsch graph

future landscape of AGT, namely the approaches of DM and Higman & Sims. This will be accomplished in Section 13.

7 Graphs $NL_g(g^2 + 3g)$: A putative DM-series

We start now a proper consideration of one of the greatest scientific achievements of DM. As usual, our goal is to preserve the spirit of the text if not the literal word. We follow DM's more detailed presentation in [160, Sec. 8, pp. 58-72], though we point to [159] as a precursory source.

Stated in modern terminology, the problem is to describe all possible parameter sets of triangle-free SRGs (that is, SRGs with $\lambda = 0$) that are simultaneously NL-graphs. A nice introduction to this branch of AGT may be found in Sec. 8 of [42].

Aiming to construct $\Gamma = NL_g(n)$ with $\lambda = 0$, DM again starts from an initial vertex α and splits the vertex set $V = S_0 \cup S_1 \cup S_2$, where $S_0 = \{\alpha\}$, $S_1 = \Gamma_1(\alpha)$, $S_2 = \Gamma_2(\alpha)$. He next considers the block decomposition of the adjacency matrix $A_1 = A(\Gamma)$ inherited from this partition.

Let C_1 denote the submatrix of A_1 whose rows and columns are indexed by S_1 and S_2, respectively. DM's Theorem 8.1 asserts that (i) C_1 is the incidence matrix of a BIBD $\mathcal{C}$ with the parameters $\widehat{v} = k$, $\widehat{b} = l$, $\widehat{r} = k - \lambda - 1$, $\widehat{k} = \mu$, $\widehat{\lambda} = \mu - 1$, and (ii) each block of $\mathcal{C}$ is disjoint from at least $k - \mu$ other blocks. (Here the symbol $\widehat{\ }$ is used by DM to distinguish BIBD parameters from those of an SRG.)

The proof of Theorem 8.1 boils down to analyzing submatrix products in the matrix formulation of an SRG: $A_1^2 = kI + \lambda A_1 + \mu(J - I - A_1)$. The product $C_1 C_1^T$ suffices for part (i) but the proof of part (ii) is quite more subtle. From the perspective of a BIBD, it is convenient to refer to vertices in S_1 as points and those in S_2 as blocks.

It follows from the definition of the graph $NL_g(n)$ that $\lambda = g^2 + 3g - n$. Thus the proposed absence of triangles yields $n = g^2 + 3g$. In particular, this implies $v = n^2 = (g^2 + 3g)^2$, $k = g(g^2 + 3g + 1)$, $l = (g^2 + 2g - 1)(g^2 + 3g + 1)$, $\mu = g(g + 1)$.

As a corollary to part (i) of Theorem 8.1, the parameters of design $\mathcal{C}$ are presented: $\widehat{v} = g(g^2 + 3g + 1)$, $\widehat{b} = (g^2 + 2g - 1)(g^2 + 3g + 1)$, $\widehat{r} = (g + 1)(g^2 + 2g - 1)$, $\widehat{k} = g(g + 1)$, $\widehat{\lambda} = g^2 + g - 1$. Moreover, part (ii) now asserts that each block of $\mathcal{C}$ is disjoint from at least $g^2(g+2)$ other blocks.

The next portion of DM's text is aimed at proving that the existence of a design $\mathcal{C}$ with properties (i), (ii) of Theorem 8.1 is not only necessary but sufficient, to establish existence of $NL_g(g^2+3g)$. For this task it is necessary to refine the partition of V still further. We choose a block $\gamma \in S_2$ and, as in Model 5, define the sets S_{ij} relative to initial vertex α and block γ. Here the cells of the previous partition split further as $S_1 = S_{11} \cup S_{12}$ and $S_2 = S_{20} \cup S_{21} \cup S_{22}$, where S_{20} is evidently $\{\gamma\}$. Note that the cardinalities of sets in this case yield suitable intersection numbers of the 2-class association scheme for which graph $NL_g(g^2+3g)$ is the first class, namely $|S_{ij}| = p^2_{ij}$. (Note that $\{\alpha, \gamma\}$ is a non-edge in this graph.)

DM's next Lemma 8.2 establishes that, in fact, each block of design $\mathcal{C}$ is disjoint from exactly $p^2_{12} = k - \mu = g^2(g+2)$ other blocks, and moreover that it intersects each remaining block in exactly g points. The proof is just an easy application of the now classical method of variance counting in designs, which is attributed to Hussain [111].

Next come Theorems 8.3 and 8.4, which together prove that a BIBD $\mathcal{C}$ satisfying the conditions of Lemma 8.2 completely determines a $NL_g(g^2+3g)$ graph. The proofs here involve skillful manipulation of matrix products conjoined with simple combinatorial arguments. Corollary 8.4.1 now formulates these findings in natural combinatorial terms. Let $\mathcal{C} = (S_1, S_2)$ be a BIBD that fulfills Lemma 8.2 and define the graph Γ with vertex set $V = \{\alpha\} \cup S_1 \cup S_2$ as follows:

(i) α is adjacent to every vertex from S_1 but to none from S_2,

(ii) $x \in S_1$ is adjacent to $y \in S_2$ provided x and y are incident in $\mathcal{C}$,

(iii) $x \in S_2$ is adjacent to $y \in S_2$ provided x and y are disjoint in $\mathcal{C}$.

Then Γ is a $NL_g(g^2+3g)$ graph.

Corollary 8.4.1 is followed by a table of parameters for Γ and $\mathcal{C}$ for the first four values of g, leading to putative SRGs on 16, 100, 324, 784 vertices. In addition, DM considers a few auxiliary incidence structures defined in terms of the sets S_{ij}. Next comes Theorem 8.5, which identifies these incidence structures as BIBDs and gives their parameters. The proof here involves a combination of clever combinatorial arguments and matrix calculus. Finally, DM gets "for free" a proof of the existence and uniqueness of $NL_1(4)$ because in this case, as noted earlier, $\mathcal{C}$ is the trivial design. He next considers the cases $g \geq 2$, which are, in DM's words, "far from trivial".

Some interesting byproducts are considered by DM toward the end of this central section. In the general case, he describes the subgraph Δ of Γ induced on S_2, which turns out to also be an SRG. Later we will elaborate on Δ in the particular case $g = 2$.

Remark 5. (a) The elements of an ingenious theory developed by DM in Section 8 of [160] are already visible in his thesis [159], though in somewhat rudimentary form. For example, the case $g = 2$ is treated there.

(b) It is more than once stressed in [160] that if the BIBD $\mathcal{C}$ is uniquely determined by its properties in the sense of Lemma 8.2, then the corresponding graph Γ is also unique. Careful analysis of DM's arguments shows also that in such case $Aut(\Gamma)$ acts transitively on V though this is never explicitly stated. (Groups were not the subject of DM's explicit interest at this stage, see Section 13.)

(c) Unfortunately, DM appears to have missed (both in [159] and [160]) that $\mathcal{C}$ is a 3-design, despite the fact that his arguments are sufficient to articulate a proof. In evident form, such observations will be formulated by his followers.

8 Graph $NL_2(10)$ via the DM-approach

Consideration of the new SRG on 100 vertices occupies pp. 131–149 of [159], while in [160] fulfillment of this task is quite more brief, viz. pp. 72–83. The reason for this reduction is clear: the preliminary job in [159] was transformed into an elegant general DM-theory in [160], the digest of which was provided above. Moreover, note that while in [159] DM remains with four possible solutions, in [160] he establishes uniqueness of $NL_2(10)$.

The genre of [160] is close to lecture notes, which allows DM the freedom to reveal his feelings and share his pedagogical views. Two striking examples follow (the first refers to the case $g = 2$, the second to his Theorem 8.6):

> “The author conjectured that the design did not exist in this case, undertook an empirical search in hopes of proving its nonexistence, and in the course of the search inadvertently constructed it.” – p. 72

> “This method of proof is reminiscent of Bhaskhara whose 1150 A.D. treatise on mathematics presented a sketch of a particularly lucid construction for the Pythagorean theorem, accompanied by the brief written proof, ‘Behold!’ ” – p. 74

Construction of the desired $NL_2(10)$ according to DM-theory is reduced to the construction of a design $\mathcal{C}$ with the parameters $v = 22$, $b = 77$, $r = 21$, $k = 6$, $\lambda = 5$ (the case $g = 2$). DM provides a solution simply by listing all 77 blocks (see Table 8.2, which occupies the entire p. 73), and asks that the reader verify all required properties. (Note that the order in which these 77 blocks are listed carries a special significance, serving as a brief “guide” to a forthcoming analysis performed by DM.) Such detailed inspection implies DM’s Theorem 8.6, thus asserting the existence of an $NL_2(10)$ graph. It is indeed a proof in the style of Bhaskhara.

This is followed by Theorem 8.7, which asserts the uniqueness of $NL_2(10)$. In a prefatory remark, DM discusses the obstacles involved in attempting to establish nonexistence or uniqueness empirically, as well as his own attempt to precariously “steer between tedium and non-proof.”

For the proof, DM first explains why it is sufficient to establish uniqueness of the underlying design $\mathcal{C}$. Recall that in addition to the established parameters for such a design, we further know that each of its blocks is disjoint from 16 other blocks, while intersecting the remaining 60 blocks in two points apiece.

Let $V = [1, 22]$ be the point set of $\mathcal{C}$, and assume $\gamma = [1, 6]$ to be a block. DM now splits V into $S_{11} \cup S_{12}$, where $S_{11} = [1, 6]$. Similarly, he splits the block set into $\{\gamma\} \cup S_{21} \cup S_{22}$, where S_{21} consists of those blocks disjoint from γ. He is next able to reveal some auxiliary designs:

(a) a symmetric BIBD $\mathcal{F} = (S_{12}, S_{21})$ with 16 blocks of size 6,

(b) a design $\mathcal{E}$ with repeated blocks that is uniquely determined by the parameters $v = 6$, $b = 60$, $r = 20$, $k = 2$, $\lambda = 4$,

(c) a (hidden) design $\mathcal{N}$ with point set S_{12} and blocks of size 4.

Now a clever backtracking search, combined with some tedious technical arguments, establishes that the only possibility for $\mathcal{C}$ is the design depicted in Table 8.2. The reader will become more acquainted with the hidden idea behind DM’s vision of $\mathcal{C}$ in Section 11 below.

Having completed the proof of Theorem 8.7, DM discusses the subgraph of $NL_2(10)$ induced on the block set of $\mathcal{C}$ (non-neighbors of the initial vertex α), a new SRG with parameters $(77, 16, 0, 4)$, and other substructures. One such substructure (recorded in Table 8.3 on p. 82) is a beautiful symmetric square of size 6 with empty diagonal. Each entry of this square is itself a square of size 4, filled by the same four elements. This substructure, which also appeared in [159], possesses some nice orthogonality properties. We feel it is certainly deserving of special attention, however its careful consideration is beyond the scope of our text.

Perhaps the most surprising observation by DM is that $NL_2(10)$ contains 7700 subgraphs, which are incidence graphs of a symmetric BIBD with $\widehat{v} = 16$, $\widehat{k} = 6$, $\widehat{\lambda} = 2$.

Although DM freely uses quasi-symmetric properties of $\mathcal{C}$, he never makes the formal observation that $\mathcal{C}$ is a 3-design. Fortunately this alternative DM-approach, though of a more sophisticated nature, provides the modern reader with new perspectives on attacking existence of $NL_g(g^2 + 3g)$ for larger values of g.

9 History of the design $S(3, 6, 22)$

We come now to 1967, the year in which DM's article [162] finally appears. This is the first "official" announcement of DM's discovery of $NL_2(10)$, though quite unpredictably it is also the last. This was not DM's intention, e.g., see [162, p. 574] where he speculates that his discovery will be presented in future papers. What then, are the events that could have mitigated this change?

As previously mentioned, DM was unfamiliar at this time with the notion of a 3-design. He constructed his design $\mathcal{C}$ using very clever *ad hoc* arguments, and proved its uniqueness by a sophisticated brute force attack *without* the aid of a computer. In contrast, at the same moment a rather wide mathematical audience was already well acquainted with this notion, and even with the actual construction. We now attempt to provide an historical perspective on the object in question, the Steiner design $S(3, 6, 22)$.

The story properly begins in the mid-19$^{\text{th}}$ century with E. Mathieu, a mathematician who was nearly a century ahead of his time. Even today the mystery is not completely solved as to how he was able to discover five highly unusual groups that now bear his name, the Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}. A brief, though very nice, historic account of this great accomplishment may be found in [2].

It is the group M_{22} that occupies center stage for us, as its automorphism group (viz. $Aut(M_{22}) \cong M_{22}.\mathbb{Z}_2$) coincides with $Aut(\mathcal{C})$. Nevertheless, all five groups have some relevance to our discussion, especially M_{23} and M_{24} as they correspond to 4- and 5-designs that are successive one-point extensions of $S(3, 6, 22)$. Indeed, anyone who is already aware of the design $S(5, 8, 24)$ well understands all aspects of the design $\mathcal{C}$ on 22 points.

Though one may find early attributions to T. Skolem, we may only refer to p. 42 of [71] as being the first (to our knowledge) prior announcement of a result by Carmichael [43] relating a similar design construction to M_{11}. Actual consideration of $S(5, 8, 24)$ appears in [43], as well as later on in the book [44] where Carmichael in fact alludes to the existence of $S(3, 6, 22)$, though without giving explicit parameters or a construction.

Remark 6. The authors hold the book [44] in very high regard, a feeling that is shared by several of our colleagues. In a definite sense it was ahead of its time. The text contains a plethora of perfect exercises that challenge the reader to work with many concrete combinatorial and geometric structures, as well as their symmetries. (Note that the term automorphism was not in use at that time.) One explicit feature of the exercises in [44] is their vast range of difficulty. Many problems are trivial while others are reasonably sophisticated. However, occasionally there is a pearl. Only the strongest and most committed reader, working independently, will be rewarded with its solution, which in turn will reveal some deeply hidden treasures. It seems that Carmichael's failure to stratify his exercises was not an oversight but a brilliant strategy on his part.

One additional text essential to our presentation is the article [229] of H. Zassenhaus. This is a landmark paper in the development of the method of transitive extension, e.g., see [132]. An actual construction of M_{22} appears as *Satz 7* in [229] with a clear group theoretic proof. In [223], E. Witt refers to [229] as the origin of this newly created method. In turn [223] serves as motivation for [224], which is written in a purely combinatorial spirit. In particular, *Satz 4* in [224] outlines a proof of the existence and uniqueness of the Steiner system $S(3, 6, 22)$.

Unfortunately, the latter paper [224] of Witt went virtually unnoticed for over a decade, until R. G. Stanton, a Ph.D. student of Richard Brouwer, prepared his 1948 thesis at the University of Toronto. In Stanton's subsequent publication [208] one finds reference to $S(3, 6, 22)$ with credits to the two papers of Witt. As a result of this, Witt designs and their groups would come to gain deserved acceptance; in fact W_{22} serves as a modern alternative to the more traditional notation $S(3, 6, 22)$. We cite [168, 215, 76, 214, 107, 3, 216] as only a sample of publications from that era that were already recognizing the results of Witt, and which today are gaining a lot of traction.

This was also a time when links between groups and geometries were becoming transparent, e.g., Tits in [214] was already citing Segre [189]. Nevertheless, some nice considerations of finite geometries were still living on a separate island. A bright example of this is the article [69], in which W. L. Edge prepared in a beautiful and surprisingly translucent manner an analysis of the geometry $PG(2, 4)$ and its symmetry group, probably never anticipating that in a few years his results would have special significance for group theorists.

However, this bucolic picture of the mathematical world would drastically change due to the events of one single evening in 1967. The results achieved over the course of this evening would form the content of [101], a publication that would have profound influence on the fields of group theory, combinatorics, geometry and computer algebra, and which would stimulate fruitful interdisciplinary links that nowadays look traditional and longstanding.

The propagation of waves from this breakthrough was surprisingly quick. To illustrate, we mention the book [63] of P. Dembowski, which nowadays is viewed by many as forming an unbreakable bond between finite geometries and group theory. At the time of preparation of [63], the result of Higman-Sims was still in preprint form; yet the telescopic eyes of Dembowski had already witnessed existence of the breakthrough, see his footnote 2 on p. 91 with credits to C. Hering. (In fact, one also finds in [63] references to a few papers of DM, although it appears Dembowski was not aware of [162].) Similarly, [150, 151] were also influenced by [101], either implicitly or explicitly, with Hering again receiving acknowledgment.

We close this section with a discussion of the uniqueness of the Witt design W_{22}. Nowadays, most authors follow the same basic mode of proof: use of the projective plane $PG(2, 4)$ and its hyperovals. The first clear exposition in this mode was formulated in [151] by H. Lüneburg, who exploited the approach of Edge [69] who in turn was influenced by Segre. Nice expositions of this method in English appear in [17, 13, 42]. However, we prefer to follow Theorem 6.6.D on p. 200 of [64], which we outline below. First we paraphrase the theorem statement as follows:

Up to isomorphism, there exists a unique Steiner system $S(3, 6, 22) \cong W_{22}$.

Moreover, $Aut(W_{22})$ acts 3-transitively on the point set of W_{22}.

For the proof, we start with $\Pi = (\mathcal{P}, \mathcal{L}) = PG(2,4)$. Note that Π has 168 (hyper)ovals each consisting of six points. Define an equivalence relation on the set $\mathcal{O}$ of ovals as follows: $O_1 \equiv O_2 \iff |O_1 \cap O_2|$ is even. Next show that there are precisely three equivalence classes of ovals, each of size 56, preserved by the transitive action of $PSL(3,4)$ on $\mathcal{O}$. Fix one such equivalence class $\mathcal{H}$. Now for $\alpha \notin \mathcal{P}$, form the sets $\mathcal{P}^* = \mathcal{P} \cup \{\alpha\}$ and $\mathcal{B}^* = \mathcal{L}^* \cup \mathcal{H}$, where $\mathcal{L}^* = \{l \cup \{\alpha\} \mid l \in \mathcal{L}\}$. Finally, check that $(\mathcal{P}^*, \mathcal{B}^*)$ is a Steiner $S(3,6,22)$-design with incidence defined by inclusion.

The proof of uniqueness depends on the uniqueness of Π plus some nice properties of hyperovals. (We still believe [69] is the best source for such information.) Uniqueness of W_{22} implies the transitivity portion of the theorem: $H = Aut(W_{22})$ acts 3-transitively on $\mathcal{P}^*$ because $PSL(3,4)$ acts 2-transitively on $\mathcal{P}$. This also proves $|H| \geq 22 \cdot |PSL(3,4)| = 443520$, which is the precise order of the group M_{22}.

Remark 7. (a) In fact $|H| = 2 \cdot |M_{22}|$, as H is a transitive extension of the group $P\Sigma L(3,4) \cong PSL(3,4).\mathbb{Z}_2$ acting on $\mathcal{P}$. Such vision becomes absolutely clear if we consider the group $M_{24} = Aut(W_{24})$, where W_{24} is the unique $S(5,8,24)$-design. Indeed, in the action of M_{24} on the point set of W_{24}, the stabilizer of two points β, ζ yields a copy of M_{22} while the stabilizer of the set $\{\beta, \zeta\}$ yields H.

(b) Note that the modern way to view this entire picture is to interpret W_{24} as the set of octads of the Golay code of length 24. The reader is strongly encouraged to consult [113] for a comprehensive treatment that follows this approach.

10 Higman-Sims graph and the sporadic simple group HS

The discovery of the sporadic simple group HS by D. G. Higman and C. C. Sims stands as one of the most fascinating stories in modern group theory. Details of this historic event are well documented, e.g., see [104, 10].

The story begins at a 1967 conference in Oxford, where Marshall Hall has just delivered his talk, "A search for simple groups of order less than one million". Higman and Sims are two of the many in attendance. Hall has just described the construction of a new sporadic simple group (Hall-Janko group) as a rank 3 permutation group of degree 100. Higman and Sims are immediately inspired to think along the lines delineated in Hall's talk. Crucial observations are made during the night of Saturday, September 2 through the morning of Sunday, September 3, 1967. The end result is a new sporadic simple group HS. A manuscript is submitted fairly quickly (received by the editors on November 20, 1967, published as [101] in 1968), yet news of their discovery spreads even faster.

The group HS was discovered via the procedure of "rank 3 extension". A graph Γ was constructed from an initial vertex $*$ in such a way that the stabilizer of $*$ in the proposed group would have orbit sizes 1, 22, 77, and would contain the Mathieu group M_{22}. The role of the unique Witt design W_{22} became crucial in this construction, particularly since $Aut(W_{22})$ contains M_{22} as a subgroup of index 2. In the end, the construction of Γ is in every detail identical to the construction of $NL_2(10)$ performed by DM. Of course neither Higman nor Sims was aware of this accomplishment by DM. Their great advantage was a formidable knowledge of the diverse properties of W_{22}. (Relevant attributions in [101] are given to [224, 216].)

Higman and Sims denote by $\overline{G}$ the full automorphism group of the constructed graph Γ. To now obtain their group HS, they note that $\overline{G}$ is transitive, contains odd permutations, and that the stabilizer of $*$ in $\overline{G}$ is $Aut(M_{22})$. This alone allows them to conclude that $\overline{G}$ contains a simple group of index 2. They select a vertex α adjacent to $*$ in Γ and show that its stabilizer too is isomorphic to $Aut(M_{22})$. Finally they are able to explicitly construct HS as a subgroup of $\overline{G}$ generated by two permutations of respective orders 2 and 7. It is fair to say that at many critical junctures in their proof they rely on purely combinatorial arguments.

There are amazing numerological coincidences at work here. Sims recalls (see [104]), "If it were not the case that we use the decimal system and that $100 = 10^2$, I am not sure we would have asked this question." DM's notion of $NL_g(n)$ results in a graph with n^2 vertices. Many pages of [159] are devoted to the cases $4 \leq n \leq 9$, including a discussion of the exceptional case $n = 6$ (there is no field with six elements). In Sec. 3.3 of [159], where the graph $NL_2(10)$ is presented for the first time, DM's opening remark is, "Since no Galois field of order 100 exists, the method of Section 3.2 cannot be applied here. The scheme would seem to have some special interest because of its possible connection with the unsolved question of the existence of orthogonal 10×10 Latin squares." (Keep in mind that DM's remark is more than a half-century old.)

There is an interesting spin-off to our main story of the group HS. In late 1967, Graham Higman described at the Urbana Group Theory Symposium a certain geometry on 176 points, having a simple doubly-transitive group of automorphisms (momentarily, we denote this group as GH). The results were published in [102]. Originally, G. Higman worked with 176 points and 1100 conics as his objects, but at the suggestion of D. R. Hughes he realized the existence of a symmetric BIBD with blocks of size 50 (called quadrics). The paper [102] is filled with beautiful combinatorial arguments relying on exceptional properties of the symmetric groups S_n, $n \in \{6, 7, 8\}$.

Almost immediately, Sims realized that the groups GH and HS were in fact isomorphic. Sims' proof [200] was published in the legendary collection of papers of the Symposium on Theory of Finite Groups at Harvard, 1968. During the same short period, proofs of this isomorphism were also obtained by J. H. Conway [53], and D. Parrott & S. K. Wong [169]. (The latter of these papers relies essentially on the uniqueness theorem of D. Wales [218] characterizing rank 3 graphs with the parameters $(100, 22, 0, 6)$.) A new proof of this same fact was later presented in [203, 204], with again the uniqueness of W_{22} as a key ingredient.

Since its discovery, the group HS has continued to be the subject of special attention, e.g., see [119, 153, 75]. The same is true of the Higman-Sims graph Γ; over the past four decades diverse investigations related to the geometry and symmetry of Γ have resulted in numerous publications. Of these we mention only a handful [23, 27, 58, 94, 95] that correlate strongly with the spirit of our presentation.

We return once more to DM. It is 1968, and by now he has learned of the result of Higman & Sims. (Dale has discussed with us his conversations with J. J. Seidel from around this time period, see [117, 118].) One can speculate, probably with great accuracy, the myriad of emotions he is experiencing over this news. For the next seven years DM would not publish even a single paper. Over the course of his long career, never again would he undergo such a prolonged period of silence.

His current state of inactivity is broken in 1974 with his joint paper [144] with E. Kramer, where one observes DM as a mathematician with revitalized energy and renewed confidence. A deep familiarity with Steiner systems, acquaintance with the papers of Witt, references to papers of Frobenius and Wielandt — these are the ingredients of his research interests. From a pure statistician with a penchant for matrix multiplication to a mature expert in AGT and design theory, he is now cognizant of the treasures available through group theory and finite geometries. The world of mathematics is forever remaining bright and attractive in his eyes, yet still he is not receiving the recognition he deserves for the discovery he made 12 years prior to his contemporaries.

11 Mesner's vision of the case $g = 2$ revisited

We are now in a position to introduce the main ideas of the initial construction of $NL_2(10)$ as it originally appeared in [159], and later, in more polished form, in [160]. No attempt is made to preserve the original terminology or notation. Indeed, our motivation is to present the construction in a manner that is readily accessible to modern researchers.

Throughout, we rely heavily on the use of the computer packages GAP [186] and COCO [72]. (To provide full credits, we also mention the GRAPE subpackage of GAP [205], which in turn relies on the use of nauty [158].)

We begin by introducing certain relevant auxiliary structures.

11.1 Generalized quadrangle of order 2

This famous configuration goes back to J. J. Sylvester (1844), who used the language of duads and synthemes for its description. Starting from a 6-element set Ω_1, the *duads* are the 15 2-element subsets of Ω_1, and the *synthemes* are the 15 partitions of Ω_1 each into three duads. Incidence is containment. It is convenient to think of this configuration in terms of the complete graph K_6 with vertex set Ω_1, in which duads are edges and synthemes are 1-factors.

One can readily check that the resulting incidence geometry $\mathfrak{S}$ satisfies all axioms of $GQ(2)$, e.g., see [174]. This geometry is self-dual; letting $\Gamma_{\mathfrak{S}}$ denote its incidence graph, we have $Aut(\mathfrak{S}) \cong S_6$ while $Aut(\Gamma_{\mathfrak{S}}) \cong Aut(S_6)$. Thus $\Gamma_{\mathfrak{S}}$ gives a natural way to visualize the exceptional outer automorphisms of S_6.

Now the concept of a *spread* becomes essential, that is, a subset of lines of a geometry that partitions its point set. Clearly, each spread of $\mathfrak{S}$ corresponds to a 1-factorization of the graph K_6. There are six such pairwise isomorphic structures, see [42] for details. Denote by Ω_2 the set of such structures.

In such manner, we obtain an action (S_6, Ω_2) which differs fundamentally from the natural action (S_6, Ω_1). Indeed, while (S_6, Ω_1) and (S_6, Ω_2) are equivalent with the aid of outer automorphisms of S_6, the stabilizers in these actions are not conjugate in S_6. Namely, in the case of (S_6, Ω_1) a stabilizer is a copy of S_5 acting 5-transitively of degree 5, while a stabilizer arising from (S_6, Ω_2) is an S_5 acting 2-transitively of degree 6. (Note that in both cases, we are regarding S_5 as a subgroup of the natural action of (S_6, Ω_1).)

A philosophical discussion of this unusual occurrence (due to Tarski) goes back to [165]. We also refer to [175, 103] for additional details in two extrema: nice and convincing visual images in the first cited article, and rigorous considerations on the border of groups and geometries in the second. One additional reference of significance here is [167], where both actions of S_6 are investigated with detailed attention to their geometric origins.

We have introduced (S_6, Ω_2) in terms of spreads of $\mathfrak{S}$, but hasten to point out that (S_6, Ω_1) has a similar such realization. A spread in the dual geometry $\mathfrak{S}^T$ is a collection of five duads each of which contains a common element $a \in \Omega_1$. Clearly the action of S_6 on the six spreads of $\mathfrak{S}^T$ is equivalent to the natural action (S_6, Ω_1). Thus our two discussed actions both arise in a natural way in terms of spreads of generalized quadrangles of order 2.

11.2 The exceptional isomorphism $A_8 \cong PSL(4, 2)$

Of the many presentations of this classical result, we again prefer the one by Edge [68], which in turn discusses an old paper [55] by G. M. Conwell. This result proceeds in conjunction with another exceptional isomorphism: $S_6 \cong PSp(4, 2)$. In fact, the structure $\mathfrak{S}$ introduced above is the smallest case of an infinite family of symplectic generalized quadrangles.

Note that S_6 naturally embeds in A_8 as the subgroup $(S_6 \times S_2)^{\text{pos}}$ of all even permutations in $S_6 \times S_2$. Concurrently, one has an embedding of $U = E_{2^4} \rtimes S_6$ into $N_{S_{16}}(V) \cong E_{2^4} \rtimes A_8$, where $N_{S_{16}}(V)$ is the normalizer in S_{16} of a 4-dimensional vector space $V \cong E_{2^4}$ over $GF(2)$ acting regularly on 16 points. Our aim is to become more familiar with the geometry of group U, especially with regard to how to recognize its homomorphic image $U/V \cong S_6$ in these same geometric terms.

11.3 The nicest biplane on 16 points

We are interested in BIBDs with the parameters $v = b = 16$, $k = r = 6$, $\lambda = 2$, a particular case of biplanes. According to the classical result of Hussain [110] there are exactly three such designs.

A very friendly and quite elementary exposition on the result of Hussain may be found in [109]. Note that while DM's thesis [159] does not refer to [110], his later paper [160] does so, and quite essentially.

Of these three BIBDs, we denote by $\mathcal{D}$ the one with the highest degree of symmetry. Indeed, $Aut(\mathcal{D})$ is a 2-transitive group of order 11520. The history of $\mathcal{D}$ can be traced back to the 19^{th} Century during which time it acquired such diverse names as the Jordan design and Kummer's quartic surface (see Section 13 for historic details). Thus we concur with [179] in calling $\mathcal{D}$ the "nicest biplane on 16 points".

Our path to $\mathcal{D}$ relies on the reader's familiarity with the Clebsch graph $\square_5$, see Section 6. Recall that $\square_5$ is a Cayley graph over E_{2^4} with connection set of size 5. We also know that $Aut(\square_5) \cong E_{2^4} \rtimes S_5$. We shall again refer to the Clebsch graph $\square_5$ depicted in Fig. 6, only this time with one slight modification: Each binary sequence serving as a vertex label will be substituted by its decimal equivalent i, $0 \le i \le 15$.

The following simple device is well known and goes back to [183]. It applies to any SRG for which $\lambda = \mu - 2$, in particular to $\square_5$:

Let Γ be an SRG with parameters $(v, k, \mu - 2, \mu)$ defined on the vertex set Ω. For each $x \in \Omega$, form $B_x = \{x\} \cup \Gamma_1(x)$ where $\Gamma_1(x)$ is the set of neighbors of x. Set $\mathcal{B} = \{B_x \mid x \in \Omega\}$. Then $(\Omega, \mathcal{B})$ is a BIBD with parameters $(v, k+1, \mu)$.

Taking into account that $Aut(\square_5)$ contains a subgroup E_{2^4} acting regularly on points, we observe that $\mathcal{D}$ may be additionally obtained with the aid of a difference set over E_{2^4}, for example $B = \{0, 1, 2, 4, 8, 15\}$ fulfills this role. Thus we get $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P} = [0, 15]$ and $\mathcal{B}$ consists of the following blocks (6-element subsets of $[0, 15]$):

$\{0,1,2,4,8,15\}$	$\{0,1,3,5,9,14\}$	$\{0,2,3,6,10,13\}$	$\{1,6,10,12,14,15\}$
$\{0,4,5,6,11,12\}$	$\{0,7,8,9,10,12\}$	$\{2,5,9,12,13,15\}$	$\{1,6,8,9,11,13\}$
$\{1,4,5,7,10,13\}$	$\{0,7,11,13,14,15\}$	$\{3,4,8,12,13,14\}$	$\{3,4,9,10,11,15\}$
$\{2,5,8,10,11,14\}$	$\{1,2,3,7,11,12\}$	$\{2,4,6,7,9,14\}$	$\{3,5,6,7,8,15\}$

Clearly $\mathcal{D}$ has the parameters $v = b = 16$, $k = r = 6$, $\lambda = 2$. We omit simple arguments that show that $U := Aut(\mathcal{D})$ acts 2-transitively on $\mathcal{P}$ and that $U \cong E_{2^4} \rtimes S_6$.

11.4 Anatomy of the design $\mathcal{D}$

Recall that an *oval* O is a subset of points of a design such that $|O \cap B| \in \{0, 2\}$ for every block B of the design. According to the theory presented in [42], ovals in our nicest biplane $\mathcal{D}$ should all have size 4.

We can describe the ovals of $\mathcal{D}$ with the aid of the following simple procedure applied to our canonical copy of $\square_5$:

Fix any edge $\{a,b\}$ of $\Delta = \square_5$, and consider the set $\overline{\Delta(a)} \cap \overline{\Delta(b)}$ (mutual non-neighbors of a and b). The subgraph of Δ induced on the six vertices of this set is a 1-factor consisting of three edges, say $\{c,d\}$, $\{e,f\}$, $\{g,h\}$. In this case one obtains $\{a,b,c,d\}$, $\{a,b,e,f\}$, $\{a,b,g,h\}$ as ovals in $\mathcal{D}$. Varying our choice of initial edge, we obtain $\frac{16\cdot 5\cdot 3}{2\cdot 2} = 60$ distinct ovals in this manner, thus exhausting the entire set $\mathcal{O}$ of ovals in $\mathcal{D}$. Moreover, the incidence structure $(\mathcal{P}, \mathcal{O})$ is another BIBD invariant under G.

Recall that $Aut(\square_5)$ is a subgroup of index 6 in $Aut(\mathcal{D})$. This implies that there are exactly six different copies of the Clebsch graph that produce design $\mathcal{D}$ via the procedure outlined in Subsection 11.3. Denote by $\widetilde{\Omega}_1$ the collection of these six Clebsch graphs.

The easiest way to observe the members of $\widetilde{\Omega}_1$ is to consider the six blocks of $\mathcal{D}$ that contain the point 0. Each such block (with the point 0 excluded) serves as a connection set for a Cayley graph over E_{2^4}, and the six Cayley graphs so obtained are none other than our six Clebsch graphs forming $\widetilde{\Omega}_1$.

Furthermore, as $\mathcal{D}$ is a symmetric design (in fact, self-dual), we know that any two blocks of $\mathcal{D}$ intersect in exactly two points. This means that each pair of connection sets above have exactly one point $x \neq 0$ in common, and consequently each pair of Clebsch graphs from $\widetilde{\Omega}_1$ intersect in precisely a 1-factor of E_{2^4}, namely the Cayley graph over E_{2^4} with connection set $\{x\}$. In this manner we obtain $\binom{6}{2} = 15$ such 1-factors, each yielding a graphical representation of an involution from E_{2^4}. Thus it will be convenient to identify 1-factors with involutions as follows: the 1-factor arising from the involution $x \in E_{2^4}$ will be denoted by $m = m_x$, where $m \in [1, 15]$ is the decimal equivalent of x, regarding x as a binary number. Let us denote by Σ_1 the set of all such 1-factors.

At this stage we are interested in spreads of the new design $(\mathcal{P}, \mathcal{O})$, that is partitions of $\mathcal{O}$ into four ovals. GAP now informs us that there are two orbits of such spreads of respective sizes 15 and 90. Clearly the smaller orbit is preferable because it corresponds to a resolution $\mathcal{R}$ of $(\mathcal{P}, \mathcal{O})$ into 15 parallel spreads. In principle, one would like to arrive at this resolution without the aid of computer-generated data. Indeed this can be done, however not within the framework of our current presentation. At present, suffice it to say that the $15 + 90 = 105$ spreads of $(\mathcal{P}, \mathcal{O})$ correspond in a natural way to the 105 Klein-4 subgroups of E_{2^4}, and below we exhibit this correspondence explicitly for the 15 spreads in our chosen resolution $\mathcal{R}$.

One relevant Klein-4 subgroup of E_{2^4} here is $\langle 0001\rangle \times \langle 1010\rangle$, which contains the three involutions $1, 10, 11$ (in decimal). The Cayley graph over E_{2^4} with connection set $\{1, 10, 11\}$ is $4 \circ K_4$ (four disjoint copies of the complete graph K_4), and the resulting partition of the vertex set is given by $\{0,1,10,11\}$, $\{2,3,8,9\}$, $\{4,5,14,15\}$, $\{6,7,12,13\}$. To see that this yields a spread in $(\mathcal{P}, \mathcal{O})$, one need only verify that each 4-element subset in this partition is indeed an oval of $\mathcal{D}$.

We denote by Σ_2 the set of 15 spreads comprising $\mathcal{R}$. Just as we earlier identified 1-factors with involutions, we here find it convenient to identify spreads with triples of involutions (again, the underlying connection sets). We may now list the members of Σ_2 in a most compact form:

$\{1,10,11\}$	$\{2,9,11\}$	$\{1,12,13\}$	$\{1,6,7\}$	$\{5,10,15\}$
$\{3,8,11\}$	$\{4,10,14\}$	$\{4,9,13\}$	$\{2,5,7\}$	$\{2,12,14\}$
$\{6,9,15\}$	$\{3,12,15\}$	$\{5,8,13\}$	$\{3,4,7\}$	$\{6,8,14\}$

11.5 $GQ(2)$ by way of the nicest biplane

We are now ready to introduce a new incidence structure $\mathcal{M} = (\Sigma_1, \Sigma_2)$ with point set Σ_1 and line set Σ_2. This configuration may be viewed simultaneously at several levels. For example, incidence in $\mathcal{M}$ is easiest described in terms of the group E_{2^4}: points are involutions, lines are triples of involutions (coming from the 15 aforementioned "special" Klein 4 subgroups of E_{2^4}), and incidence is containment.

Simple routine inspection allows us to conclude that $\mathcal{M}$ is the same $GQ(2)$ (cf. Subsec. 11.1) in new clothing. To see this it is convenient to interpret points once more as 1-factors, that is, pairwise intersections of Clebsch graphs from $\widetilde{\Omega}_1$. Thus each point is a duad of Clebsch graphs, and each line is a triple of disjoint duads, that is, a syntheme of $\widetilde{\Omega}_1$. Thus one sees the evident relationship between $\mathcal{M}$ and the model $\mathfrak{S}$ of Sylvester.

Moreover, $\widetilde{\Omega}_1$ is nothing more than a spread in the dual geometry $\mathcal{M}^T$ of $\mathcal{M}$. Indeed, the action $(S_6, \widetilde{\Omega}_1)$ coincides with the natural action (S_6, Ω_1) encountered in the context of the dual geometry $\mathfrak{S}^T$.

It remains for us to identify six more objects that are for the moment quite hidden from view, and revealed to us only through the use of GAP.

Recall that the elements of Σ_2 comprise a resolution $\mathcal{R}$ in $(\mathcal{P}, \mathcal{O})$. As it turns out, the data generated by GAP facilitate a description of these six hidden objects in terms of six sub-resolutions $\mathcal{S}_i$ of $\mathcal{R}$, $1 \leq i \leq 6$, each consisting of five spreads. These sub-resolutions are easiest represented in terms of connection sets, that is to say, we represent each $\mathcal{S}_i$ as a collection of five triples of involutions, where each such triple is the connection set of a spread that occurs in $\mathcal{S}_i$. Specifically, we get

$$\begin{array}{l}
\mathcal{S}_1 : \{\{1,10,11\},\{2,5,7\},\{3,12,15\},\{4,9,13\},\{6,8,14\}\} \\
\mathcal{S}_2 : \{\{1,10,11\},\{2,12,14\},\{3,4,7\},\{5,8,13\},\{6,9,15\}\} \\
\mathcal{S}_3 : \{\{1,6,7\},\{2,9,11\},\{3,12,15\},\{4,10,14\},\{5,8,13\}\} \\
\mathcal{S}_4 : \{\{1,12,13\},\{2,9,11\},\{3,4,7\},\{5,10,15\},\{6,8,14\}\} \\
\mathcal{S}_5 : \{\{1,12,13\},\{2,5,7\},\{3,8,11\},\{4,10,14\},\{6,9,15\}\} \\
\mathcal{S}_6 : \{\{1,6,7\},\{2,12,14\},\{3,8,11\},\{4,9,13\},\{5,10,15\}\}
\end{array}$$

This representation of six hidden objects turns out to be much more than a notational convenience. Indeed, from it we readily observe that each $\mathcal{S}_i$ is a set of five elements from Σ_2 (viewed as triples of involutions) that partitions Σ_1 (viewed as a set of involutions). That is to say, each $\mathcal{S}_i$ is nothing else but a spread in the design $\mathcal{M}$. Let $\widetilde{\Omega}_2$ denote the set of these six spreads $\mathcal{S}_i$, $1 \leq i \leq 6$. Clearly, $(S_6, \widetilde{\Omega}_2)$ and (S_6, Ω_2) are equivalent actions, as may be observed directly.

The fact that our six hidden objects turn out to be nothing more than spreads in $\mathcal{M}$ helps to remove some of their mystique. However, it is still a mystery as to why these specific sub-resolutions are the ones that appear. It seems that there are just two obstacles to a computer-free interpretation of the results thus far presented in this section: the one alluded to above, and the earlier choice of resolution of $(\mathcal{P}, \mathcal{O})$. No doubt, all clues required to solve this conundrum are living in the nicest biplane on 16 points.

11.6 Assembling the pieces

In modern terms, the initial construction of DM can be compactly expressed by the intersection diagram in Fig. 8. The monicker of "non-edge model" refers to the fact that the construction is starting from an initial pair P^*, B^* of objects that are to be nonadjacent vertices in the graph $NL_2(10)$. Note that including P^* and B^* there are

$$2 + |\mathcal{P}| + |\mathcal{B}| + |\mathcal{O}| + |\widetilde{\Omega}_2| = 2 + 16 + 16 + 60 + 6 = 100$$

available objects, which evidently will comprise the vertex set Ω of the forthcoming graph.

There are two especially important observations to be made here. First, all indicated sets of objects are manifestly connected to the nicest biplane on 16 points (viz. points,

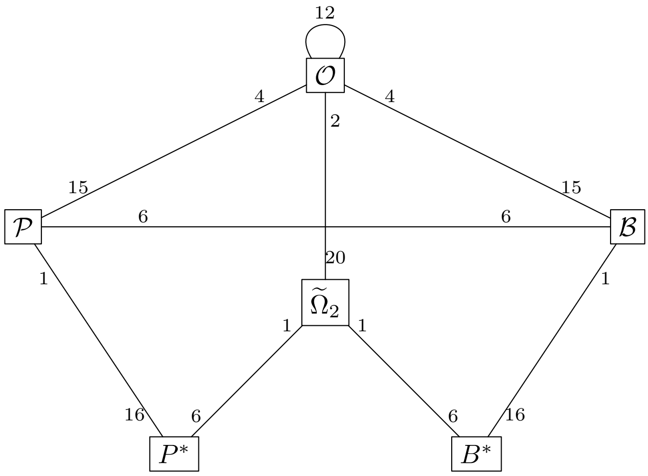


Figure 8. DM intersection diagram

blocks, ovals, sub-resolutions). Second, each set of objects admits a natural action by the group $G := Aut(\mathcal{D}) \cong E_{2^4} \rtimes S_6$.

The final task is performed by the computer package COCO. Generators are first obtained for the intransitive action (G, Ω) having six orbits of respective lengths 1, 1, 6, 16, 16 and 60. Next, COCO constructs the coherent configuration $\mathfrak{A} = (\Omega,\ 2\text{-orb}(G, \Omega))$ in the sense of D. G. Higman. It turns out that $\mathfrak{A}$ has rank 51 (i.e., $|2\text{-orb}(G, \Omega))| = 51$). At the next stage, COCO computes all intersection numbers of $\mathfrak{A}$, mergings of $\mathfrak{A}$ that yield associations schemes, and automorphism groups of these resulting association schemes. In the present case we get two copies of the graph $NL_2(10)$ (compare this with the situation of the Shrikhande graph in Sec. 2). Note that the two obtained copies of $NL_2(10)$ are interchanged by a suitable involution in $AAut(\mathfrak{A}) \cong E_{2^2}$, a fact revealed to us by GAP. Below we provide a description of one such copy.

11.7 Adjacencies revealed

As indicated by the intersection diagram in Fig. 8, vertex P^* is adjacent to every vertex in $\mathcal{P}$. Similarly, B^* is adjacent to every vertex in $\mathcal{B}$.

The link adjoining $\mathcal{P}$ to $\mathcal{B}$ is simply the incidence graph of the design $\mathcal{D}$. Likewise, the link adjoining $\mathcal{P}$ to $\mathcal{O}$ is the incidence graph of the design $(\mathcal{P}, \mathcal{O})$.

A vertex $B \in \mathcal{B}$ is adjacent to a vertex $O \in \mathcal{O}$ provided B and O are disjoint.

Two vertices from $\mathcal{O}$ are adjacent provided they are disjoint and do not occur in any common sub-resolution $\mathcal{S}_i \in \widetilde{\Omega}_2$.

Finally, a vertex $O \in \mathcal{O}$ is adjacent to $\mathcal{S}_i \in \widetilde{\Omega}_2$ provided O occurs in $\mathcal{S}_i$.

Note that the full list of 60 ovals remains hidden from the reader. Only those 15 which contain 0 are listed explicitly at the end of Subsec. 11.4.

11.8 Additional remarks

The reconstruction of DM's ideas as presented in this section requires special discussion.

Literally speaking, the texts [159, 160] of DM are focusing on a description of the incidence structure $\mathcal{C}$, whereas we are presenting the entire graph Γ. Nevertheless, DM paid great attention to explaining how the global structure of Γ could be derived from a knowledge of $\mathcal{C}$. In our eyes, the manner in which Γ is finally assembled from its component pieces truly reflects the spirit of DM's vision, although it does not strictly coincide with his presentation.

Further texts that helped shape our ideas and insights in arriving at our interpretation of DM's vision are [81, 122, 4, 22, 179]. Indeed, these strongly influenced the direction of our arranged computer algebra experimentation.

Last but not least, we mention a goal of the authors (together with M. Ziv-Av) to arrive at a new interpretation of DM's original proof of the uniqueness of Γ, which will be computer-free and rely strictly on knowledge of the design $\mathcal{C}$, its symmetries, and its substructures.

12 Further developments

In this section, we endeavor to explain how the crucial discovery by DM of a putative infinite series of parameters for $NL_g(g^2+3g)$ was recognized, interpreted and developed by his followers. Though most of what we here present may be substantiated factually, there are also a few occurrences of speculation on our part in order to fill gaps in the literature.

Recall the starting point in the construction suggested by DM, namely a 2-design $\mathcal{C}$ with parameters as given in Section 7, with the property that each block of $\mathcal{C}$ is disjoint from $g^2(g+2)$ other blocks while intersecting each remaining block in exactly g points. The kernel of our narrative will be to trace the evolution of this set of properties.

12.1 Concept of a quasi-symmetric design

We adopt the definition appearing in Sec. 48 of [49], as formulated by M. S. Shrikhande: A t-(v,k,λ)-design $\mathcal{D}$ is *quasi-symmetric* (or a QSD) with intersection numbers x, y $(x < y)$ if any two blocks of $\mathcal{D}$ intersect in either x or y points.

Given a QSD $\mathcal{D}$, we define the *block graph* $\Gamma = \Gamma(\mathcal{D})$ to have vertex set the set of blocks of $\mathcal{D}$, with two vertices B_i, B_j adjacent if $|B_i \cap B_j| = y$. Of central importance to us is the fact that Γ is an SRG in this case, its parameters being readily expressible in terms of the parameters of $\mathcal{D}$. The origin of this concept goes back to S. S. Shrikhande [195] (the father of M. S. Shrikhande), but the term "quasi-symmetric" does not appear until [210] where it is coined by R. G. Stanton and J. C. Kalbfleisch. Prior to this point, the concept was further discussed in [209, 198], see also Thm. 10.3.4 of [176] and related references.

12.2 Emergence of J. J. Seidel

As was the case with Bose, J. J. Seidel started out as an expert in geometry. He submitted his first AGT-related paper [149] on December 18, 1965, however it is in Seidel's next publication [190] where one witnesses his "seduction" to AGT. (Indeed, 14 of the 15 references in [190] are linked to the notion of an SRG.) It is also apparent from [190] the degree to which Seidel was influenced by the seminal paper [18] of Bose.

The first formal link between Seidel and DM may be found in [84], where reference is made to [161] (specifically to the term "pseudo-cyclic", which would later become the subject of careful investigations by Seidel and others, cf. our discussion of [161] in Sec. 5). From this moment on, the name of DM would be imprinted on Seidel's mathematical consciousness. (Recall that a year later they would meet for the first time in Lincoln NE.)

The first proper and systematic investigation of QSDs was initiated by Seidel *et al* at approximately the dawn of the Higman-Sims era in AGT. In evident form this was accomplished in the extended abstract [85] with a much more comprehensive presentation being reached in [86]. Both texts refer to $\mathrm{SRG}(100, 22, 0, 6)$ as being "first constructed by Higman and Sims, while discovering the simple group which carries their name."

Reference is made to [101] here. (Credits in [86] are also given to two papers of Gewirtz, in particular to [81] for the proof of the uniqueness of the Higman-Sims graph. Finally, we mention [192], where once more construction is attributed to Higman and Sims and uniqueness to Gewirtz.)

Curiously, in the very same paper [86] in which Seidel accords priority to Higman and Sims for the discovery of $\mathrm{SRG}(100, 22, 0, 6)$, he cites [162] and accredits DM with construction of the graphs $NL_g(n)$ for prime powers n.

Remark 8. We are forced here to speculate on two puzzles. First, we are guessing that Seidel, upon first exposure to the remark in [162] on the existence of $NL_2(10)$, "did not believe his own eyes" and so intentionally decided to give priority for the discovery to Higman and Sims. However, there is concrete evidence that after his first personal meeting with DM during the following year his mind was forever changed: in all subsequent publications Seidel conspicuously omitted the word "first" when referencing the construction of Higman and Sims. Regardless, we can offer no insight into our second, more perplexing puzzle: why Seidel never took the opportunity in any of his future articles or public presentations to disseminate DM's discovery to the mathematical world. We fear this second puzzle may never be given a satisfactory explanation.

12.3 Influence of DM

Here we digress ever so briefly, to emphasize DM's growing influence at this time on his colleagues in the field of statistics.

In [50] W. S. Connor cites [159] as one of the motivational sources for his studies.

In [196] S. S. Shrikhande provides new proofs of certain results from [159], in addition establishing that all L_2-type graphs on 16 vertices are known.

In his seminal paper [18] R. C. Bose accredits DM [159], along with Shrikhande and Bruck, as laying the foundations for the theory of Latin square type SRGs, which strongly influenced his techniques.

In [14] one finds reference to [160, 162], as well as specific mention (pp. 365-366) of DM's main result regarding the existence of an $NL_g(g^2 + 3g)$-graph given the existence of a QSD with suitable parameters. As an example, uniqueness of the case of $NL_1(4)$ is demonstrated. In addition, further exploiting DM's techniques from [160], the authors prove nonexistence of any $\mathrm{SRG}(28, 9, 0, 4)$.

In retrospect, one would have to consider [14] to have been a very natural place to mention the result in [160] about the existence of $NL_2(10)$. Indeed, it is a pity that this was not the case. (Note that [101] was clearly out of the scope of the authors at the time their paper [14] was written.)

Finally, we draw attention to the paper [193] of Mohan S. Shrikhande, which we believe to be the earliest text in which attribution is paid to the work of DM surrounding the graph Γ. In his paper Shrikhande refers to the graphs $NL_g(g^2 + 3g)$ introduced in [160, 162], and provides a complete proof of his Theorem 2.1 in which a construction of $NL_g(g^2 + 3g)$ is described in terms of a suitable quasisymmetric design $\mathcal{D}$. More specifically, this theorem asserts that the existence of design $\mathcal{D}$ is both necessary and sufficient for the existence of $NL_g(g^2 + 3g)$.

In his concluding remarks, M. S. Shrikhande mentions only that a suitable design $\mathcal{D}$ for the case $g = 2$ had been discovered by Witt in [224] and that the resulting graph is exactly the one obtained by Higman and Sims in [101]. It is a somewhat delicate issue that while credits are given to DM for his discovery of the family $NL_g(g^2 + 3g)$ there is still no evident wording in [193] that reflects the construction of $NL_2(10)$ by DM himself. Thus, although the author MK has been carrying a copy of [193] since 1977, for the great majority of this time it was not possible for us, nor for our contemporaries, to understand from [193] a clear message about the priority of DM.

12.4 Two significant lemmas

The first lemma we wish to discuss is due to K. N. Majindar (= K. N. Majumdar), a statistician by training.

Lemma 1 (Majindar [156])**.** A given block in a BIBD with parameters v, b, k, r, λ can never have more than $b-1-\frac{(r-1)^2k}{r-\lambda-k+k\lambda}$ blocks disjoint from it. If some block has that many, then $\frac{r-\lambda-k+k\lambda}{r-1}$ is a positive integer and each of the non-disjoint blocks has $\frac{r-\lambda-k+k\lambda}{r-1}$ varieties common with it.

The proof is a brilliant half-page exercise in the art of variance counting, a main tool among statisticians. Starting from the design $\mathcal{C}$ used by DM (for which one has the parameters $(22, 77, 6, 21, 5)$), the reader may easily verify that we are getting an extremal case in the sense of Lemma 1, namely $b-1-\frac{(r-1)^2k}{r-\lambda-k+k\lambda} = 16$ and $\frac{r-\lambda-k+k\lambda}{r-1} = 2$. (This latter equation is expressed as $y = 2$ in the language of QSDs.)

This property was first observed on an empirical level by DM in his thesis [159], which predated [156] by several years. In contrast DM's Lemma 8.2 of [160] (a special case of Majindar's Lemma stated above) appeared after [156]. Yet DM makes no mention of [156] in [160]. While we may safely presume that Majindar was unaware of DM's thesis, the presumption that DM was unaware of [156] is tenuous at best. The only explanation we can offer is speculative yet entirely consistent with DM's character. We believe that DM may have felt that by referencing [156] in such close proximity to his own earlier work [159], his actions might be construed as a challenge to priority, something that DM by his very nature would never abide.

Still, it is unfortunate that only a modest number of scholars recognize the depth of Majindar's contributions (e.g., see [63, 13, 94, 112]). It is indeed a pity that for the vast majority of modern day design theorists his name remains in relative obscurity.

With or without the aid of Lemma 1, we may now better understand what was outlined in Section 7: the crucial design $\mathcal{C}$ should be a QSD with the parameters $x = 0$ and $y = g$. It is still not clear, however, why this QSD should be a 3-design. We need one more auxiliary result for this.

Lemma 2 (Cameron [39])**.** For a 2-design $\mathcal{D}$, any two of the following imply the third:

(i) $\mathcal{D}$ is a 3-design;

(ii) $\mathcal{D}$ is a QSD with $x = 0$;

(iii) $\mathcal{D}$ has $\frac{v(v-1)}{k}$ blocks.

Surprisingly, there is no proof of this result nor any attribution given in [39]. This causes us to speculate that at the time the paper was written the author regarded this result as folklore. In fact, many proofs will appear much later in diverse formulations and contexts, e.g., see [5, 155, 194, 42]. Nonetheless, our own attempts to detect in the literature any hint of Lemma 2 prior to 1973 (the year that [39] was submitted) met with utter failure. Hopefully, active players in AGT will help to shed some light on this small but intriguing puzzle.

Application of Lemma 2 gives the reader a clear picture of the critical fact that escaped DM's awareness at the time: that design $\mathcal{C}$ is a 3-design. Had DM realized this fact at any time prior to 1968 his fate might have been changed dramatically.

12.5 Extensions of designs

We discussed in Section 11 the DM-approach used in the construction of the design $\mathcal{C}$ responsible for the existence of $NL_g(g^2+3g)$. Unlike DM, Higman and Sims were granted this design for free by Witt, who obtained it from the projective plane of order 4 via the method of transitive extension. This explains why an understanding of the result of Higman and Sims promoted immediate strong interest in the direction of extensions of designs.

The main driving force in this area was P. J. Cameron, who relied on significant input from Hughes (see [107]) and Lüneburg [151]. In the short span of 2-3 years, Cameron removed all traces of mystery (at least at the level of feasible parameters) in his series of papers [36, 38, 39]. Indeed, the classical theorem of Cameron [36] describes all possible parameters for a symmetric design to admit a transitive extension, namely those which are occurring in (i) Hadamard designs, (ii) two sporadic cases on 111 and 495 points, and (iii) an infinite series of designs with $v = (\lambda+2)(\lambda^2+4\lambda+2)$ points.

The first sporadic case (projective plane of order 10) was disposed of with the aid of a computer [145], while the second sporadic case has yet to be resolved. Hadamard designs form their own classical branch of combinatorics. The remaining infinite family leads, via extension, to the 3-designs necessary for obtaining an $NL_g(g^2+3g)$-graph. If $\lambda = 1$, we start with a projective plane of order 4, and hence obtain the Witt design W_{22} upon extending. (The case $\lambda = 0$ may be regarded here as degenerate, with the extended design being the trivial design on five points.)

In fact Cameron's results may be considered in the broader context of SRGs with no triangles. (See [42] which provides a bright self-contained introduction to the subject.) The preprint [16] attempts to revive interest in this most appealing, albeit difficult, area of AGT (see also our Subsec. 13.7 below).

Finally, we mention ongoing efforts to prove nonexistence of 3-QSDs with suitable parameters, see [172, 173] as samples of this activity.

12.6 The case $g = 3$

As was mentioned by S. S. Shrikhande in [197], the case $g = 3$ provides the next possibility to construct an SRG of NL_g-type from the DM-family. Here one needs a biplane on 56 points (i.e., $k = 11$) which extends to a 57-point 3-design. We refer the reader to [109, 37] for a helpful introduction to biplanes.

For the longest time only four biplanes on 56 points were known (e.g., see [184]) until a fifth one was discovered in [120]. About three years later, it was asserted that no biplane on 56 points could be extended to a 3-design (see [6], as well as the corrigendum in [7] by the same author). The amended proof was believed to be correct, e.g., see [42], until a fatal flaw was detected by A. E. Brouwer, see [128] for details. Finally, all matters were put to rest by an exhaustive computer search (316 machines running in parallel for two months), and the results established in [127] as follows: The five known biplanes on 56 points comprise a complete listing of such objects, and none admits an extension. In particular, there is no $NL_3(18)$ making $g = 4$ the smallest unresolved case.

It is quite surprising that the particular result about nonexistence of the graph $NL_3(18)$ on 324 points was obtained by A. Gavrilyuk and A. Makhnev in [79] on a purely theoretical level, prior to the appearance of [127]. Both texts are using a different terminology, and neither refers to the work of DM.

The reader is welcome to approach the case $NL_4(28)$ by starting from a suitable symmetric BIBD on 115 points.

13 A more wide panorama

The initial goal of this project was to provide a reasonably self-contained historic narrative that focused on the origins of $\mathrm{SRG}(100, 22, 0, 6)$ and emphasized the important “hidden” contributions of Dale Marsh Mesner within these frames. It was only upon extensive investigation of the literature that we came to realize, and appreciate, the true scope of the task we were undertaking. Many surprises slowly revealed themselves to us, and due to their depth and sheer number we were forced many times to redefine our objectives. All the while, we felt a strong obligation to provide the reader with a full panoramic view of those events which were to play such an important role in the future development of AGT.

As a consequence, the purpose of this final section is to house all extra material that in our judgment may have created too burdensome a load on our earlier exposition. Of course, the reader has complete freedom to decide what will be his/her level of comprehension in pursuing various portions of this surplus material. Any attempt to synthesize new scientific leads, based on an appreciation of the ideas of our accomplished predecessors, will be sufficient reward for the authors.

13.1 Scientific ingredients

All facets of research relevant to the discovery of $NL_2(10)$ and its related structures, including investigations, interpretations and consequent applications, can only be adequately described in a broad interdisciplinary framework. In this fashion we distinguish eight different branches, which, although clearly not existing in full isolation, may be characterized separately. To each such branch we devote a few sentences, and suggest a couple of noteworthy texts that will hopefully allow the interested reader to make a few independent steps in a new direction. In each case the texts have been chosen to complement one another in terms of style, time of publication, intended audience, and presumed background.

- *Design of experiments.* The initial sentences of [11] explain the role of combinatorial designs in helping statisticians to answer two questions:

  (i) What is the best way of choosing subsets of treatments to allocate to the blocks, given the resource constraints?

  (ii) How should the data from the experiment be analyzed?

  R. C. Bose and his collaborators at the Calcutta Institute of Statistics were a driving force in this area. The book [176] remains today a valuable comprehensive source of diverse examples accumulated over decades of development of the subject. In contrast, [11] offers a more fresh approach from an author with great pedagogical skills. Its aim is successfully reached: to give the reader a quicker and more accessible entrance into the subject.

- *Permutation group theory.* We were fortunate to witness that DM was fond of permutation groups, though he claimed to be an amateur in this area. Oppositely, Higman and Sims belonged to a handful of experts who creatively shaped this part of group theory. The texts [64, 41] reflect the views of evident leaders in this area, although their target audiences and expository styles are quite different. Of course one should also keep in mind the short pioneering book [220] of Wielandt.

- *Finite geometries.* The reader has already been made aware of the classic text [63]. A trilogy by Hirschfeld and Thas (covering projective geometries, projective spaces and general Galois geometries) is an unusually robust work housing the accumulated treasures of thousands of researchers. Here we refer to the particular volume [103] of the trilogy that treats the Mathieu groups and Witt designs.

- *Algebraic graph theory.* The name AGT was coined by Biggs [15] in 1974. In fact, the techniques exploited in AGT are so wide that one could justifiably consider AGT to be a collection of diverse tools, ideas, and even philosophy encompassing many adjacent mathematical areas with the common goal of enumerating and classifying graphs with symmetry-related properties. Our own experience suggests that the text [83] may be enjoyed by a wide spectrum of audiences ranging from university students to specialists. Recently published introductory lecture notes [130] might also be helpful for the beginning reader.
- *Design theory.* The text [109] is a friendly place to start, although by no means is it trivial or easy. Of a more comprehensive nature is [13], especially the section entitled "The Higman-Sims group" (pp. 230-236), which is an absolute gem in our eyes.
- *Combinatorial matrix and spectral graph theory.* Variations in the name reflect variations of tools mobilized from linear algebra to investigate symmetry of graphs. Clearly DM was very strong in this area, belonging to the cohort of statisticians who created its modern theory and applications. Links of Higman and Sims to this area were of a more exotic nature, via that branch of representation theory treating centralizer algebras of permutation groups. The brief introduction to SRGs found in Sec. 5.2 of [28], as well as the attention paid to Higman in [59], should immediately convince the reader of the relevance of both sources. Finally, we again mention [26].
- *Coherent configurations and association schemes.* This branch is commonly referred to as "algebraic combinatorics" as coined by E. Bannai and T. Ito in [9]. However, in more recent times the term has grown to incorporate a much wider range of combinatorial areas. Nonetheless, our attention is dogmatically restricted to the line pursued in [9], reflecting those objects (association schemes) developed by R. C. Bose *et al* for the purpose of experimental design. The more general notion of a coherent configuration was developed mainly through the efforts of D. G. Higman, although a similar concept carrying the name of "cellular algebra" can be traced to an even earlier period in Moscow (B. Weisfeiler and A. Leman). At present, a comprehensive treatment of coherent configurations has yet to be written. (Indeed, among widely accepted textbooks it seems that only [41, Chapter 3] attempts such a treatment.) Nevertheless, the lecture notes [100] of Higman serve as one of the most serious introductions to the subject.
- *Diagram geometries.* Of all branches related to our paper either explicitly or implicitly, this is clearly the most modern. Early publications of F. Buekenhout served as an initial impetus. Presentations in [170, 113] provide a wide background in which our central design W_{22} is just one of hundreds of illuminating examples. To give the reader just a small taste of the exposition, we refer to [113, Lemma 4.10.17 (p.189)] which simultaneously serves as a definition of the Higman-Sims group. Specifically, it appears as the stabilizer of a suitable induced subgraph of $\overline{\Lambda}_3$, which in turn is defined in terms of the famous Leech lattice Λ.

13.2 More on historical origins

In their initial role, association schemes served as auxiliary objects, subsidiary to the investigation of PBIBDs. The latter were a natural generalization of BIBDs, formulated as applicative tools for the study of statistical design of experiments. At exactly this time the creation of catalogues such as [46] emerged, starting what would become a well established tradition in modern combinatorics. This flavor is already observed in [159], which focuses much of its attention on PBIBDs. In contrast, the seminal paper [19] is the first to treat association schemes as objects of independent interest, a vision that would be supported and strengthened in each successive publication of DM.

As previously mentioned, two influential creators of the theory of SRGs, Bose and Seidel, both entered into combinatorics by way of geometry. In fact, attention to certain types of combinatorial structures was already an established tradition in 19$^{\text{th}}$ Century algebraic geometry; indeed, our favorite example of the Clebsch graph was just one of many attractive objects inherited from this “old-fashioned” field. The ability to analyze such structures by means of their coordinate presentations, and to describe their internal symmetries in strict group theoretic terminology, was a clear advantage at the dawn of modern combinatorics and AGT.

Seeing great potential in this geometric–combinatorial synergy, more and more experts in algebraic geometry shifted their attention to finite geometries, simultaneously honing their skills in group theory. (We cite J. A. Todd and W. L. Edge as two striking examples.) By the 1960s, especially due to the efforts of Reinhold Baer and his followers, the marriage between group theory and geometry was already well established. This was a great advantage to experts like Higman and Sims, however its impact on DM would not be felt for many years.

Starting from the 1950s, permutation group theory was enjoying a revival through the efforts of H. Wielandt and other followers of Issai Schur. This approach was enthusiastically picked up by a younger generation of researchers, with Higman and Sims serving as bright leaders. The publications [98, 99, 199] served to strongly fertilize the ground on the edge between combinatorics and group theory. This explains how, miraculously, an exotic flower came to bloom on September 3, 1967 from the seeds sown by two colleagues on just the previous evening.

13.3 Kummer’s quartic surface

This is in fact the title of a book [106] written at the turn of the century and reprinted in 1990. The main subject is Kummer’s 16_6 configuration (having today a few alternate names), which had already attracted the attention of Camille Jordan who viewed it as an incidence structure, enjoying the 2-transitive action of its automorphism group on 16 points.

Needless to say, the “nicest biplane on 16 points” discussed in Sec. 11 is none other than 16_6. Correspondingly, the substructures forming our sets $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$ in Subsec. 11.5 were known to algebraic geometers already at the time of Ronald W. H. T. Hudson.

Despite the fact that algebraic geometry has experienced a couple of revolutionary changes in paradigms throughout the years, one may still observe how the 16_6 configuration serves as an inspiration to each new generation of algebraic geometers. Occasionally, it even stimulates the modern researcher to take a trip into an unfamiliar past in order to observe the beauty and symmetry of the structure in its more natural historic habitat. However, we recognize the hazards of such a journey, particularly when a deeper comprehension is desired. Thus we just supply the reader with some guidelines. For an initial exposure to the language of linear complexes, associated congruences, and apolar pentagons we refer to [122] with credits to [68] and [55]. Some additional papers of Edge may also be helpful, viz. [65, 66, 67, 70], however the main ingredient will be perseverance on the part of the individual who decides to brave the journey.

13.4 Six levels of description of $NL_2(10)$

The graph $\Gamma = NL_2(10)$ is a rank 3 graph, in other words, its automorphism group acts transitively on three naturally defined sets (vertices, oriented edges, and oriented non-edges). Nowadays, with the aid of the classification of finite simple groups (CFSG), all rank 3 graphs are characterized, e.g., see the references in [42].

Clearly all rank 3 graphs are SRGs but not conversely. The smallest SRG that is not a rank 3 graph occurs on 16 vertices and is commonly called the *Shrikhande graph.* (In fact, this is exactly the graph appearing in our Example 3 of Section 2.) The Shrikhande graph already shows up in DM's thesis [159]. See also a very nice depiction of the Shrikhande graph on the cover of the text [28].

Based on CFSG, one splits all rank 3 graphs into two categories: classical and sporadic. The classical ones may be described in terms of geometries over finite fields. All requisite information in this case may be found in the framework of geometric algebra, see [1, 212, 90] which collectively reflect the evolution of this subject over the last half-century.

The situation for sporadic rank 3 graphs is more sophisticated. Such objects generally arise via *ad hoc* constructions, and consequently they may be viewed at many different levels. Unlike the case for classical graphs, a complete comprehension is truly achieved only by constantly adjusting our looking glass, striving to uncover hidden secrets at each successive stage. We illustrate this phenomenon below for the graph Γ, imagining that we are honing in on its "dwelling" with the aid of a very powerful telescopic lens.

- *Leech lattice (cosmic view).* There are a lot of diverse texts to help one become acquainted with this object. For example, [181] is intended for a wide audience while [54] is a comprehensive source. The short note [222] provides a fresh elementary perspective.

  The Leech lattice was constructed by John Leech in 1965 (see [147]), and although he is often given credit for its discovery much earlier traces may be observed. Here, we again refer to Ernst Witt, this time to [225, p. 328], splitting our feelings about this monumental mathematician between deep admiration and bitter confusion (e.g., see [181, pp. 131-133]).

  The Leech lattice Λ arises in connection with the sphere-packing problem, admitting the densest packing of non-overlapping identical spheres in 24-space with centers at lattice points, see [48]. Its importance is also felt in group theory: $Aut(\Lambda)$ is the double cover of Conway's largest sporadic simple group Co_1 [53], and it contains many other sporadics, including the Higman-Sims group, occurring as stabilizers of various configurations of its vectors. Moreover, the Griess algebra, which has the Monster group as its automorphism group [89], can be constructed by means of compactifying a certain vertex algebra (that describing bosonic string theory) on the 24-dimensional quotient torus $\mathbb{R}^{24}/\Lambda$.

  The aforementioned induced subgraph of $\overline{\Lambda}_3$ in Subsection 13.1 is none other than our graph Γ. Note as well that Conway in his classic paper [52] provides an embedding of Γ into Λ.

  At this point, the Witt design W_{22} is barely visible to us. Thus we must increase its clarity in order to render more revealing views of Γ:

- *Binary Golay code (view from a mountain).* Binary and ternary Golay codes are related to Witt designs as well as the Mathieu groups. The codes were described in the one-page note [87] of M. J. E. Golay. We are here interested in the extended binary $[24, 12, 8]$-code $\mathfrak{G}_{24}$, e.g., see [42]. The group $Aut(\mathfrak{G}_{24}) \cong M_{24}$ acts transitively on codewords of weight 8, with corresponding orbit the block set of the Witt design W_{24}. Note that in a certain natural sense $\mathfrak{G}_{24}$ lives inside the Leech lattice Λ, while M_{24} lives inside Co_1.

- *Witt design W_{24} (view from a hill).* The largest Witt design W_{24} is none other than the Steiner system $S(5, 8, 24)$ with automorphism group M_{24}. As was discussed, its discovery (along with that of W_{12}) is generally accredited to Witt [223] but can be traced to the quite earlier little known paper [43] of Carmichael.

  The literature on these objects is very rich, e.g., see the legendary text [57] of R. Curtis, as well as a more recent self-contained elementary treatment [115] due to S. Iwasaki.

- *Witt design W_{22} (street-level view).* We have reached the point of absolute clarity, that of the natural embedding of W_{22} in W_{24}. It is little wonder that this is the view that fueled two independent approaches to Γ, one by Higman and Sims the other by DM. See the expository paper [12] which pays special attention to the nature of this embedding.

  However, we wish an even closer look, so we enter the dwelling:

- *Projective plane (view from the basement).* For most authors, the most natural way to arrive at W_{22} is to add a point to the projective plane $PG(2, 4)$ and use the extension procedure. This was a paradigm most clear to Higman and Sims, inherited from Witt. It was by no means the approach taken by DM.

- *The nicest biplane (ascending the staircase).* The entirety of Section 11 was devoted to this methodological device. Below we will compare it to its underground alternative, the projective plane $PG(2, 4)$ as a possible starting point. Our ultimate desire is to synthesize these two approaches.

Remark 9. There are some very interesting alternate views of $\Gamma = NL_2(10)$ and its substructures not touched upon in our treatment. A non-standard construction of Γ may be found in [154]. In [185] one encounters relevance of the non-edge decomposition of Γ in a fresh context. In [217] an embedding of W_{22} into a symmetric design on 78 points is examined. A characterization of W_{22} in terms of QSDs is given in [35]. Finally each of [129, 31] provides a construction of the Gewirtz graph; the first focuses as well on the underlying BIBD, while the second is of a decidedly more geometric flavor.

13.5 A methodological amalgam of two approaches

In this section we share with the reader our vision of the pros and cons of two independent approaches to the graph $\Gamma = NL_2(10)$, namely that of Higman-Sims and the one of DM. We next "glue" them together to form a methodological amalgam which is based quite literally on the notion of amalgam from group theory.

Transitivity is the central concept that underlies the notion of a rank 3 graph. It was a main driving force behind interest in SRGs at the dawn of CFSG. Indeed, each new rank 3 graph had the potential to lead to the discovery of a new sporadic simple group, or to at least produce a new action of a known group, thereby providing the possibility of a new computer-free construction.

The transitivity paradigm was exploited in [101] wherein the authors were building none other than a rank 3 graph. This approach was also followed by DM, however it was done so in the absence of adequate descriptive terminology. As was previously mentioned, the proof of uniqueness in [160] implies vertex transitivity of Γ (at the time DM did not know of such a term, nor did he care to use it anyway). In addition, DM's observation on p. 81 of [160] about "7700 incidence matrices of the symmetric design with $r = 6$, $\lambda = 2$" can be interpreted in only one way: $Aut(\Gamma)$ acts transitively on all oriented non-edges.

Recall that in [101] generators of $Aut(\Gamma)$ were obtained by adjoining a fixed-point-free involution t to the generators of a point stabilizer, aka $Aut(W_{22})$. This is a familiar setting in permutation group theory. On a naive level, its roots can be traced back to W. A. Manning (1921). Namely, let Δ be a connected vertex-transitive graph, G its automorphism group, $\{a, b\}$ an edge of Δ, $H = G_a$ the stabilizer in G of a, t a permutation that reverses orientation of $\{a, b\}$, e.g., see [82]. Note that this setting applies equally well to non-edges of Δ provided its complementary graph $\overline{\Delta}$ is assumed to be connected. (Indeed, non-edges of Δ are edges of $\overline{\Delta}$, and one has $Aut(\overline{\Delta}) = Aut(\Delta)$.)

A more refined formulation of the above is the following: $G = \langle G_a, G_{\{a,b\}}\rangle$ and $G_a \cap G_{\{a,b\}} = G_{a,b}$. Here one speaks of a triple of subgroups of G which forms an amalgam, more specifically the *amalgam of* G_a *with* $G_{\{a,b\}}$ *over* $G_{a,b}$.

Nowadays, the notion of an amalgam plays a crucial role on the edge between group theory and diagram geometries. The above defined amalgam of the vertex stabilizer and edge stabilizer of a connected vertex transitive graph is one of the simplest illustrations of this fruitful concept. The interested reader is referred to [114], which additionally provides interesting information about amalgams related to M_{22}.

Returning to Section 6, we apply the notion of amalgam to the Clebsch graph $\square_5$ (the simplest example among all NL_g-graphs). In this case we have: $G = Aut(\square_5) \cong E_{16} \rtimes S_5$, $\{a, b\}$ a non-edge of $\square_5$ (hence an edge of the complementary graph $\overline{\square}_5$), $G_a \cong S_5$, $G_{\{a,b\}} \cong S_2 \times D_6$, and $G_{a,b} \cong D_6$.

The next simplest case of an NL_g-graph, and the one most relevant to our exposition, is $\Gamma = NL_2(10)$. In the language of amalgams we obtain: $G = Aut(\Gamma) = Aut(HS) \cong HS \rtimes 2$, $G_a \cong Aut(M_{22}) = Aut(W_{22})$, $G_{\{a,b\}} \cong E_{32} \rtimes S_6$, and $G_{a,b} \cong E_{16} \rtimes S_6$.

In the presentation of Higman and Sims [101] only parts of this amalgam are visible, namely the entire group G, the stabilizer G_a and an involution $t \in G_{\{a,b\}} \setminus G_{a,b}$. What can be said about DM's presentation in [160]?

At first sight groups are not even visible in [160]. However, let us switch to relational language, which was adopted by DM before his counterparts. A correct formulation here is provided by the use of a Galois correspondence between relational structures and permutation groups, as described in [73]. In this context, the role of Galois-closed objects is fulfilled by so-called "Schurian configurations" (i.e., coherent configurations the relations of which are the 2-orbits of a suitable permutation group).

We now get a dual incarnation of the above amalgam. Starting with the Schurian configuration $\mathcal{W} = (\Omega, \text{2-orb}(G_{a,b}, \Omega))$, we merge its two subconfigurations

$$\mathcal{W}_1 = (\Omega, \text{2-orb}(G_{\{a,b\}}, \Omega)) \quad \text{and} \quad \mathcal{W}_2 = (\Omega, \text{2-orb}(G_a, \Omega)).$$

Then $\mathcal{M} = \mathcal{W}_1 \cap \mathcal{W}_2$ is none other than the $NL_2(10)$-association scheme the classes of which are the graphs Γ and $\overline{\Gamma}$. To get a closer picture, let us refer to the intersection diagram in Fig. 8 of Subsection 11.6, which is just a compact way of viewing $\mathcal{W}$. Here, in the role of a, b we have P^*, B^* respectively. Loosely speaking, DM establishes that $\mathcal{W}$ corresponds to the nicest biplane $\mathcal{D}$, $\mathcal{W}_1$ reflects the fact that $\mathcal{D}$ is self-dual, $\mathcal{W}_2$ is providing incidence of points and blocks in $\mathcal{C} = W_{22}$, and $\mathcal{M}$ is giving the entire graph $\Gamma = NL_2(10)$. This embodies the concept of an amalgam in pure relational language.

To summarize, in [101] one sees groups without an amalgam, while in [160] one sees an amalgam without groups. This perfectly explains what we mean by "methodological amalgam" of two approaches. We are here witnessing a real impact of the ideas of

DM with those of Higman and Sims, which converge to the notion of a mathematical amalgam.

Indeed, in Higman and Sims we have two established experts in the use of groups. Higman was, in fact, the creator of the language of coherent configurations, On the other hand, DM was one of the first experts who established new standards in the use of association schemes, a particular case of coherent configurations. Though in 1964 DM was still not aware of the notion of a coherent configuration, he was actually operating with its matrix theoretic analogue: a stable color graph. In fact, this latter terminology would not be coined for a few more years, see [219].

A rigorous analogue of the dual amalgam described above may be reflected in the procedure of transitive extension formulated in pure relational terms. One of the evident advantages of this approach is that unlike what occurs in the group case, the resulting object need not be a rank 3 graph. This of course opens the door for new discoveries. Different roots of such a procedure were developed and analyzed by DM and the present authors in [132], though work at the final stages of this paper was sadly interrupted by Dale's untimely death.

We conclude the established virtual posthumous handshake between Mesner and Higman with mixed feelings: regret that it never happened in real life but satisfaction that they are forever bonded by the threads of their collective genius.

13.6 Beyond the Higman-Sims graph

The graph $\Gamma = NL_2(10)$, also commonly denoted SRG$(100, 22, 0, 6)$, is forever historically linked to the names Higman and Sims. For the balance of this subsection we adopt this established tradition.

The Higman-Sims graph has many exceptional properties which stress its unique features as well as those of its substructures. Below we present four of what we consider to be the most striking examples.

- *Two copies of the Hoffman-Singleton graph as an induced subgraph of* $\mathbf{\Gamma}$. We denote by $HoSi$ the unique SRG$(50, 7, 0, 1)$ discovered in [105]. As well, it is a rank 3 graph and in fact the largest known Moore graph.

  It turns out that the vertex set of Γ may be partitioned into two halves such that the subgraph induced on each half is $HoSi$. This occurrence is closely related to the existence of the unique bipartite distance-regular graph Δ on 100 vertices having diameter 4 and valency 15. Merging classes of valency 15 and 7 in the metric association scheme generated by Δ yields Γ. The idea of such a decomposition was suggested by Sims in [200]. See also [27, 24, 94] for more details.

- *Highly transitive finite geometric lattices.* In [61], A. Delandtsheer was investigating geometric lattices of dimension $n \geq 3$ such that the automorphism group of the lattice acts transitively on unordered pairs of secant hyperplanes. With the exception of the evident classes (Boolean lattices, affine planes, and projective spaces) there appears only one real surprise: the planar space obtained in a natural way from the Steiner system $S(3, 6, 22)$. This generalizes an earlier result of W. M. Kantor with stronger assumptions [126].

- *Spin models.* Roughly two decades ago, the (late) mathematician F. Jaeger made a breakthrough in the theory of link invariants (see the classic paper [121] of V. R. F. Jones for definitions) by demonstrating how one could produce spin models from the Bose-Mesner algebras of formally self-dual association schemes. In particular, Jaeger constructed a new spin model from the Higman-Sims graph, see [116].[4]

- *Energy minimizing point configurations.* Over the past decade, there has been an explosion of interest in spherical point configurations that minimize potential energy. Motivation here stems from physics, discrete geometry and combinatorics. Once again the graph Γ appears, this time as a naturally defined object on 100 points of the 22-dimensional sphere, see [8].

13.7 SRGs with no triangles

Investigations of DM touched upon two families of SRGs: those of NL_g-type and those without triangles. Graph $NL_2(10)$ appears in the intersection of these two families. While there are known infinite series of NL_g-graphs, very few primitive triangle-free SRGs (briefly, tf-SRGs) have ever been constructed.

One of the impressive achievements of [159] is the construction of Table II (pp. 257-259), which gives feasible parameters for all putative SRGs on 100 or fewer vertices. Among the 101 listed parameter sets, only 9 satisfy $\lambda = 0$ (the triangle-free condition for SRGs). The first crucial input of DM was the construction of tf-SRGs on 77 and 100 vertices. In addition, two parameter sets on 28 and 64 vertices were excluded. (Note that the existence of tf-SRGs on 50 and 56 vertices was totally unknown to DM at this time, as well as to all other experts.) Such exclusion was achieved by first establishing necessary conditions for the existence of a tf-SRG based on the existence of two related BIBDs with certain properties (Theorem 2.6 of [159]), and next proving nonexistence of said designs. To do this DM used *ad hoc* tricks in variance counting to show that the value of variance should be negative. Recall that the modern way to exclude such parameter sets (e.g., see Sec. VII.11 of [49] by Brouwer) is to show that one or more Krein parameters must be negative. An interesting enterprise would be to compare the power of old and new techniques on a wider sample of feasible sets, cf. [16].

In fact, a wider family of graphs to which Γ belongs is based on a consideration of Krein parameters. We speak now of *Smith graphs*, which are primitive SRGs that meet the Krein bound, see [42]. The name refers to M. Smith, who in [202] established a two-parameter family of putative rank 3 graphs with extremal properties.

A characteristic feature of Smith graphs is that for each vertex x their first subconstituent (i.e., subgraph induced on the neighbors of x) and second subconstituent (i.e., subgraph induced on the non-neighbors of x) are both SRGs. These remarkable graphs are called 3-tuple regular in [42]; an alternate terminology used by the present authors is 3-isoregular, see [180]. In turn, the concept generalizes to k-isoregular graphs, $k \geq 2$. All 5-isoregular graphs have been classified, e.g., see [42]. A highly nontrivial result is the characterization of all feasible parameters for putative 4-isoregular graphs. In the primitive case, we get only the pentagon, the line graph $L_2(3)$, or an extremal Smith graph, see [32, 40, 42, 137].

Returning to our main discussion, we refer to [42] as a comprehensive source of information for tf-SRGs. Valuable new input is provided in [16], where Biggs gives a list of surviving feasible sets for tf-SRGs on at most 1000 vertices (there are 21 such sets), as well as a larger list of possibilities on at most 6025 vertices.

Regarding constructed tf-SRGs, our current state of knowledge surprisingly coincides with what was known in 1968, nothing more. There are just seven known tf-SRGs, each uniquely determined by its parameters. We list these as $(v, k) = (5, 2)$, $(10, 3)$, $(16, 5)$, $(50, 7)$, $(56, 10)$, $(77, 16)$, $(100, 22)$ where, as usual, v is number of vertices and k is valency.

[4]Note that DM was already aware of this self-dual property of the $NL_2(10)$-association scheme at the time of his thesis; see [24, pp. 68-71] for a self-contained treatment of this concept.

The tf-SRG with parameters $(56, 10, 0, 2)$ is known as the Sims-Gewirtz graph. Traditionally, credits are given to [200, 80], see also [81], though the most accurate attribution would be to an unpublished text of Sims. There are many nice descriptions of this graph (see [23]), in particular it is a subgraph of $\Gamma = NL_2(10)$. To observe this, one need only consider the edge decomposition of Γ, exactly like the non-edge decomposition described in Section 11. The Sims-Gewirtz graph then appears as the induced graph on the set of vertices nonadjacent to both vertices of a selected edge.

As a result, we may now make a rather striking observation:

Every known tf-SRG is a subgraph of Γ.

This raises an intriguing question: Can a tf-SRG exist independent of Γ? In our eyes, this is one of the more important and challenging open problems in modern AGT. (Note that the existence of a putative Moore graph of valency 57 would resolve this issue since such a graph would not embed in Γ.)

A local approach to triangle free NL_g-graphs was, as far as we know, first developed by DM. Here, we refer to a way of describing the entire graph in terms of its local structure with respect to an arbitrary vertex x. The first and second subconstituents of x, and all remaining adjacencies, may be described with the aid of an auxiliary QSD which turns out to be a 3-design. Such an approach does not require that the resulting graph be vertex transitive.

Further "localization" was also outlined by DM in terms of a non-edge decomposition, as was interpreted by us in Section 11. It is open to speculation as to why DM was so insistent upon preferring a non-edge decomposition to an edge decomposition. Indeed, one can even point to a small forfeiture of this behavior: DM failed to discover SRG$(56, 10, 0, 2)$ (#39 in his table of feasible parameter sets) which is immediately visible from the perspective of an edge decomposition.

A modern scholar may find an intriguing interpretation of DM's preference. A brief but striking outline of what could legitimately be called "local theory of suitable SRGs" (the Clebsch graph being one of these) is provided in Sec. 10.6 of [83], entitled "Local eigenvalues". There one finds links between the eigenspaces of a putative SRG with those of its subconstituents. The origins of this local theory go back to [77, 78, 92, 74]. In particular, it is known that for certain SRGs the second subconstituent should be distance-regular. This is fulfilled for every tf-SRG, moreover the distance-regular graph in this case has diameter at most 3, see [16]. This modern extension of the original local DM-theory opens new horizons for research in this area.

The first open case is SRG$(162, 21, 0, 3)$. Interesting information about such potential graphs, related to possible order and structure of their automorphism groups, is provided in [157]. In our eyes, existence of SRG$(162, 21, 0, 3)$ is a difficult though far from hopeless problem. A more ambitious stream of hope and challenge stems from the paper [152].

13.8 Maturation of ideas

This section contains a blend of material with one unifying feature: all presented notions and ideas, alterations to language and terminology, shifts in perspectives and paradigms can be in some way traced to seeds planted by DM, Higman and Sims, and their mathematical predecessors. As mentioned earlier in the context of algebraic geometry, there are periods when a field undergoes dramatic change due to turbulent forces both internal and external. The same sentiment applies to modern combinatorics and group theory. We here focus on the relative historic positioning of DM and Higman and Sims in the midst of such changes.

- *Graphs*. As already mentioned, DM never used the notion of a graph in his texts [159, 160] as an analogue of a class of an association scheme. Instead he operated strictly within the confines and terminology of association schemes, its classes and related PBIBDs. What is somewhat surprising is that one finds on p. 213 of [159] evidence of his familiarity with the term; indeed DM there lists all 11 graphs on four vertices, referring, in particular, to [182].

  A similar remark applies to the initial papers of Higman. Nowhere in [98], which establishes a theory of rank 3 permutation groups, does the explicit notion of a graph appear. Instead Higman operates at the level of incidence structures attributed to a permutation group (G, Ω): points are elements of Ω, while blocks are suborbits of G (i.e., orbits with respect to a stabilizer G_a, $a \in \Omega$). The seminal paper [18] of Bose marks a definite change in paradigm. From this moment on, graphs (and SRGs in particular) begin to gain acceptance in design of experiments, finite geometries and group theory, though the process is still gradual.

- *Orbitals*. Given a transitive permutation group (G, Ω) one can consider all directed graphs on Ω (without loops) on which G is acting both vertex transitively and arc transitively. Such graphs (more correctly, their arc sets) are traditionally called orbitals. Note that the number of orbitals will be $r - 1$ where r is the rank of (G, Ω). The term orbital was suggested by Sims [199], and is commonly used in modern literature, e.g., see [26, 146].

  The authors usually prefer the more universal terminology of k-orbits suggested by H. Wielandt in [221]. It is applicable to arbitrary permutation groups (G, Ω) acting on Ω^k (not just transitive permutation groups (G, Ω) acting on Ω^2). The texts [125, 73, 131, 132] adequately demonstrate the advantages of such terminology.

- *Schur rings*. Also called S-rings in the literature, this is a concept that goes back to I. Schur [187], its name attributed to Wielandt. The texts [220, 188, 143, 211] provide classical foundations for S-rings and their applications to group theory, see also [166] for combinatorial applications. In the terminology of association schemes, S-rings are sometimes called translation schemes (see [24]), that is, association schemes which admit a regular (transitive) subgroup of the full automorphism group.

  A thoughtful acquaintance with the early work of DM [159, 160] extends our understanding of combinatorial applications of S-rings. Indeed, in [160, Sec. 4] one finds evident seeds of a theory that is equivalent to the elementary use of such objects. Credits are given there to [20, 207]. These links warrant a more careful and thorough examination.

- *Diagram geometries*. A number of results in this area are based on the exceptional properties of the graph Γ and the Higman-Sims group HS, and likewise for the design-group tandem W_{22} and M_{22}. Examples of such geometries are provided in [228, 171, 148]. In particular, the diagram discussed in [148, Sec. 2] is closely related to DM's non-edge decomposition of Γ. Earlier discussions of this same diagram can be traced to [33, 108].

- *Negative Latin square graphs*. This was a favorite topic of DM, dating back to his 1956 thesis. DM's work on NL_g-graphs with $\lambda = 0$ has already been discussed in the context of triangle-free SRGs, so here we may speak less restrictively.

  Construction of an SRG$(81, 20, 1, 6)$ was presented in Sec. 3.2 of [159] (#68, in DM's table of feasible parameters). Later this graph was rediscovered a few times in diverse settings. A proof of its uniqueness and detailed information about its structure may be found in [25].

The case $NL_2(6)$ with corresponding parameters $(36, 14, 4, 6)$ marks the smallest NL_g-graph the existence of which DM was unable to settle. Nowadays, all such graphs have been classified with the aid of a computer. There are an astonishing 180 $NL_2(6)$-graphs in total, of which only one is a rank 3 graph. Of the remaining 179 graphs, only three additional ones satisfy the 4-vertex condition in the sense of Hestenes and Higman [97]. These four graphs were the subject of careful investigation in [133], where edge decompositions for all graphs were described. Some similarities to the DM-approach, though unknown to the authors at the time, support hope that further investigations of these and larger NL_g-graphs satisfying the 4-vertex condition may provide new insights into SRGs with high combinatorial symmetry.

Motivated by chemistry, I. Gutman formulated in [91] a notion of "energy of a graph". It turns out that the parameters of certain NL_g-graphs form a concrete family of putative SRGs of maximal energy. At the level of parameters, such graphs are characterized in [93] and are seen to be equivalent to a certain class of Hadamard matrices. In particular, this is relevant to the case $NL_4(10)$ (#100 in DM's table). The first five examples of 100-vertex graphs with maximal energy were constructed in [123] via the technique of switching in SRGs.

13.9 DM as a mathematician

For the first part of his career, DM should definitely be regarded as a statistician whose main area of expertise was design of experiments with clever use of association schemes. Formally, DM was never a student of Bose, however the Bose school (regarded in a very wide sense) created for DM an environment conducive to scientific exploration and collaboration through a healthy exchange of ideas. DM's famous contribution [19], which is one of the frequently cited texts in modern AGT, is in fact the product of such fruitful collaboration.

The fact that by its very definition an association scheme appears in conjunction with a suitable PBIBD created a permanent geometric vision for leading experts in experimental design. The approach to SRGs suggested in [18] had a heavily geometric flavor from the very beginning: the most interesting and significant SRGs are those arising as point graphs of incidence structures. This point of view was adopted by DM to the fullest extent. Each SRG considered by him was the subject of immediate geometric interpretation. As a consequence, DM's thesis [159] is filled with consideration of PBIBDs. As a typical example, the lattice square graph $L_2(6)$ on 36 vertices is accompanied by a list of 16 feasible parameter sets for its corresponding PBIBD, 10 of which are supported by actual constructions.

By 1964, the concept of isomorphism of combinatorial structures was available to DM but only on a rather empirical level. For example, in the formulation of Theorem 8.7 in [160] DM writes that the $NL_2(10)$ scheme is unique up to "permutation of objects". The concept of an automorphism group is never discussed by him explicitly, although, as previously mentioned, there is evidence that he understood in naive terms that $Aut(\Gamma)$ is acting transitively on the vertex set. In our eyes, this is the main reason why the Higman-Sims group was not discovered in either of [159, 160]. One can indeed agree with the remark on p. 139 of [181] that DM "laboured for years" to discover his graph Γ. However this discovery came at a time when the scientific community was not yet prepared for the rapidly approaching explosion of attention to sporadic simple groups.

No doubt, in a practical sense DM made the best of what was available to him at the time.

At the next stages of his career DM became quite comfortable with the concept of a group. For example, the notion of Kramer-Mesner matrices has no meaning if not preceded by a group.

Below we share with the reader an excerpt from one of Dale's letters to us, dated August 25, 2007. It was written during the preparation of our paper [131], and fairly well describes his vision of the extension procedure:

> "If such a group was available I was happy to use it to construct the [association scheme on triples] but to me the group action was only a means, not an end. Bhattacharya had a stronger background in group theory ... and may have given more thoughts to transitive groups, but I don't think it showed up in our 1990 or 1994 papers ..."

Here DM is referring to the two joint papers [163, 164] with P. Bhattacharya.

This is a confession of great significance. The process outlined by DM in [159], and rigorously described in [160], is the typical procedure of combinatorial extension. In principle, it doesn't depend at all on the knowledge of a group, although the use of a known group would drastically reduce the size of the search space.

At a time when all rank 3 graphs are known courtesy of CFSG, the real challenge for new discoveries starts when transitivity assumptions are dropped. In this fashion the DM-approach, as it resurfaces after 50 years in the shadows, still remains fresh, significant, and well arranged algorithmically. Of course a modern researcher may derive even stronger benefits by interpreting the DM-approach in the frames of coherent configurations, thus enriching DM's vision with Higman's formalism.

Last but not least, we come to DM's prophetic understanding of a dual association scheme. This concept was formalized in the 1973 thesis of Delsarte [62], yet its practical use requires only an acquaintance with spectral invariants of commutative association schemes, a favorite ingredient of DM's methodology as far back as 1956 [159]. As a consequence, many traces of duality appear in his text both implicitly and explicitly. In particular, connections and distinctions between variance counting and the Krein bound are still awaiting careful clarification by modern researchers.

14 About our project

14.1 Evolution of the project

Both authors were actively working with the graph $\Gamma = NL_2(10)$ before realizing its evident relevance to DM, e.g., see [123, 140, 227] as a representative sample.

Beginning in 2005 we were fortunate to have the opportunity to collaborate with DM on an extended project that resulted in the papers [131, 132].

In December 2008, the author MK prepared a short note as a private communication to R. Griess, who at the time was working on [10] and had an interest in clarifying some of the history surrounding DM's discovery of Γ. The original note was never intended for publication, however it conveyed the definite hope that many more details could be clarified in a future project. This was to be accomplished jointly with DM, who expressed a desire to continue our collaboration.

After the sudden death of DM these hopes were forever dashed. Our focus at this point became singularly aimed at making the mathematical community aware of DM's contributions, some of which had been almost completely hidden from view.

Quite soon it became evident that we had to restart the project virtually from scratch in order to give it its due justice. Moreover, we both realized that the entire story could be traced to a very specific date, namely Friday, October 21, 2005.

Both authors were visiting the mathematics department at the University of Nebraska at Lincoln, awaiting their participation in the Special Session *Association Schemes and Related Topics* at the 1011th Meeting of the AMS. (Thanks go out to coorganizer Sung-Yell Song for extending them the invitation to speak.) The talks were scheduled to begin on Saturday, October 22. Having arrived one day early, we met Dale in his office and arranged to have dinner with him.

Our meeting with Dale was precipitated by a prior correspondence with the author MK, who had requested a copy of the DM's thesis [159]. This request was literally fulfilled by 200%, as Dale had prepared in advance two copies of his thesis, distributing one to each of us.

There was something very special, even a bit ceremonial, in this first meeting of all three parties. (The author AW had met Dale many years earlier while still a graduate student at The Ohio State University.)

Dale was a quite shy and reserved individual. However, on this particular evening one could detect definite traces of pleasure and satisfaction in his facial expressions. This was the satisfaction of an elderly scholar who, after spending much time in the shadows, had come to realize the deep admiration had for him by two of his younger colleagues. Clearly, they would read his thesis with heightened care and interest, conspicuously aware of the treasure newly provided to them.

In the development of further events, the influence of Robert Jajcay would be critical. There is something symbolic in the fact that he appeared as a coauthor of the text [117] where DM's discovery would be announced for the first time, as well as serving as witness to the authors' personal acquaintance with Dale. The initial note of MK was shared with a few colleagues, Jajcay among them, and it was he who encouraged the authors to expand their starting note to its present form.

This process took a lot of time. A preliminary version [139] was published in *BICA*, the same journal in which [118] had earlier appeared.

In fact, the full story is quite more complicated. In 2010 a rough draft of the current text, regarded as a privately distributed preprint, was prepared and sent to a number of colleagues. The job over this draft involved, in particular, several months of intensive investigation of the relevant literature. After that, the task we had chosen to follow further required our reaction to several obtained responses. In addition, wishing to conform to the format and style of *BICA*, we prepared a version of the preprint that was greatly reduced in size and mathematical content. This was exactly the aforementioned [139].

At some moment another draft was prepared, jointly with Matan Ziv-Av. This version was strongly influenced by [141], a report urgently published in a collection that is not even reflected in MathSciNet. This huge draft (85 pp.) was rather artificially conceived by gluing together the current story about DM and his discovery of $NL_2(10)$ with the known primitive triangle-free SRGs, their properties and mutual embeddings.

Fortunately, we soon realized that this attempt was counterproductive to our goals, and decided to concentrate more concretely on the careful extension and polishing of our preprint as it existed in 2010. As it turns out, the amount of time devoted to this task was comparable to that of the preparation of the preprint itself. The end result is the text here provided.

In a strong sense, the genre of this article is simultaneously an essay and an expository survey, paying equal attention to the psychology of scientific discovery and to historic events that helped to shape the current mathematical landscape. However, within these frames lies a deeper intended purpose: to expose the reader to a plethora of interesting perspectives and useful ideas that have remained in the underground for a good half-century yet have emerged in modern mathematics in various elaborate forms and guises.

Moreover, it is an article about DM, about his mathematics, about NL_g-graphs, and their special subclass of tf-SRGs. The main objects of our presentation are the graph $\Gamma = N_2(10)$, its related substructures, and the Higman-Sims group.

14.2 Possible updates

By our initial assumptions, everything essentially linked to the first and second appearances of Γ, its origins, development, and further advancement of relevant ideas, is of potential interest to the reader. However, by no means do we pretend that our presentation exhaustively touches all that is known about this graph. Indeed such an attempt would result in a volume of encyclopedic size, with hundreds if not thousands of references. Nevertheless, should such an ambitious task ever come to fruition we would be delighted to learn that in some small way our current text played an initial influential role.

Speaking about less ambitious tasks, we recall that this article is also regarded by us as a kind of dynamic survey. In our eyes this genre does not mandate that each forthcoming update should become a physical extension of a previous version. Rather, there are realistic chances that over time, new and deeper perspectives may become visible which would warrant a fresher, expanded view of the material. One possibility may be based on the aforementioned draft with Ziv-Av, carefully fashioned into a more refined and thoughtful treatment of the subject matter.

During Fall of 2016, the author MK immersed himself in the study of certain concepts related to algebraic geometry (AG) in order to understand potential links between AG and AGT. As he was basically a novice in this area, one could argue that this was not the best decision. (In fact, it delayed the appearance of this paper for about a year.)

Nevertheless, one of the pleasant byproducts of this extended activity is the fruitful collaboration with a few younger colleagues — S. Balagopalan and E. Shamovich, to name a couple — whose interests are quite close to AG. In fact, in a talk recently delivered at the conference ACA 2017 in Jerusalem (joint with Eli Shamovich) some definite potential is demonstrated at the crossroads of AG and AGT, expressed via a fresh discussion of the famous Clebsch graph.

Of course it is foolish to make concrete promises about what lies ahead. However, after roughly 12 years of deep reflection on DM's sundry contributions, the authors cannot help but look to the future with great optimism and high expectations.

14.3 Epilogue: On every branch there are many twigs

Sometimes it is difficult to set a precise line of demarcation, and then resist all temptations to cross it. The present article is no exception.

Under the general heading of "objects related to the graph $NL_2(10)$" we encountered numerous observations, remarks and clarifications that could have easily made their way into our paper had we so desired. However, we also recognized the need to not lead the reader (and ourselves) too far astray. Facing a hard and fast deadline to submit our paper to the editors of Acta UMB also contributed to our restraint. In brief, we confined our efforts to the main branches but not to all of the twigs.

Nevertheless, we describe below a portion of material that didn't make its way into our text yet occupied an area very close to our defined boundary. Although we do not give precise references in every case, we provide the appropriate attributions.

- *Breaking the silence of DM's discovery:* The text [117] was definitely the first one. Short remarks are made in [11, p. 351] and [94]. One finds a delicate discussion in [118]. Further references are found at Wikipedia and the homepage of A. E. Brouwer [23]. Recent citations are made by S. S. Magliveras, M. S. Shrikhande, J. Moori, W. Knapp and their coauthors, and so on.
- *Some constructions and nice decompositions of the graph* Γ*:* A beautiful interpretation of Γ is made by I. Shimada (2014) in terms of AG. A nice preprint of T. Vis (2007) emphasizes certain subgraphs of Γ. Decompositions of Γ and the Hoffman-Singleton graph $HoSi$ are provided by Magliveras *et al* (2012).
- *Attention of peers of AGT:* A very interesting account can be found on the well known blog of Peter Cameron (dated 23.11.2011). This was influenced by [117] and an earlier draft of the present text, as well as the initial version [139]. Two further preprints of N. L. Biggs (2010, 2011) report on a failed attempt to construct new triangle-free SRG's with $\lambda = 0$, $\mu = 2$, with moving credits to W. Edge [69].
- *On the edge between AGT and EGT:* A pioneering paper of H. Nozaki (2015) presents the known primitive triangle-free SRG's as exceptional objects from the perspective of EGT (Extremal Graph Theory). Further results in this direction were obtained by S. Cioabă, W. Li and others at the University of Delaware.
- *Codes related to the Higman-Sims group* HS*:* Early origins of this work can be traced back to R. Calderbank and D. Wales (1982), who relied on the 2-transitive action of HS of degree 176. Among other texts on this topic, the one by A. Cossidente and A. Sonnino (2012) is a nice combination of generous credit to DM and a beautiful computer-aided investigation of the quadric $Q^{-}(9, 2)$.
- *Other relevant objects:* These include generating sets and regular maps of the group HS (initiated by the author AW and subsequently investigated by M. Conder, G. Jones and others), a triality rank 5 AS on 150 points (E. van Dam *et al*, 2013), pentagonal geometries, on the edge between incidence geometries and AG (S. Ball *et al*, 2013; K. Stokes *et al*, 2016), and group theoretical extensions of HS in new clothes (S. Koshitaki *et al*, 2013; Y. Yang and S. Lin, 2014).

We shall stop here, reiterating our hope and optimism for the future.

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References

[1] E. Artin, “Geometric Algebra”, Interscience Publishers, Inc., New York, 1957.

[2] M. Aschbacher, “Sporadic Groups”, Cambridge Univ. Press, Cambridge, 1994.

[3] E. F. Assmus, Jr. and H. F. Mattson, Jr., Perfect codes and the Mathieu groups, *Arch. Math.* 17 (1966), 122–135.

[4] E. F. Assmus, Jr. and C. J. Salwach, The $(16, 6, 2)$-designs, *Internat. J. Math. & Math. Sci.* 2 (1979), 261–281.

[5] A. Baartmans and M. S. Shrikhande, Designs with no three mutually disjoint blocks, *Discrete Math.* 40 (2-3) (1982), 129–139.

[6] B. Bagchi, No extendable biplane of order nine, *J. Combin. Theory Ser. A* 49 (1988), 1–12.

[7] B. Bagchi, Corrigendum, *J. Combin. Theory Ser. A* 57 (1991), p.162.

[8] B. Ballinger, G. Blekherman, H. Cohn, N. Giansiracusa, E. Kelly and A. Schürmann, Experimental study of energy-minimizing point confgurations on spheres, *Experiment. Math.* 18 (3) (2009), 257–283.

[9] E. Bannai and T. Ito, “Algebraic Combinatorics I. Association Schemes”, Benjamin/Cummings, Menlo Park, 1984.

[10] E. Bannai, R. L. Griess, Jr., C. E. Praeger and L. Scott, The mathematics of Donald Gordon Higman, *Michigan Math. J*, 58 (1) (2009), 3–30.

[11] R. A. Bailey, “Association Schemes: Designed Experiments, Algebra and Combinatorics”, Cambridge University Press, Cambridge, 2004.

[12] D. J. Bergstrand, New uniqueness proofs for the $(5, 8, 24), (5, 6, 12)$ and related Steiner systems, *J. Combin. Theory Ser. A* 33 (3) (1982), 247–272.

[13] T. Beth, D. Jungnickel and H. Lenz, “Design Theory”, Cambridge Univ. Press, New York, London, 1993.

[14] Bhagwandas and S. S. Shrikhande, Seidel-equivalence of strongly regular graphs, *Sankhyā: The Indian Journal of Statistics* Series A (1961-2002) Vol. 30, No. 4 (1968), 359-368.

[15] N. L. Biggs, “Algebraic Graph Theory”, Second edition (First edition 1974), Cambridge University Press, Cambridge, 1993.

[16] N. L. Biggs, Strongly regular graphs with no triangles. Research Report, September 2009, arXiv:0911.2160v1.

[17] N. L. Biggs and A. T. White, “Permutation Groups and Combinatorial Structures”, London Math. Soc. Lec. Note Series 33, Cambridge Univ. Press, London, 1979.

[18] R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* 13 (2) (1963), 389–419.

[19] R. C. Bose and D. M. Mesner, On linear associative algebras corresponding to association schemes of partially balanced designs, *Ann. Math. Statist.* 30 (1) (1959), 21–38.

[20] R. C. Bose and K. R. Nair, Partially balanced incomplete block designs, *Sankhyā* 4 (1939), 337–372.

[21] R. C. Bose and T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *J. Amer. Statist. Assoc.* 47 (1952), 151–184.

[22] A. E. Brouwer, The uniqueness of the strongly regular graph on 77 points, Math. Centr. Report ZW147, Amsterdam (Nov. 1980), *J. Graph Th.* 7 (1983), 455–461.

[23] A. E. Brouwer, A slowly growing collection of graph descriptions, accessed July-2016, `http://www.win.tue.nl/~aeb/graphs/index.html`.

[24] A. E. Brouwer and A. M. Cohen, A. Neumaier, "Distance-Regular Graphs", Springer-Verlag, New York, 1989.

[25] A. E. Brouwer, W. H. Haemers, Structure and uniqueness of the (81,20,1,6) strongly regular graph, *Discrete Math.*106/107 (1992), 77–82.

[26] A. E. Brouwer, W. H. Haemers, "Spectra of Graphs", Springer, Berlin, 2012.

[27] A. E. Brouwer and J. H. van Lint, *Strongly regular graphs and partial geometries*, in: D. M. Jackson, S. A. Vanstone (eds.), "Enumeration and Design", Academic Press, London, 1984, 85–122.

[28] R. A. Brualdi and H. J. Ryser, "Combinatorial Matrix Theory", Encyclopedia of Mathematics, Cambridge Univ. Press, Cambridge, 1991.

[29] R. H. Bruck, Finite nets. I. Numerical invariants, *Canad. J. Math.* 3 (1951), 94–107.

[30] R. H. Bruck, Finite nets. II. Uniqueness and imbedding. *Pacific J. Math.* 13 (1963), 421–457.

[31] A. A. Bruen and J. W. P. Hirschfeld, Applications of line geometry over finite fields, II. The Hermitian surface, *Geom. Dedicata* 7 (3) (1978), 333–353.

[32] J. M. J. Buczak, "Some topics in group theory", Ph.D. thesis, Oxford University, 1980.

[33] F. Buekenhout, Diagrams for geometries and groups, *J. Combin. Theory Ser. A* 27 (1979), 121–151.

[34] F. C. Bussemaker, W. H. Haemers, R. Mathon and H. A. A. Wilbrink, $(49, 16, 3, 6)$ strongly regular graph does not exist, *European J. Combin.* 10 (5) (1989), 413–418.

[35] A. R. Calderbank and P. Morton, Quasi-symmetric 3-designs and elliptic curves, *SIAM J. Discrete Math.* 3 (2) (1990), 178–196.

[36] P. J. Cameron, Extending symmetric designs, *J. Combin. Theory Ser. A* 14 (2) (1973), 215–220.

[37] P. J. Cameron, Biplanes, *Math. Zeitschrift* 131 (1973), 85–101.

[38] P. J. Cameron, Locally symmetric designs, *Geom. Dedicata* 3 (1974), 65–76.

[39] P. J. Cameron, Partial quadrangles, *Quart. J. Math. Oxford* 26 (1975), 61–73.

[40] P. J. Cameron, 6-transitive graphs, *J. Combin. Theory Ser. B* 28 (1980), 168–179.

[41] P. J. Cameron, "Permutation Groups", London Mathematical Society Student Texts, 45, Cambridge Univ. Press, Cambridge, 1999.

[42] P. J. Cameron and J. H. van Lint, "Designs, Graphs, Codes and their Links", Cambridge Univ. Press, Cambridge, 1991.

[43] R. D. Carmichael, Tactical configurations of rank two, *Amer. J. Math.* 53 (1931), 217–240.

[44] R. D. Carmichael, "Introduction to the Theory of Groups of Finite Order", Ginn and Co., 1937.

[45] W. H. Clatworthy, Partially balanced incomplete block designs with two associate classes and two treatments per block, *J. Res. Nat. Bureau Stds.* 54 (1955), 177–190.

[46] W. H. Clatworthy, Tables of two-associate partially balanced designs, National Bureau of Standards, Washington D.C., 1973.

[47] A. Clebsch, Über die Flachen vierter Ordung, welche eine Doppelcurve zweiten Grades besitzen, *J. für Math.* 69 (1868), 42–84.

[48] H. Cohn and A. Kumar, The densest lattice in twenty-four dimensions, *Electron. Res. Announc. Amer. Math. Soc.* 10, S 1079-6762(04)00130-110, 2004.

[49] C. J. Colbourn and J. H. Dinitz, "Handbook of Combinatorial Designs", 2nd edition, Chapman & Hall/CRC, 2007.

[50] W. S. Connor, The uniqueness of the triangular association scheme, *Ann. Math. Stat.* 29 (1958), 262–266.

[51] W. S. Connor, W. H. Clatworthy, Some theorems for partially balanced designs, *Ann. Math. Stat.* 25 (1954), 100–112.

[52] J. H. Conway, A perfect group of order 8,315,553,613,086,720,000 and the sporadic simple groups, *Proc. Nat. Acad. Sci. U.S.A.* 61 (1968), 398–400.

[53] J. H. Conway, A group of order 8,315,553,613,086,720,000, *Bull. London Math. Soc.* 79 (1) (1969), 79–88.

[54] J. H. Conway and N. J. A. Sloane, "Sphere Packings, Lattices and Groups", 3rd edition, Springer-Verlag, New York, Berlin, 1999.

[55] G. M. Conwell, The 3-space $PG(3,2)$ and its group, *Ann. Math.* 11 (1910), 60–76.

[56] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* 56 (1950), 413–455.

[57] R. T. Curtis, A new combinatorial approach to M_{24}, *Math. Proc. Cambridge Philos. Soc.* 79 (1979), 25–42.

[58] R. T. Curtis, Symmetric generation of the Higman-Sims group, *J. Algebra* 171 (1995), 567–586.

[59] D. M. Cvetković, M. Doob and H. Sachs, "Spectra of Graphs", 3rd edition, Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

[60] E. R. van Dam and M. Muzychuk, Some implications on amorphic association schemes, *J. Combin. Theory Ser. A* 117 (2) (2010), 111–127.

[61] A. Delandtsheer, Finite geometric lattices with highly ransitive automorphism groups, *Arch. Math.* 42 (1984), 376–383.

[62] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res. Rep. Suppl. No. 10, 1973.

[63] P. Dembowski, "Finite Geometries", Springer-Verlag, New York, 1968.

[64] J. D. Dixon and B. Mortimer, "Permutation Groups", Graduate Texts in Mathematics, vol. 163, Springer, Berlin, Heidelberg, New York, 1996.

[65] W. L. Edge, The geometrical construction of Maschke's quartic surfaces, *Proc. Edinburgh Math. Soc.* Series 2, Vol. 7, Part II (1945), 93–103.

[66] W. L. Edge, Conics on a Maschke surface, *Proc. Edinburgh Math. Soc.* Series 2, Vol. 7 Part III (1946), 153–161.

[67] W. L. Edge, The Kummer quartic and the tetrahedroid based on the Maschke forms, *Proc. Cambridge Philos. Soc. London* A 189 (1947), 326–358.

[68] W. L. Edge, The geometry of the linear fractional group $LF(4,2)$, *Proc. London Math. Soc.* 3-4 (1954), 317–342.

[69] W. L. Edge, Some implications of the geometry of the 21-point plane, *Math. Zeitschrift* 87 (1965), 348–362.

[70] W. L. Edge, A new look at the Kummer surface, *Canad. J. Math.* 19 (1967), 952–967.

[71] A. Emch, Triple and multiple systems, their geometric configurations and groups, *Trans. Amer. Math. Soc.* 31 (1929), 25–42.

[72] I. A. Faradžev and M. H. Klin, *Computer package for computations with coherent configurations*, in: S. M. Watt (ed.), “Proceedings ISSAC-91”, Bonn, ACM Press, 1991, 219–223.

[73] I. A. Faradžev, M. H. Klin and M. E. Muzichuk, Cellular rings and groups of automorphisms of graphs, in: I. A. Faradžev, A. A. Ivanov, M. H. Klin, A. J. Woldar (eds.), “Investigations in Algebraic Theory of Combinatorial Objects”, Kluwer Publ., Dordrecht, 1994.

[74] M. A. Fiol and E. Garriga, On outindependent subgraphs of strongly regular graphs, *Linear and Multilinear Alg.* 54 (2) (2006), 123–140.

[75] J. S. Frame, Computation of characters of the Higman-Sims group and its automorphism group, *J. Algebra* 20 (1972), 320–349.

[76] D. Garbe and J. Mennicke, Some remarks on the Mathieu groups, *Canad. Math. Bull.* 7 (2) (1964), 201–212.

[77] A. Gardiner, Antipodal covering graphs, *J. Combin. Theory Ser. B* 16 (1974), 255–273.

[78] A. D. Gardiner, C. D. Godsil, A. D. Hensel and G. F. Royle, Second neighbourhoods of strongly regular graphs, *Discrete Math.* 103 (2) (1992), 161–170.

[79] A. L. Gavrilyuk and A. Makhnev, On Krein graphs without triangles, *Doklady Math.* 72 (1) (2005), 591–594.

[80] A. Gewirtz, The uniqueness of $g(2,2,10,56)$, *Trans. New York Acad. Sci.* 31 (1969), 656–675.

[81] A. Gewirtz, Graphs with maximal even girth, *Canad. J. Math.* 21 (1969), 915–934.

[82] G. Glauberman, Normalizers of p-subgroups in finite groups, *Pacific J. Math.* 29 (1) (1969), 137–144.

[83] C. Godsil and G. Royle, “Algebraic Graph Theory”, Springer-Verlag, New York, 2001.

[84] J. M. Goethals and J. J. Seidel, Orthogonal matrices with zero diagonal, *Canad. J. Math.* 19 (1967), 1001–1010.

[85] J. M. Goethals and J. J. Seidel, Quasisymmetric block designs, in: R. Guy (ed.), “Combinatorial Structures and Their Applications”, 111–116, Gordon-Breach, New York, 1970.

[86] J. M. Goethals and J. J. Seidel, Strongly regular graphs derived from combinatorial designs, *Canad. J. Math.* 22 (1970), 597–614.

[87] M. J. E. Golay, Notes on digital coding, *Proc. IEEE* 37 (1949), p.657.

[88] R. E. Greenwood and A. M. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* 7 (1955), 1–7.

[89] R. L. Griess, Jr., The Friendly Giant, *Inventiones Mathematicae* 69 (1982), 1–102.

[90] L. C. Grove, “Classical Groups and Geometric Algebra”, Graduate Studies in Mathematics Vol. 39, Amer. Math. Soc., Providence, R.I., 2002.

[91] I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (eds.),“Algebraic Combinatorics and Applications”, Springer, Berlin, 2001, 196–211.

[92] W. H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra & Applic.*, Vol. 226-228 (1995), 593–616.

[93] W. H. Haemers, Strongly regular graphs with maximal energy, *Linear Algebra & Applic.* 429 (11-12) (2008), 2719–2723.

[94] P. R. Hafner, On the graphs of Hoffman-Singleton and Higman-Sims, *Electron. J. Combin.*11 (1), Research Paper 77, 2004.

[95] P. de la Harpe, Spin models for link polynomials, strongly regular graphs and Jaeger's Higman-Sims model, *Pacific J. Math.* 162 (1994), 57–96.

[96] A. Heinze and M. Klin, Loops, Latin squares and strongly regular graphs: An algorithmic approach via algebraic combinatorics, in (M. Klin et al eds.) "Algorithmic Algebraic Combinatorics and Gröbner Bases", Springer-Verlag Berlin, 2009, pp.3-65.

[97] M. D. Hestenes and D. G. Higman, Rank 3 groups and strongly regular graphs, *SIAM Amer. Math. Soc. Proc.* 4 (1971), 141–160.

[98] D. G. Higman, Finite permutation groups of rank 3, *Math. Zeitschrift* 86 (1964), 145–156.

[99] D. G. Higman, Intersection matrices for finite permutation groups, *J. Algebra* 6 (1967), 22–42.

[100] D. G, Higman, "Lectures on Permutation Representations", notes taken by Wolfgang Hauptmann Vorlesungen aus dem Mathematischen Institut Giessen, Heft 4. Mathematisches Institut Giessen, Giessen, 1977, 106 pp.

[101] D. G. Higman, C. C. Sims, A simple group of order $44,352,000$, *Math./ Zeitschrift* 105 (1968), 110–113.

[102] G. Higman, On the simple group of D. G. Higman and C. C. Sims, *Illinois J. Math.* 13 (1969), 74–80.

[103] J. W. P. Hirschfeld, "Projective Geometries over Finite Fields", 2nd edition, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1998.

[104] G. Hiss, Die sporadischen Gruppen, *Jahresber. Deutsch. Math.-Verein.* 105 (2003), 169–193.

[105] A. J. Hoffman and R. R. Singleton, On Moore graphs of diameter two and three, *IBM J. Res. Develop.* 4 (1960), 497–504,

[106] R. W. H. T. Hudson, "Kummer's Quartic Surface", Cambridge 1905, revised reprint: Cambridge Univ. Press, Cambridge, 1990.

[107] D. R. Hughes, On t-designs and groups, *Amer. J. Math.* 87 (1965), 761–778.

[108] D. R. Hughes, Semi-symmetric 3-designs, in: C. A. Baker, L. M. Batten (eds.), "Finite Geometries", Lec. Notes in Pure and Appl. Math., Vol. 82, Marcel Dekker, New York, 1982, 223–235.

[109] D. R. Hughes and F. C. Piper, "Design Theory", Cambridge Univ. Press, Cambridge, 1985.

[110] Q. M. Hussain, On the totality of the solutions for the symmetrical incomplete block designs with $\lambda = 2$, $k = 5$ or 6, *Sankhyā* 7 (1945), 204–208.

[111] Q. M. Hussain, Structure of some incomplete block designs, *Sankhyā* (1948), 381–383.

[112] Y. J. Ionin and M. S. Shrikhande, "Combinatorics of Symmetric Designs", Cambridge Univ. Press, Cambridge, 2006.

[113] A. A. Ivanov, "Geometry of Sporadic Groups I. Petersen and Tilde Geometries", Cambridge Univ. Press, Cambridge, 1999.

[114] A. A. Ivanov and S. V. Shpectorov, "Geometry of Sporadic Groups II. Representations and Amalgams", Cambridge Univ. Press, Cambridge, 2002.

[115] S. Iwasaki, An elementary and unified approach to the Mathieu-Witt systems II: The uniqueness of W_{22}, W_{23}, W_{24}, *Hokkaido Mat. J.* 21 (1992), 239–250.

[116] F. Jaeger, Strongly regular graphs and spin models for the Kauffman polynomial, *Geom. Dedicata* 44 (1) (1992), 23–52.

[117] T. B. Jajcayová and R. Jajcay, On the contributions of Dale Marsh Mesner, *Bull. InstĊombin. Appl.* 36 (2002), 46–52.

[118] T. B. Jajcayová, R. Jajcay and E .S. Kramer, The Life and Mathematics of Dale Marsh Mesner 1923 - 2009, *Bull. Inst. Combin. Appl.* 59 (2010), 9–30.

[119] Z. Janko and S. K. Wong, A characterization of the Higman-Sims simple group, *J. Algebra* 13 (1969), 517–534.

[120] Z. Janko and T. van Trung, A new biplane of order 9 with a small automorphism group, *J. Combin. Theory Ser. A* 42 (2) (1986), 305–309.

[121] V. F. R. Jones, On knot invariants related to some statistical mechanical models, *Pacific J. Math.*, 137 (1989), 311–334.

[122] W. Jónsson, On the Mathieu groups M_{22}, M_{23}, M_{24} and the uniqueness of the associated Steiner systems, *Math. Zeitschrift* 125 (1972), 193–214.

[123] L. K. Jørgensen and M. Klin, Switching of edges in strongly regular graphs. I. A family of partial difference sets on 100 vertices. *Electronic J. Combin.* 10 (2003), Research Paper 17, 31 pp.

[124] J. G. Kalbfleisch and R. G. Stanton, On the maximal triangle-free edge-chromatic graphs in three colors, *J. Combin. Theory* 5 (1968), 9–20.

[125] L. A. Kalužnin and M. H. Klin, On certain maximal subgroups of symmetric and alternating groups, *Mat. Sbornik* 87 (129) (1972), 91–121. (in Russian.)

[126] W. M. Kantor, Dimension and embedding theorems for geometric lattices, *J. Combin. Theory A* 17 (1974), 173–195.

[127] P. Kaski and P. R. J. Östergård, There are exactly five biplanes with $k = 11$, *J. Combin. Designs* 16 (2008), 117–127.

[128] J. Key and V. Tonchev, Computational results for the known biplanes of order 9, in: J. W. P. Hirschfeld, S. S. Magliveras, M. J. de Resmini (eds.), "Geometry, Combinatorial Designs and Related Structures", London Math. Soc. Lecture Note Ser. vol. 245, 1997, 113–122.

[129] R. A. Kingsley and R. G. Stanton, A survey of certain balanced incomplete block designs, *Proc. Third Southeastern Conf. Combin.* (1971), 359–368.

[130] M. Klin and Š. Gyürki, "Selected Topics from Algebraic Graph Theory", Lecture notes, Belianum, Vydavatel'stvo Univerzity Mateja Bela, Banská Bystrica, 2015, xi+209 pp.

[131] M. Klin, D. Mesner, and A. Woldar, A combinatorial approach to transitive extensions of generously unitransitive permutation groups, *J. Combin. Designs* 18 (5) (2010), 369–391.

[132] M. Klin, D. Mesner and A. Woldar, On the combinatorics of certain classes of transitive extensions, preprint.

[133] M. Klin, M. Meszka, S. Reichard and A. Rosa, The smallest non-rank 3 strongly regular graphs which satisfy the 4-vertex condition, *Bayreuth. Math. Schr.* 74 (2005), 145–205.

[134] M. Klin, M. Muzychuk, C. Pech, A. Woldar, and P.-H. Zieschang, Association schemes on 28 points as fusions of a half-homogeneous coherent configuration, *Europ. J. Combin.* 28 (7) (2007), 1994–2025.

[135] M. Klin, M. Muzychuk and M. Ziv-Av, Higmanian rank-5 association schemes on 40 points, *Mich. Math. J.* 58 (1) (2009), 255–284.

[136] M. Klin, C. Pech, S. Reichard, A. Woldar and M. Ziv-Av, Examples of computer experimentation in algebraic combinatorics, *Ars Mathematica Contemporanea* 3 (2010), 237-258.

[137] M. Klin, R. Pöschel and K. Rosenbaum, "Angewandte Algebra für Mathematiker und Informatiker. Einführung in gruppentheoretisch-kombinatorische Methoden", VEB Deutscher Verlag der Wissenschaften. Berlin, 1988. (in German.)

[138] M. Klin, C. Rücker, G. Rücker and G. Tinhofer, "Algebraic combinatorics in mathematical chemistry. Methods and algorithms. I. Permutation groups and coherent (cellular) algebras", Match (40) (1999), 7-138.

[139] M. Klin and A. Woldar, Dale Mesner, Higman & Sims, and the strongly regular graph with parameters (100,22,0,6), *Bull. ICA* 63 (2011), 13–35.

[140] M. Klin and M. Ziv-Av, Computer algebra experimentation with Higmanian rank 5 association schemes on 40 vertices and related combinatorial objects, Technical report, 2007.

[141] M. Klin and M. Ziv-Av, Computer algebra investigation of known primitive triangle-free strongly regular graphs, in: L. Kovacs and T. Kutsia (eds.) SCSS 2013 (EPIC Series, vol. 15), pp. 108–123.

[142] M. Klin and M. Ziv-Av, A non-Schurian coherent configuration on 14 points exists, *Designs Codes and Cryptography* 84 (2017), 203–221.

[143] R. Kochendorfer, "Lehrbuch der Gruppentheorie unter besonderer BerÃijcksichtigung der endlichen Gruppen", Akad. Verlagsges, Geest & Pertig, Leipsig, 1966.

[144] E. S. Kramer and D. M. Mesner, Intersections among Steiner systems, *J. Combin. Theory Ser. A* 16 (1974), 273–285.

[145] C. W. H. Lam, L. Thiel and S. Swiercz, The non-existence of finite projective planes of order 10, *Canad. J. Math.* 41 (1989), 1117–1123.

[146] J. Lauri and R. Scapellato, "Topics in Graph Automorphisms and Reconstruction", London Math. Soc. Student Texts 54, Cambridge Univ. Press, Cambridge, 2003, Second edition 2016.

[147] J. Leech, Notes on sphere packings, *Canad. J. Math.* 19 (1967), 251–267.

[148] D. Leemans, Two rank six geometries for the Higman-Sims sporadic group, *Discrete Math.* 294 (1-2) (2005), 123–132.

[149] J. H. van Lint and J. J. Seidel, Equilateral point sets in elliptic geometry, *Indagationes Mathematicae* 28 (1966), 335–348.

[150] H. Lüneburg, Über die gruppen von Mathieu, *J. Algebra* 10 (1968), 194–210.

[151] H. Lüneburg, "Transitive Erweiterungen endlicher Permutationsgrupper", Lecture Notes in Mathematics 84, Springer-Verlag, Berlin, 1969.

[152] M. Mačaj and J. Širáň, Search for properties of the missing Moore graph, *Linear Algebra & Applic.*, 432 (9) (2010), 2381–2398.

[153] S. S. Magliveras, The subgroup structure of the Higman-Sims simple group, *Bull. Amer. Math. Soc.* 77 (4) (1971), 535–539.

[154] R. Mathon and A. P. Street, Partitions of sets of two-fold triple systems, and their relation to some strongly regular graphs, *Graphs and Combin.* 11 (1995), 347–366.

[155] V. C. Mavron, and M. S. Shrikhande, On designs with intersection numbers 0 and 2, *Arch. Math.* 52 (1989), 407–412.

[156] K. N. Majindar, On the parameters and intersection of blocks of balanced incomplete block designs, *Ann. Math. Statist.* 33 (1962), 1200–1205.

[157] A. A. Makhnev and V. V. Nosov, On automorphisms of strongly regular graphs with $\lambda = 0$ and $\mu = 3$, *Algebra i Analiz,* 21 (5) (2009), 138–154.

[158] B. D. McKay, "nauty User's Guide (Version 1.5)", Technical Report TR-CS-90-02. Computer Science Department, Australian National University, 1990.

[159] D. M. Mesner, "An investigation of certain combinatorial properties of partially balanced incomplete block experimental designs and association schemes, with a detailed study of designs of Latin square and related types", Ph.D. thesis, Michigan State University, 1956.

[160] D. M. Mesner, Negative Latin square designs, Institute of Statistics, UNC, NC Mimeo series, 410, 1964.

[161] D. M. Mesner, A note on the parameters of PBIB association schemes, *Ann. Math. Stat.* 36 (1965), 331–336.

[162] D. M. Mesner, A new family of partially balanced incomplete block designs with some Latin square design properties, *Ann. Math. Stat.* 38 (1967), 571–581.

[163] D. M. Mesner and P. Bhattacharya, Association schemes on triples and a ternary algebra, *J. Combin. Theory, Ser. A* 55 (2) (1990), 204–234.

[164] D. M. Mesner, P. Bhattacharya, A ternary algebra arising from association schemes on triples, *J. Algebra* 164 (3) (1994), 595–613.

[165] A. Mostowski, Axiom of choice for finite sets, *Fund. Math.* 33 (1945), 137–168.

[166] M. Muzychuk, M. Klin and R. Pöschel, The isomorphism problem for circulant graphs via Schur ring theory, *DIMACS Ser. Discrete Math. & Theoret. Comput. Sci.* vol. 56 (2001), 241–264.

[167] M. G. Oxenham and L. R. A. Casse, On a geometric representation of the subgroups of index 6 in S_6, *Discrete Math.* 92 (1991), 251–259.

[168] L. J. Paige, A note on the Mathieu groups, *Canad. J. Math.* 9 (1957), 15–18.

[169] D. Parrott and S. K. Wong, On the Higman-Sims simple group of order 44,352,000, *Pacific J. Math.* 32 (2) (1970), 501–516.

[170] A. Pasini, "Diagram Geometries", Clarendon Press, Oxford, 1994.

[171] A. Pasini, A quarry of geometries, *Rend. Sem. Matematico e Fisico di Milano* 65 (1995), 179–247.

[172] R. M. Pawale, Inequalities and bounds for quasi-symmetric 3-designs, *J. Combin. Theory Ser. A* 60 (2) (1992), 159–167.

[173] R. M. Pawale, Non-existence of triangle free quasi-symmetric designs, *Designs, Codes and Cryptography* 37 (2) (2005), 347–353.

[174] S. E. Payne and J. A. Thas, "Finite Generalized Quadrangles", Research Notes in Mathematics 110, Pitman Publ. Inc., London, 1984.

[175] B. Polster, "A Geometrical Picture Book", Springer-Verlag, New York, 1998.

[176] D. Raghavarao, "Constructions and Combinatorial Problems in Design of Experiments", Wiley, New York, 1971.

[177] D. K. Ray-Chaudhuri, On the application of the geometry of quadrics to the construction of PBIB designs and error correcting codes, Institute of Statistics, UNC, Chapel Hill, NC Mimeo Series 230, 1959.

[178] D. K. Ray-Chaudhuri, Some results on quadrics in finite projective geometry based on Galois fields, *Canad. J. Math.* 15 (1962), 129–138.

[179] N. N. Rognelia and S. S. Sane, Classification of $(16, 6, 2)$-designs by ovals, *Discrete Math.* 51 (1984), 167–177.

[180] S. Reichard, Strongly regular graphs with the 7-vertex condition, *J. Algebraic Combin.* 41 (3) (2015), 817–842.

[181] M. Ronan, "Symmetry and the Monster", Oxford University Press, Oxford, 2006.

[182] I. C. Ross and F. Harary, On the determination of redundancies in sociometric chains, *Psychometrika* 15 (1952), 195–208.

[183] A. Rudvalis, (v, k, λ)-graphs and polarities of (v, k, λ)-designs, *Math Z.* 120 (1971), 224–230.

[184] C. J. Salwach and J. A. Mezzaroba, The four biplanes with $k = 9$, *J. Combin. Theory Ser. A* 24 (1978), 141–145.

[185] S. S. Sane, An extremal property of the 4-$(23, 7, 1)$ design, Proc. XVI Southeastern Internat. Conf. Combin., Graph Theory and Comp. (Boca Raton, Fla., 1985), *Congr. Numer.* 49 (1985), 191–193.

[186] M. Schönert et al. "GAP – Groups, Algorithms, and Programming", Lehrstuhl D für Mathematik, Rheinisch-Westfälische Technische Hochschule, Aachen, 1995.

[187] I. Schur, Zur Theory der einfach transitiven Permutationsgruppen, *S.-B. Preus Akad. Wiss. Phys. Math. Kl.* (1933), 598–623.

[188] W. R. Scott, "Group Theory", Prentice-Hall, Englewood Cliffs NJ, 1964 (Dover Publ., New York, 1987).

[189] B. Segre, Le geometrie di Galois, *Ann. Mat. Pura Appl.* 48 (4) (1959), 2–97.

[190] J. J. Seidel, Strongly regular graphs of L_2 type and of triangular type, *Indagationes Mathematicae* 29 (1967), 188–196.

[191] J. J. Seidel, Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3, *Linear Algebra & Applic.* 1 (1968), 281–298.

[192] J. J. Seidel, Strongly regular graphs, in: W. T. Tutte (ed.), "Recent Progress in Combinatorics", Academic Press, New York, 1969, 185–197.

[193] M. S. Shrikhande, On a class of negative Latin square graphs, *Utilitas Mathematica* 5 (1974), 293–303.

[194] M. S. Shrikhande and S. S. Sane, "Quasi-Symmetric Designs", London Math. Soc. Lec. Notes Ser. 164, Cambridge Univ, Press, 1991.

[195] S. S. Shrikhande, On the dual of some balanced incomplete block designs, *Biometrics* 8 (1952), 66–72.

[196] S. S. Shrikhande, The uniqueness of the L_2 association scheme, *Ann. Math. Statist.* 30 (1959), 781–798.

[197] S. S. Shrikhande, Strongly regular graphs and symmetric 3-designs, Survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971), 1973, 403–409.

[198] S. S. Shrikhande and D. Bhagwandas, Duals of incomplete block designs, *J. Indian Statist. Assoc. Bull.* 3 (1965), 30–37.

[199] C. C. Sims, Graphs and finite permutation groups, *Math. Zeitschrift* 95 (1967), 76–86.

[200] C. C. Sims, *On the isomorphism of two groups of order 44,352,000*, in: "Theory of Finite Groups" (Symposium, Harvard Univ., Cambridge MA, 1968), Benjamin, New York, 1969, 101–108.

[201] J. Singer, A theorem in finite projective geometry and some applications to number theory, *Trans. Amer. Math. Soc.* 43 (3) (1938), 377–385.

[202] M. S. Smith, On rank 3 permutation groups, *J. Algebra* 33 (1975), 22–42.

[203] M. S. Smith, On the isomorphism of two simple groups of order 44352000, *J. Algebra* 41 (1976), 172–174.

[204] M. S. Smith, A combinatorial configuration associated with the Higman-Sims group, *J. Algebra* 41 (1976), 175–195.

[205] L. H. Soicher, *GRAPE: a system for computing with graphs and groups*, in: L. Finkelstein, W. M. Kantor (eds.), "Groups and Computation", Vol. 11 DIMACS Ser. Discrete Math. & Theoret. Comput. Sci., Amer. Math. Soc., 1993, 287–291.

[206] E. Spence, Regular two-graphs on 36 vertices, *Linear Algebra & Applic.* 226/228 (1995), 459–497.

[207] D. A. Sprott, Some series of partially balanced incomplete block designs, *Canad. J. Math.* 7 (1955), 369–381.

[208] R. G. Stanton, The Mathieu groups, *Canad. J. Math.* 3 (1951), 164–174.

[209] R. G. Stanton and D. A. Sprott, Block intersections in balanced incomplete block designs. Canad. Math. Bull. 7, 1964, 539-548.

[210] R. G. Stanton and J. G. Kalbfleisch, Quasi-symmetric balanced incomplete block designs, *J. Combin. Theory* 4 (1968), 391–396.

[211] O. Tamaschke, Schur-Ringe, BI-Hochschulskripten 735, Bibliographisches Institut, Manheim, 1970.

[212] D. E. Taylor, “The Geometry of the Classical Groups”, Heldermann Verlag, Berlin, 1992.

[213] G. Tinhofer and M. Klin, Algebraic Combinatorics in Mathematical Chemistry. Methods and Algorithms. Iii. Graph Invariants and Stabilization Methods, Technische Univerität München, Report TUM–M9902, 1999.

[214] J. Tits, Sur les systèmes de Steiner associés aux trois ‘grands’ groupes de Mathieu, *Rend. Mat. e Appl.* 23 (5) (1964), 166–184.

[215] J. A. Todd, On representations of the Mathieu groups as colineation groups, *J. London Math. Soc.* 34 (1959), 406–416.

[216] J. A. Todd, A representation of the Mathieu group M_{24} as a colineation group, *Ann. Mat. Pura Appl.* 71 (4) (1966), 199–238.

[217] V. D. Tonchev, Embedding of the Witt-Mathieu system $S(3,6,22)$ in a symmetric 2-$(78,22,6)$ design, *Geom. Dedic.* 22 (1987), 49–75.

[218] D. Wales, Uniqueness of the graph of a rank 3 group, *Pacific J. Math.* 30 (1969), 271–276.

[219] B. Weisfeiler and A. A. Leman, A reduction of a graph to a canonical form and an algebra arising during this reduction, Nauchno-Techn. Inform. Ser. 2, 9, 1968.

[220] H. W. Wielandt, “Finite Permutation Groups”, Academic Press, New York, 1964.

[221] H. Wielandt, Permutation groups through invariant relations and invariant functions, Ohio State Univ. Lecture Notes, Columbus, 1969.

[222] R. A. Wilson, Octonions and the Leech lattice, *J. Algebra* 322 (6) (2009), 2186–2190.

[223] E. Witt, Die 5-Fach transitiven Gruppen von Mathieu, *Abh. Math. Sem. Univ. Hamburg* 12 (1938), 256–264.

[224] E. Witt, Uber Steinersche systeme, *Abh. Math. Sem. Hansischen Univ.* 12 (1938), 265–275.

[225] E. Witt, “Collected Papers”, Gesammelte Abhandlungen, Springer-Verlag, Berlin, New York, 1998.

[226] A. Woldar, A combinatorial approach to the character theory of split metabelian groups, *J. Combin. Theory Series A* 50 (1) (1989), 100-109.

[227] A. J. Woldar, The symmetric genus of the Higman-Sims group HS and bounds for Conway’s groups Co_1, Co_2, *Ill. J. Math.* 36 (1) (1992), 47–52.

[228] S. Yoshiara, A locally polar geometry associated with the group HS, *Europ. J. Combin.* 11 (1990), 81–93.

[229] H. Zassenhaus, Über transitive Erweiterungen gewisser Gruppen aus Automorphismen endlicher mehrdimensionaler Geometrien, *Math. Ann.* 111 (1935), 748–759.

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Some bounds on the maximum induced matching numbers of certain grids

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Abstract

An induced matching M in a graph G is a matching in G that is also the edge set of an induced subgraph of G. That is, any edge not in M must have no more than one incident vertex saturated by M. The maximum size $|M|$ of an induced matching M of G is maximum induced matching number of G, which is denoted by $\mathrm{Max}(G)$. In this article, we obtain upper bounds for $\mathrm{Max}(G)$, for $G = G_{n,m}$, grids with $n, m \geq 9$, $m \equiv 1 \mod 4$ and nm odd.

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1 Introduction

Let G be a graph with edge and vertex sets $E(G)$ and $V(G)$, respectively, and for any $u, v \in V(G)$, let $d(u, v)$ be the distance between u and v. A matching is a set of edges with no shared vertices. The vertices incident to an edge of a matching M are said to be saturated by M, the other vertices are unsaturated by M. A subset $M \subseteq E(G)$ of G is an induced matching of G if for any two edges $e_1 = u_iu_j$ and $e_2 = v_iv_j$ in M, then $d(u_i, v_i) \geq d(u_i, v_j) \geq 2$ and $d(u_j, v_i) \geq d(u_j, v_j) \geq 2$. In other words, M is an induced matching of G if for any two edges e_1, e_2 in M, there is no edge in G incident to both e_1 and e_2. Equivalently, an induced matching is a matching which forms an induced subgraph.

Introduced by Stockmeyer and Vazirani [9] as a special case of the well known matching problem, the concept finds applications, among others, in cryptology where certain communication channels between two ends are classified [2].

The size $|M|$ of an induced matching M is the number of edges in the induced matching. Denote by $\mathrm{Max}(G)$ the maximum size of an induced matching in G. A maximum induced matching M in G is an induced matching with $\mathrm{Max}(G)$ edges. We refer to $\mathrm{Max}(G)$ as the maximum induced matching number (or strong matching number) of G. Unlike in the case with finding the maximum matching number of a graph, which can be obtained in polynomial time [4], obtaining $\mathrm{Max}(G)$ in general, is $NP-$hard, even for classes of graphs such as the regular bipartite graphs [3].

In [10], it was observed that for any G with maximum degree $\Delta(G)$,

$$\text{Max}(G) \geq \frac{|V(G)|}{2(2\Delta(G)^2 + 2\Delta(G) + 1)} \tag{1.1}$$

The above bound is certainly not a sharp one and therefore, Joos in [6], presented the following bound:

$$\text{Max}(G) \geq \frac{|V(G)|}{(\lceil \frac{\Delta(G)}{2} \rceil + 1)(\lfloor \frac{\Delta(G)}{2} \rfloor + 1)} \tag{1.2}$$

Joos also showed that it holds for $\Delta(G) = 1000$ and this, according to him, could be reviewed down to 200. In the end, he conjectured that this bound holds for $\Delta(G) \geq 3$ with the exception of certain graphs that he listed. Inspired by the work in [6], Nguyen [8] showed that the conjecture is true for $\Delta(G) = 4$ as long as G is not one of the excepted graphs.

The maximum induced matching number of many graphs can be obtained efficiently just as in the cases of chordal graphs [2], bounds of bounded cliques width, intersection graphs [1], circular arc graphs [5] among others.

A grid $G_{n,m}$ is obtained by the Cartesian product of any two paths of lengths n and m, where $n, m \geq 2$ are integers, representing rows and columns of the grid, respectively. We introduce an odd grid as a grid whose path factors are of odd order. Marinescu-Ghemaci in [7], obtained Max(G) values for all grids with even nm, and some cases where nm is odd. She also gave useful lower and upper bounds. Particularly, she showed that for any odd grid $G_{n,m}$, $\text{Max}(G_{n,m}) \leq \lfloor \frac{nm+1}{4} \rfloor$.

This paper improves Marinescu-Ghemachi's upper bound for $G_{n,m}$, n, m odd, $m \equiv 1 \mod 4$. The results provide new upper bounds for some cases whose lower bounds are established in [7] and thus, in a number of situations, precise values of Max($G_{n,m}$) were obtained. These may also prove useful in probing some of the unresolved conjectures made in [7].

2 Definitions and Preliminary Results

Grid, $G_{n,m}$, as defined in this work, is the Cartesian product of paths P_n and P_m with n and m being positive integers, where P_n and P_m have disjoint vertex sets $V(P_n) = \{u_1, u_2, \cdots, u_n\}$ and $V(P_m) = \{v_1, v_2, \cdots, v_m\}$, respectively. Unless explicitly stated, n is any odd integer, $m \equiv 1 \mod 4$ and $2 \leq n \leq m$. We introduce the following notations: $V_i = \{u_1v_i, u_2v_i, \cdots, u_nv_i\} \subset V(G_{n,m})$ and $U_i = \{u_iv_1, u_iv_2, \cdots, u_iv_m\} \subset V(G_{n,m})$; for edge set $E(G_{n,m})$ of $G_{n,m}$, if $u_iv_j\ u_kv_j \in E(G_{n,m})$ and $u_iv_j\ u_iv_k \in E(G_{n,m})$, we write $u_{\{i,k\}}v_j \in E(G_{n,m})$ and $u_iv_{\{j,k\}} \in E(G_{n,m})$ respectively.

Recall that a vertex v is said to be saturated by an induced matching M if it is a member of an edge in M and unsaturated by M, otherwise. We say v is saturable if either v is saturated by M or v is unsaturated by M, but satisfies $d(v, u) \geq 2$ for every saturated vertex u in $V(G)$. This implies that an unsaturated vertex can become saturated if there is no saturated vertex within distance 2. If v can not be saturated, then we say v is an isolated vertex. A boundary vertex is a vertex on any of V_1, V_n, U_1, U_m. A saturable vertex v in subgraph G_a of G, which is not saturated by induced matching M_a of G_a can still be saturated by M of G in case v is on the boundary of G_a. However, if v is not on the boundary of G_a, then v is isolated. The sets of all saturated vertices, saturable vertices and isolated vertices in a graph G are denoted by $V_s(G), V_{sb}(G)$ and $V_{is}(G)$, respectively. Clearly, $|V_s(G)|$ is even and $V_s(G) \subseteq V_{sb}(G)$.

The following results about grid $G_{n,m}$ are from [7]:

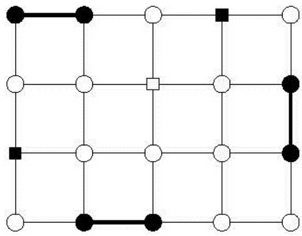

Figure 1. Saturable vertices as black squares and an isolated vertex as white square in an induced matching of $G_{4,5}$

Lemma 1. *Let $m, n \geq 2$ be two positive integers.*

1. *If $m \equiv 2 \mod 4$ and n odd, then, $|V_{sb}(G_{n,m})| = \frac{mn+2}{2}$ and $|V_{sb}(G_{n,m})| = \frac{mn}{2}$ otherwise;*
2. *For $m \geq 3$, m odd, $|V_{sb}(G_{n,m})| = \frac{nm+1}{2}$, for $n \in \{3, 5\}$.*

Theorem 2. *For $G_{n,m}$ where $2 \leq n \leq m$, let $|M| = Max(G_{n,m})$. Then, for n even, $Max(G_{n,m}) = \lceil \frac{mn}{4} \rceil$, for $n \in \{3, 5\}$, $m \equiv 1 \mod 4$, $Max(G_{n,m}) = \frac{n(m-1)}{4} + 1$ and for $m \equiv 3 \mod 4$, $Max(G_{n,m}) = \frac{n(m-1)+2}{4}$.*

Remark 3. For $m \equiv 1 \mod 4$, $|V_{sb}(G_{3,m})| = 2(\text{Max } (G_{3,m})) = |V_s(G_{n,m})|$.

Theorem 4. *For m, n odd integers, $Max(G_{n,m}) \leq \lfloor \frac{mn+1}{4} \rfloor$.*

The obvious implication of Theorem 4, based on the proof, is that $|V_{sb}G_{n,m}| \leq \frac{nm+1}{2}$, for n, m odd.

3 Results

We start our results by stating a few observations.

Remark 5. Let $G_{3,3}$ be a 3×3 grid with induced matching M. Clearly, by Lemma 1, $|V_{sb}(G_{3,3})| = 5$. Suppose $u_{\{1,2\}}v_2 \in M$, then $|M| = 1$. However, there are non-adjacent saturable vertices u_3v_1 and u_3v_3.

Lemma 6. *Let M be an induced matching of $G_{3,m}$. If $u_{\{1,2\}}v_2 \in M$. Then, $|M| \neq Max(G_{3,m})$.*

Proof. Suppose $u_{\{1,2\}}v_2 \in M$ and let $G_a = G_{3,m-3} \subset G_{3,m}$, be a subgrid of $G_{3,m}$ induced by $V(G_{3,m}) \setminus \{V_1, V_2, V_3\}$, where $V_1, V_2, V_3 \subset V(G_{3,m})$, $m \geq 3$.

Case I: Suppose $m \equiv 3 \mod 4$, $m \geq 7$. Then $m = 4k + 3$ for some positive integers k. By Lemma 1 and Theorem 2, $|V_s(G_a)| = 6k$. Now, with $u_{\{1,2\}}v_2 \in M$, u_3v_1 is saturable, by its position and $V_s(G_a) = V_{sb}(G_a)$. Since $m - 3$ is even, then either u_3v_3 remains isolated (or unsaturated) or if it is forced to be saturated with u_3v_4, a saturable vertex in $V(G_a)$ becomes isolated (or unsaturated). So without loss of generality, we may assume u_3v_3 is unsaturated. Therefore $G_{3,m}$ contains $3 + 6k$ saturable vertices and then, $|M| \leq 3k + 1$, which is a contradiction since by Theorem 2, $\text{Max}(G_{3,m}) = 3k + 2$. Note that $m = 3$, has been covered by Remark 5.

Case II: Suppose $m \equiv 1 \mod 4$. From Lemma 1, Theorem 2 and following similar argument as in Case I, we see that if $u_{\{1,2\}}v_2 \in M$ then one vertex in $V_{sb}(G_3, m)$ will become isolated and therefore, $|V_{sb}(G_{3,m})| = 6k+1$. Hence $|M| \leq 3k$ and a contradiction since $\text{Max}(G_{3,m}) = 3k + 1$ if $m = 4k + 1$.

Case III: If $m \equiv 0 \mod 4$, then $m-3 \equiv 1 \mod 4$, and by Remark 3, $G_{3,m-3} = G_a$, $|V_s(G_a)| = |V_{sb}(G_a)|$. Now $|V_s(G_a)| = \frac{3(m-3)+1}{2} = 6k-4$. By following the argument in the previous cases, we see that $|V_{sb}(G_a)| = 6k-1$. Therefore, $|M| \leq 3k-1$, which is less than $3k$.

Case IV: If $m \equiv 2 \mod 4$, then $m-3 \equiv 3 \mod 4$ and therefore, $|V_{sb}(G_a)| = \frac{3(4k+2-3)+1}{2} = 6k-1$, which is odd. Therefore, there exists a saturable vertex in $V(G_a)$, which can pair with u_3v_3 and thus the two vertices become saturated. This way, we have the total saturable vertices in $G_{3,m}$, which has $u_{\{1,2\}}v_2 \in M$, to be $6k+3$ which implies that $|M| \leq 3k+1$. But from the results in Theorem 2, if $m \equiv 2 \mod 4$, $\text{Max}(G_{3,m}) = 3k+2$. This is a contradiction and the claim holds. □

Remark 7. Suppose we have $u_{\{1,2\}}v_2 \in M$ in a grid $G_{3,m}$, then the following holds for $|V_{sb}(G_{3,m})|$, from Lemma 6.

m	$\lvert V_{sb}(G_{3,m})\rvert$
$4k$	$6k-1$
$4k+1$	$6k+1$
$4k+2$	$6k+3$
$4k+3$	$6k+3$

Lemma 8. *If $m \equiv 1 \mod 4$ and there exists M, an induced matching of $G_{3,m}$, such that $u_{\{1,2\}}v_j, u_{\{1,2\}}v_{j+2} \in M$, then $|M| \neq Max(G_{3,m})$.*

Proof. Let $u_{\{1,2\}}v_j, u_{\{1,2\}}v_{j+2} \in M$, where M is an induced matching of $G_{3,m}$.

Case I: Suppose that $j+1 \equiv 3 \mod 4$, then $m-(j+1) \equiv 2 \mod 4$. Since $u_{\{1,2\}}v_j \in M$, by Lemma 6, suppose there exist an induced matching M' in $G_a = G_{3,j+1}$, induced by $V_1, V_2, \cdots, V_{j+1}$, with $u_{\{1,2\}}v_j \in M'$, then $|M'| \neq \text{Max}(G_a)$. By Remark 7, given a non-negative integer l, $|V_{sb}(G_a)| = 6l+3$, which being odd, contains a saturable vertex $v' = u_3v_{j+1}$ which is not a member of $V_s(G_a)$. In fact, $v' \in V_{is}(G_a)$, since $u_{\{1,2\}}v_{j+1} \in M$. Since $m-(j+1) \equiv 2 \mod 4$, then given a subgrid $G_b = G_{3,m-(j+1)}$, induced by $V_{j+2}, V_{j+3}, \cdots, V_m$ $|V_s(G_b)| = 6(k-l)-2$. Therefore, $|V_{sb}(G_{3,m})| = 6k$ and hence, $|M| \leq 3k$, which is less that $3k+1$.

Case II: If $j+1 \equiv 1 \mod 4$, by following the argument in Case I, $v' \in V_{is}(G_{3,m})$. Now, $j+1 = 4l+1$, and by the isolation of v', and Remark 7, $|V_{sb}(G_{3,m})| = 6l + |V_{sb}(G_b)|$. Meanwhile, $m-(j+1) \equiv 0 \mod 4$ and therefore, $|V_{sb}(G_b)| = 6k$. Thus, $|M| \leq 3k$.

Case III: If $j+1$ is even, we follow similar arguments as the earlier cases. □

Lemma 9. *Suppose that $m \equiv 1 \mod 4$ and M is an induced matching of $G_{3,m}$. If $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+3} \in M$, then $|M| \neq Max(G_{3,m})$.*

Proof. For some positive integers k, let $m = 4k+1$. Now suppose that $j \equiv 3 \mod 4$. This implies that $j+3 \equiv 2 \mod 4$ and $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+3} \in M$. Let $G_a = G_{3,j+1}$, $G_b = G_{3,m-(j+1)} \subset G_{3,m}$, induced by $V_1, V_2, \cdots, V_{j+1}$ and $V_{j+2}, V_{j+3}, \cdots, V_m$, respectively. By earlier result and remark, $|V_{sb}(G_a)| = 6l-1$ since $j+1 = 4l$. Also, $|V_{sb}(G_b)| = 6(k-l)+1$, since $m-(j+1) = 4(k-l)+1$. Certainly, $|V_{sb}(G_{3,m})| = 6k$. Thus $|M| \leq 3k$. Therefore, $|M| \neq 3k$. For $j \equiv 1 \mod 4$, $j+3 \equiv 3 \mod 4$. Let $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+3} \in M$. By earlier lemma and result, we have that $|V_{sb}(G_a)| = 6l+3$. Since $j+1 = 4l+2$. Also, $|V_{sb}(G_b)| = 6[(k-l)-1]+3$. Therefore, $|V_{sb}(G_{3,m})| = 6k$ and therefore, $|M| = 3k$, which is a contradiction. □

Remark 10. By following similar argument as in the last result, it is easy to see that if M contains $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+4}$, then $|M| \neq \text{Max}(G_{3,m})$. Therefore, suppose M' is a maximum induced matching of $(G_{3,m})$ and $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+8} \in M'$, then there is no $u_{\{1,2\}}u_{j+k} \in M$, such that $2 \leq k \leq 6$.

Theorem 11. *For $m = 4k+1$, there exist at least $2k$ saturated vertices in $U_1 \subset V(G_{3,m})$.*

Proof. Let $G_a = G_{2,m} \subset G_{3,m}$, induced by U_2, U_3, a subgrid of $G_{3,m}$. From earlier results, $|V_{sb}(G_a)| = 4k+2$. Now, $|V_{sb}(G_{3,m})| = 6k+2$. Therefore, V_1 has at least $2k$ saturable vertices. □

Theorem 12. *Suppose $u_{\{1,2\}}u_j, u_{\{1,2\}}v_{j+8} \in M$, in a $G_{3,m}$ and $m \equiv 1 \mod 4$.*

(a) There exists four other saturated vertices from u_1v_j to u_1v_{j+8}.

(b) There exists at most one saturated vertex between u_2v_j and u_2v_{j+8}.

(c) There exists at most five saturated vertices from v_3v_j to u_3v_{j+8}.

Proof. (a) From Remark 10, let $|M| = \text{Max}(G_{3,m})$, and $u_{\{1,2\}}u_j, u_{\{1,2\}}v_{j+8} \in M$, then there is no $u_{\{1,2\}}v_k \in E(G_{3,m})$, $1 \leq k \leq 7$ such that $u_{\{1,2\}}v_k \in M$. Thus, suppose there exists another saturated vertex such that $u_1v_{\{j+k,j+k-1\}} \in M$. Then there exist at least four saturated vertices from u_1v_1 to u_1v_m. Now suppose that there is no other saturated vertex on U_1, then by the Theorem 11, the grid $G_a = G_{2,m} \subset G_{3,m}$, induced by vertices $\{u_2v_j, u_2v_{j+1}, \cdots, u_2v_{j+8}\}$ and $\{u_3v_j, u_3v_{j+1}, \cdots, u_3v_{j+8}\}$ must contain ten saturated vertices (including u_2v_j and u_2v_{j+8}). Clearly, vertices $u_2v_{j+1}, u_2v_{j+7}, u_3v_j$ and u_3v_{j+8} can not be saturated by G_a. It is clear, therefore, that G_a only has eight saturable vertices, which is a contradiction. Thus, there exists two more saturated vertices in U_1, and hence the claim.

Parts (b) and (c) follow from (a). □

Remark 13. (a) Since there are five saturated vertices between u_3v_j and u_3v_{j+8} and there exist only one saturated vertex between u_2v_j and u_2v_{j+8}, then there are edges $e_1, e_2 \in E(G_b)$, G_b induced by $u_3v_j, u_3v_{j+1}, \cdots u_3v_{j+8}$.

(b) Suppose $m = 4k+1$, k being positive integers, then, at $G_c = G_{1,m} \subset G_{3,m}$, induced by either V_1 or V_3, there exists at least k edges of G_c in the maximum induced matching M of $G_{3,m}$.

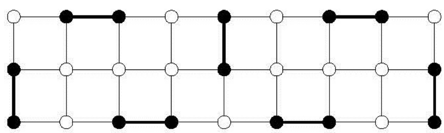

Figure 2. A $G_{3,9}$ grid with $Max(G_{3,9}) = 7$

Next, we consider the grid $G_{4,m}$, $m \equiv 1 \mod 4$.

Lemma 14. *Let $|M| = Max(G_{4,m})$ and let U_4 contain $\frac{m-1}{2}$ saturated vertices. Then, for any edge $e_1 \in E(G_a)$, $G_a = G_{1,m} \subset G_{4,m}$, induced by $U_4 \subset V(G_{4,m})$, $e_1 \notin M$.*

Proof. Let $u_4v_i, u_4v_{i+1} \in U_4 \subset V(G_{4,m})$, be saturated vertices. By hypothesis, there are $\frac{m-5}{2}$ other saturated vertices in U_4. Suppose $G_a = G_{2,m} \subset G_{4,m}$, induced by U_1, U_2. The $|V_s(G_a)| = |V_{sb}(G_a)| = m+1$. Let $G_b = G_{2,m} \subset G_{4,m}$, induced by U_3, U_4. Since $|V_s(G_a)| = m+1$, then $|V_s(G_b)| \leq m-1$ where $|V_s(G_{4,m})| = 2m$. By hypothesis, suppose there exist $\frac{m-1}{2}$ saturated vertices in U_4, then there are also at most $\frac{m-1}{2}$ saturated vertices in U_3. Without loss of generality, suppose for all the $\frac{m-5}{2}$ other saturated vertices in U_s, there exist adjacent saturated vertices in U_3, then we have that $|V_s(G_b)| \geq m-3$.

Claim: There are at most $\frac{m-5}{2}$ saturated vertices in U_3.

Reason: Since there are at most $\frac{m-1}{2}$ saturated vertices in U_4, then suppose v_k is saturable in U_3, v_k is not incident to a saturable vertex in U_4. Also, since $V_s(G_a) = V_{sb}(G_a)$, then there is no saturable vertex in U_2 to which v_k is incident to form an edge in M. Thus, suppose there exists a vertex v_{k-1}, adjacent to $v_k \in U_3$, v_{k-1} will be adjacent to a saturated vertex in U_2 since there can not be two adjacent vertices both of which are not saturated in U_2. Thus, v_k is isolated.

Finally, $|V_s(G_{4,m})| \leq 2m-2$ which implies that $|M| \leq m-1$. However by Theorem 2, $\text{Max}(G_{4,m}) = m$. □

Corollary 15. *Let $G_{n,m}$ be a grid with $n \equiv 0 \mod 4, m \equiv 1 \mod 4$ and U_n contains $\frac{m-1}{2}$ saturable vertices, with $|M| = Max(G_{n,m})$, then no two saturated vertices, say, $v', v'' \in U_n$ such that $v'v'' \in M$.*

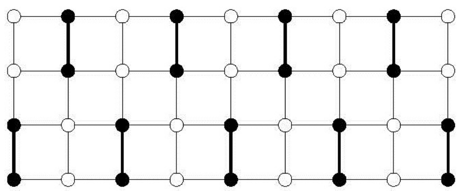

Figure 3. A $G_{4,9}$ grid with $Max(G_{4,9}) = 11$

Next we observe some results in the grid $G_{5,m}$, where $m \equiv 1 \mod 4$.

Lemma 16. *Let $G_{5,5}$ be a grid and suppose that $|M| = Max(G_{5,5})$, and $u_{\{1,2\}}v_1 \in M$, then*

- *(a) there exists at least another saturated vertex in V_1 (either u_4v_1, u_5v_1 or both).*
- *(b) suppose $G_a = G_{5,5} \backslash V_1$, and M_a is some induced matching in G_a, then $|M_a| \neq Max(G_a)$.*
- *(c) there exists some $e \in M$ with $e = u_iu_j$, such that $u_i, u_j \in V_5$.*

Proof. (a) By Theorem 2, $|M| = 6$. Suppose that $u_{\{1,2\}}v_1 \in M$ and that no other vertex in V_1 is saturated. Clearly, u_1v_2 and u_2v_2 can not be saturated. Therefore, there can only be two saturated vertices in V_2. Suppose $u_{[4,5]}v_2 \in M$. Now we show that in $G_{5,5} \backslash \{V_1, V_2\}$, only three edges belong to M. Let vertices u_1v_3, u_1v_4, u_1v_5 and u_2v_3, u_2v_4, u_2v_5, induce $G_b = G_{2,3} \subset G_{5,5}$. Now, G_b has a maximum of four saturable vertices. Also let G_c be a subgraph of $G_{5,5}$, induced by u_3v_3, u_3v_4, u_3v_5; u_4v_4, u_4v_5 and u_5v_4, u_5v_5. The subgraph G_c also has a maximum of four saturable vertices. However, if we consider the positions G_b and G_c, it is clear that at least a saturable vertex in G_b is adjacent to a saturable vertex in G_c, which implies that $|V_s(G_b \cup G_c)| \leq 6$. Thus $|V_s(G_{5,5})| \leq 10$, and therefore $|M| \neq \text{Max}(G_{5,5})$ and hence a contradiction. By following similar argument, it can be seen also that $|M| = 5$ if we consider $u_{\{3,4\}v_2} \in M$.

(b) Suppose that $G_a = G_{5,5} \backslash \{V_1\} \subset G_{5,5}$. Assume that V_1 contains only three saturated vertices. There exists a saturated vertex $u_2 \in V_2$ such that given some vertex $u_1 \in V_1$, $v_1v_2 \in M$. Now, it is clear that $v_1v_2 \notin E(G_a)$ therefore, $|V_{sb}(G_a)\backslash v_2| = 9$, implying that $|M_a| \leq 4$, while $\text{Max}(G_a) = 5$.

(c) By (a) and (b) above, $|M_a| = 4$. Suppose that $u_3v_{\{2,3\}} \notin M$, then, for $G_{5,5}\backslash\{V_1, V_2\}$, there must be at least two saturated vertices on V_5. Let $G'' = G_{5,2} = G_{5,5}\backslash\{V_1, V_2, V_5\} \subset G_{5,5}$. Clearly $G_{5,2}$ will contain at most six saturated vertices. Suppose that u_a, u_b are saturated in V_5, and $u_au_b \notin E(G_{5,5})$, then u_a and u_b form two edges with adjacent vertices $v_a, v_b \in V_4$. However, it can be seen that if this is so, there would be, at most, only four saturable vertices in G'', apart from v_a and v_b, and at least one of which is an isolated vertex. Thus, there could only be four saturated vertices in G'', which is a contradiction. Suppose $u_3v_{\{2,3\}} \in M$. It is easy to see by observation that it is impossible to have the matching in subgraph $G_f \subset G_{5,5}$, induced by $\{u_1v_3, u_1v_4, u_1v_5, u_2v_4, u_2v_5, u_3v_5, u_4v_4, u_4v_5, u_5v_3, u_5v_4, u_5v_5\} \in V(G_{5,5})$ since it contains at most three edges in M without any of them being made up of adjacent vertices in V_5. □

Lemma 17. *Let M be a maximum induced matching of $G_{5,9}$. If $v_{\{1,2\}}u_1 \in M$, and there is at most one more saturated vertex v_1 on V_1. Then, $v_1 = u_5v_1$.*

Proof. It is obvious that if $u_{\{1,2\}}v_1 \in M$, then $u_3v_1 \notin V_s(G_{5,9})$. Suppose that $v_1 = u_4v_1$, then since it is the only lone saturated vertex on V_1, then, $u_4v_{\{1,2\}} \in M$. For G_a, some $G_{5,2}$ grid, induced by V_1, V_2, clearly, there is no other saturable vertex in V_2. Now, let $G_b = G_{5,9}\backslash G_a$. The subgrid G_b is a $G_{5,7}$ grid and $\text{Max}(G_b) = 8$. Thus, $|M| = 10$, and hence, not $\text{Max}(G_{5,9})$. Hence a contradiction. □

Lemma 18. *Suppose that there are at most two saturated vertices on $V_1 \subset V(G_{5,5})$, and that they are adjacent. Then, there are at least three saturated vertices on $V_5 \subset V(G_{5,5})$, two of which are adjacent.*

Proof. Suppose M is the maximal induced matching of $G_{5,5}$, and that $u_au_b \in M$, where $u_jv_1, u_{j+1}v_1 \in U_1$. (It should be noted that since there are two saturated vertices in V_1 and are adjacent, then by Lemma 16(a), none of j and $j+1$ is either 1 or 5). Suppose that $G_b \subset G_{5,5}$, induced by V_2, V_3, V_4 and that V_5 only contains two saturated vertices $u_iv_5, u_{i+1}v_5$ and are adjacent, (meaning also that neither i nor $i+1$ is 0 or 5.) Let $j \neq i$ and obviously, without loss of generality, set $i = 2$, while j clearly becomes 3. Let $G_d, G_e \subset G_b$ be two subgraphs of G_b induced by vertex sets $\{u_1v_2, u_1v_3, u_1v_4, u_2v_3, u_2v_4\}$ and $\{u_3v_3, u_4v_2, u_4v_3, u_5v_2, u_5v_3, u_5v_4\}$. $E(G_d), E(G_e)$ can only have a member each in M and if $u_{\{3,4\}}v_3 \in M$, then no member of $E(G_d)$ belongs M. Then $|M| \leq 4$, which is a contradiction. If $i = j = 1$, let G_d be induced by $\{u_1v_2, u_1v_3, u_1v_4, u_2v_3\}$ and G_e by $\{u_2v_3, u_4v_2, u_4v_3, u_4v_4, u_5v_2, u_5v_3, u_5v_4\}$. Following similar argument as above, $E(G_d)$ and $E(G_e)$ have maximum of three members in M, and thus $|M| = 5$, which is a contradiction. Suppose V_5 contains three saturated vertices, such that none is adjacent to another. Clearly the saturated vertices are u_1v_5, u_3v_5 and u_5v_5. Since they are saturated, then $u_1v_{\{3,4\}}, u_3v_{\{3_4\}}, u_5v_{\{3,4\}} \in M$. Therefore by this, only two vertices on V_3 is saturable. Obviously $|M| \leq 5$. □

Remark 19. Let $G_a \subseteq G_{n,m}$ be a $G_{5,9}$ grid induced by $V_1, V_2, \cdots, V_9$, an induced matching M. From Lemma 16 and Lemma 18, we see that if $u_{\{1,2\}}v_i \in M$, it is possible to have some $M_a \subseteq M$, for which $|M_a| = \text{Max}(G_a)$ as seen in the grid above. Also,

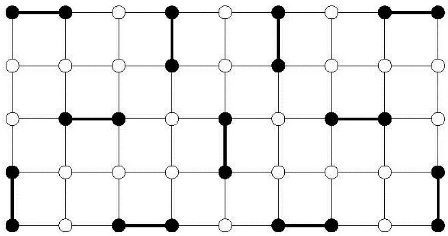

Figure 4. A $G_{5,9}$ grid with $Max(G_{5,9}) = 11$

from Lemma 18, since V_1 has three saturated vertices with two of them being adjacent, then for M_a to be maximal, V_9 will also have three saturated vertices. It is easy to determine, however, that if this scheme extends to $m \geq 13$, then $|M| \neq \text{Max}(G_{n,m})$ since $|M| \leq 11 + 4j + 5k$, $j \geq k$, and $j - k \leq 1$.

Lemma 20. *Let M be a matching of $G_{5,m}$, $m \equiv 1 \mod 4$, $m \geq 13$, with $u_{\{1,2\}}v_1 \in M$, then $|M| \neq Max(G_{5,m})$.*

Proof. Let $m - 10 = q$. Clearly, $q \equiv 3 \mod 4$. For $q \geq 7$, let $G_{5,q} = G_a$, induced by $V_{11}, V_{12}, \cdots V_m$. We already know that $|V_{sb}(G_a)| = \frac{5q+1}{2}$, while $|V_s(G_a)| = 2\left(\frac{5(q-1)-2}{4}\right)$. Therefore, there are two saturable vertices, say, v_1, v_2 on $V(G_a)$. Suppose that $v_1, v_2 \in V_{11}$. Note that v_1, v_2 are not adjacent, else $v_1v_2 \in M$. Let $G_b \subset G_{5,m}$ be a $G_{5,9}$, induced by $V_1, V_2, \cdots, V_9$. Suppose that $u_{\{1,2\}}v_1 \in M$, then there are two adjacent saturated vertices in V_9 and at least a saturated boundary vertex, by Lemma 18 and Remark 19. This implies that there are two adjacent saturable vertices on V_{10}, say u_1, u_2. Now, there could be at most one edge in M from $\{u_1, u_2, v_1, v_2\}$. Hence, $|M| = \text{Max}(G_b) + \text{Max}(G_a) + 1 = \frac{45+5q}{4}$. Since $q = m - 10$, we have $|M| = \frac{5(m-1)}{4}$, which is less than $\text{Max}(G_{5,m})$ by an edge, and hence a contradiction. For $q = 3$, the result is similar to that obtained by careful observation of the positions u_1, u_2 and the possible isolated vertex or vertices on G_a. □

Corollary 21. *Let M be a matching of $G_{5,m}$, $m \geq 13$, with $u_{\{1,2\}}v_i \in M$, $1 \leq i \leq m$, then $|M| \neq Max(G_{5,m})$.*

Lemma 22. *Let $U_1 \subset V(G_{5,m})$. Then there are at least $\frac{m+1}{2}$ saturable vertices in U_1.*

Proof. This follows similar arguments as in the proof of Theorem 11. □

Remark 23. We note that for $m = 4k+1$, $\frac{m+1}{2}$ is odd, also from Lemma 20, the number of saturated vertices on U_1 is even and there cannot be any isolated vertex on U_1 if M is a maximal induced matching of $G_{5,m}$. Therefore, for $G_1 \subset G_{5,m}$, induced by U_1, there exists at least $k+1$ edges of G_1 in M, for $m \geq 13$. For $m = 9$, there are at least k edges in G_1 as seen in the last figure.

Theorem 24. *Let $G_{n,m}$ be a grid with $m \equiv 1 \mod 4$, $m \geq 13$ and let M be the maximum induced matching of $G_{n,m}$. Then*

$$Max(G_{n,m}) \leq \begin{cases} \left\lfloor \frac{2mn-m-1}{8} \right\rfloor & \text{if } n \equiv 1 \mod 4; \\ \left\lfloor \frac{2mn-m+3}{8} \right\rfloor & \text{if } n \equiv 3 \mod 4 \end{cases}$$

Proof. Let $n = 4l + 1$, l a positive integer, and let $r = n - 5$, that is, $r \equiv 0 \mod 4$. Suppose G_a is a $G_{r,m}$ induced by $V_1, V_2, \cdots, V_r$. By Theorem 22, at least there are $\frac{m+1}{2}$, saturated vertices on U_r and these are $u_r v_2, u_r v_4, \cdots, u_r v_{m-1}$. Let G_b be a $G_{5,m}$ grid, induced by $V_{r+1}, V_{r+2}, \cdots, V_n$ and G_c be a $G_{1,m}$ grid, induced by U_{r+1}. By Remark 23, there are $k + 1$ edges of $E(G_c)$ in M. Clearly a saturated vertex induced by M on each of the $k + 1$ edges is adjacent to some saturated vertices in U_r, implying that only one of the two saturated vertices belongs to $V_{sb}(G_{n,m})$. Hence, $V_{sb}(G_{n,m}) \leq \frac{nm+1}{2} - k + 1$. Now, $k+1 = \frac{m-1}{4} + 1$. Therefore, $V_{sb}(G_{n,m}) \leq \frac{2nm-m-1}{4}$ and hence, $|M| \leq \left\lfloor \frac{2nm-m-1}{8} \right\rfloor$. For $n = 4l + 3$, set $s = n - 3$, that is, $s \equiv 0 \mod 4$. By Remark 23, and following the argument above, $V_{sb}(G_{n,m}) = \frac{mn+1}{2} - k$ with $k = \frac{m-1}{4}$, $\mathrm{Max}(G_{n,m}) \leq \left\lfloor \frac{2mn-m+3}{8} \right\rfloor$. □

Remark 25. For the grid $G_{n,m}$, it should be noted that the results in the last theorem depend mainly on the value of m.

Remark 26. Following similar argument as Theorem 24, for $m = 9$, $\mathrm{Max}(G_{n,9}) \leq \left\lfloor \frac{18n-6}{8} \right\rfloor$.

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References

[1] K. Cameron. Induced matching in intersection graphs. *Discrete Math.* **278** (2004), 1–9.

[2] K. Cameron, R. Sritharan and Y. Tang. Finding a maximum induced matching in weakly chordal graphs. *Discrete Math.* **266** (2003), 133-142.

[3] K. K. Dabrowski, M. Demange and V. Lozin. New results on maximum induced matchings in bipartite graphs and beyond. *Theoretical Computer Science,* **478** (2013), 33–40.

[4] J. Edmonds, Paths, trees, and flowers, *Canad. J. Math.* **17** (1965), 449–467 .

[5] M. C. Golumbic and R.C Laskar, Irredundancy in circular arc graphs. *Discrete Applied Math.* **44** (2013), 79–89.

[6] F. Joos, Induced matching in graphs of bounded maximum degree. *arXiv:* (2014) 1406.2440.

[7] R. Marinescu-Ghemeci, Maximum induced matchings in grids. *Optimization Theory, Decision Making and Operations Research Applications. Springer New York*, (2013) 177–187.

[8] V. H. Nguyen, Induced matching in graphs of degree at most 4. *arXiv:* (2014) 1407.7604.

[9] L. J. Stockmeyer and V.V. Vazirani. $NP-$completeness of some generalizations of the maximum matching problem. *Information Processing Letters*, **15**(1) (1982), 14–19.

[10] M. Zito, "Induced matching in regular graphs and trees". Lecture Notes in Computer Sci. 1665, Springer, Berlin, 1999

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Coefficient inequality for transforms of bounded turning functions

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Abstract
The objective of this paper is to obtain sharp upper bound for the second Hankel functional associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of normalized analytic function $f(z)$ when it belongs to bounded turning functions, defined on the open unit disc in the complex plane, with the help of Toeplitz determinants.

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1 Introduction

Let A denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined in the open unit disc $E = \{z : |z| < 1\}$, satisfying the conditions that $f(0) = 0$ and $f'(0) = 1$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function, its n^{th} Taylor coefficient is bounded by n (see [2]). The bounds for the coefficients of these functions give the information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by

*corresponding author

the bound of its second coefficient. The k^{th} root transform for the function f given in (1.1) is defined as

$$F(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.2}$$

Now, we introduce the Hankel determinant for the k^{th} root transform for the function f, for $q, n, k \in \mathbb{N} = \{1, 2, 3, ...\}$, defined as

$$[H_q(n)]^{\frac{1}{k}} = \begin{vmatrix} b_{kn} & b_{kn+1} & \cdots & b_{k(n+q-2)+1} \\ b_{kn+1} & b_{k(n+1)+1} & \cdots & b_{k(n+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k(n+q-2)+1} & b_{k(n+q-1)+1} & \cdots & b_{k[n+2(q-1)-1]+1} \end{vmatrix} (b_k = 1).$$

In particular for $k = 1$, the above determinant reduces to the Hankel determinant defined by Pommerenke [9] for the function f given in (1.1). For the values $q = 2, n = 1$ and $q = 2, n = 2$, the above Hankel determinant simplifies respectively to

$$[H_2(1)]^{\frac{1}{k}} = \begin{vmatrix} b_k & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix} = b_{2k+1} - b_{k+1}^2$$

$$\text{and} \qquad [H_2(2)]^{\frac{1}{k}} = \begin{vmatrix} b_{2k} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix} = b_{2k} b_{3k+1} - b_{2k+1}^2. \tag{1.3}$$

Ali et al. [1] obtained sharp bounds for the Fekete-Szegö functional denoted by $|b_{2k+1} - \mu b_{k+1}^2|$ associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of the function f given in (1.1) and belonging to certain subclasses of S. We refer to $[H_2(2)]^{\frac{1}{k}}$ as the second Hankel determinant for the k^{th} root transform associated with the function f. In the present paper, we consider the Hankel determinant given by $[H_2(2)]^{\frac{1}{k}}$ and obtain sharp upper bound to the functional $|b_{k+1} b_{3k+1} - b_{2k+1}^2|$ for the k^{th} root transform of the function f when it belongs to certain subclass denoted by $\Re$ of S, consisting of functions whose derivative has a positive real part, defined as follows.

Definition 1. Let f be given by (1.1). Then $f \in \Re$, if it satisfies the condition

$$Re f'(z) > 0, \qquad \forall z \in E.$$

The subclass $\Re$ was introduced by Alexander in 1915 and a systematic study of properties of these functions was conducted by MacGregor [7] in 1962, who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part (also called Bounded turning functions).

Some preliminary Lemmas required for proving our result are as follows:

2 Preliminary Results

Let $\mathcal{P}$ denote the class of functions consisting of p such that

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}$$

which are analytic (regular) in the open unit disc E and satisfy $\text{Re}p(z) > 0$, for any $z \in E$. Here $p(z)$ is called a Caratheódory function [3].

Lemma 2 ([8], [10]). *If $p \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 3 ([4]). *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, $\sum_{k=1}^{m} \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in (see [4]) is due to Carathe*ó*dory and Toeplitz. Without loss of generality, in view of Lemma 2, we consider $c_1 > 0$. On using Lemma 3, for $n = 2$ and $n = 3$ respectively, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix}$$

On expanding the determinant, we get

$$D_2 = [\ 8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4 \mid c_1 \mid^2\ ] \geq 0,$$

Applying the fundamental principles of complex numbers, the above expression is equivalent to

$$2c_2 = c_1^2 + y(4 - c_1^2), \text{ for some complex value of } y \text{ with } |y| \leq 1. \tag{2.2}$$

$$\text{and} \quad D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

Simplifying the relations (2.2) and (2.3), we obtain

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta\} \tag{2.4}$$

for some complex values y and ζ with $|y| \leq 1$ and $|\zeta| \leq 1$ respectively.
To obtain our main result, we refer to the classical method developed by Libera and Zlotkiewicz [6], which has been used widely (see [11, 12, 13, 14, 15]).

3 Main Result

Theorem 4. *If $f \in \Re$ and F is the k^{th} root transformation of f given by (1.2) then*

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \frac{4}{9k^2}$$

and the inequality is sharp.

Proof. For $f \in \Re$, by virtue of Definition 1, we have

$$f'(z) = p(z), \qquad \forall z \in E. \tag{3.1}$$

Using the series representation for f and p in (3.1), upon simplification, we obtain

$$a_{n+1} = \frac{c_n}{n+1}, \quad n \in \mathbb{N}. \tag{3.2}$$

For the function f given in (1.1), on computing, we have

$$\left[f(z^k)\right]^{\frac{1}{k}} = \left[z^k + \sum_{n=2}^{\infty} a_n z^{nk}\right]^{\frac{1}{k}} = z + \frac{1}{k}a_2 z^{k+1} + \left\{\frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2\right\}z^{2k+1}$$
$$+ \left\{\frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1} + \cdots. \tag{3.3}$$

From the equations (1.2) and (3.3), we get

$$b_{k+1} = \frac{1}{k}a_2 \ ; \quad b_{2k+1} = \frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2 \ ;$$
$$b_{3k+1} = \frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3. \tag{3.4}$$

Simplifying the expressions (3.2) and (3.4), we get

$$b_{k+1} = \frac{c_1}{2k} \ ; \quad b_{2k+1} = \frac{c_2}{3k} - \frac{(k-1)}{8k^2}c_1^2 \ ;$$
$$b_{3k+1} = \frac{c_3}{4k} - \frac{(k-1)}{6k^2}c_1c_2 + \frac{(k-1)(2k-1)}{48k^3}c_1^3. \tag{3.5}$$

Substituting the values of b_{k+1}, b_{2k+1} and b_{3k+1} from (3.5) in the functional $|b_{k+1}b_{3k+1} - b_{2k+1}^2|$, which simplifies to give

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| = \frac{1}{576k^4}\left|(72c_1c_3 - 64c_2^2)k^2 + 3(k^2-1)c_1^4\right|. \tag{3.6}$$

Substituting c_2 and c_3 values from (2.2) and (2.4) respectively, on the right-hand side of the expression (3.6), we have

$$576k^4|b_{k+1}b_{3k+1} - b_{2k+1}^2| = \Big|\Big[72c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)y - c_1(4-c_1^2)y^2$$
$$+2(4-c_1^2)(1-|y|^2)\zeta\} - 64 \times \frac{1}{4}\{c_1^2 + y(4-c_1^2)\}^2\Big]\, k^2 + 3(k^2-1)c_1^4\Big|.$$

Then applying the triangle inequality and using the fact $|\zeta| < 1$, will give

$$576k^4|b_{k+1}b_{3k+1} - b_{2k+1}^2| \le \big|(5k^2-3)c_1^4 + 36k^2c_1(4-c_1^2) + 4k^2c_1^2(4-c_1^2)|y|$$
$$+2(c_1+2)(c_1+16)k^2(4-c_1^2)|y|^2\big|. \tag{3.7}$$

Choosing $c_1 = c \in [0,2]$, noting that $(c_1+a)(c_1+b) \ge (c_1-a)(c_1-b)$, where $a, b \ge 0$, applying the triangle inequality and replacing $|y|$ by μ on the right-hand side of (3.7), we obtain

$$\begin{aligned} 576k^4|b_{k+1}b_{3k+1} - b_{2k+1}^2| \;\le\; & \Big[\big(5k^2-3\big)c^4 + 36k^2c(4-c^2) + 4k^2c^2(4-c^2)\mu \\ & + 2(c-2)(c-16)k^2(4-c^2)\mu^2\;\Big] \\ & = F(c,\mu),\ \text{for } 0 \le \mu = |y| \le 1. \end{aligned} \tag{3.8}$$

$$\begin{aligned} \text{Here}\ \ F(c,\mu) = & \big(5k^2-3\big)c^4 + 36k^2c(4-c^2) + 4k^2c^2(4-c^2)\mu \\ & + 2(c-2)(c-16)k^2(4-c^2)\mu^2. \end{aligned} \tag{3.9}$$

Next, we need to find the maximum value of the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. For this, let us suppose that there exists a maximum value at any point (c,μ) in the interior of the closed region $[0,2]\times[0,1]$ for the function $F(c,\mu)$. Differentiating $F(c,\mu)$ in (3.9) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 4k^2\big\{c^2 + (c-2)(c-16)\mu\big\}(4-c^2). \tag{3.10}$$

For $0<\mu<1$, for fixed c with $0<c<2$, from (3.10), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point (c,μ) in the interior of the closed region $[0,2]\times[0,1]$. The maximum value of $F(c,\mu)$ occurs on the boundary only i.e., when $\mu=1$. Therefore, for fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) = G(c). \tag{3.11}$$

In view of (3.11), replacing μ by 1 in (3.9), we get

$$G(c) = -(k^2+3)c^4 - 40k^2c^2 + 256k^2, \tag{3.12}$$

$$G'(c) = -4(k^2+3)c^3 - 80k^2c. \tag{3.13}$$

From the expression (3.13), we observe that $G'(c)\le 0$ for all values of c in the interval $[0,2]$ and for every k. Therefore, $G(c)$ is a monotonically decreasing function of c in the interval $[0,2]$ and hence it attains the maximum value at $c=0$ only. From (3.12), the maximum value $G(c)$ at $c=0$ is given by

$$\max_{0\le c\le2} G(0) = 256k^2. \tag{3.14}$$

Simplifying the relations (3.8) and (3.14), we obtain

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \le \frac{4}{9k^2}. \tag{3.15}$$

Choosing $c_1=c=0$ and selecting $y=1$ in (2.2) and (2.4), we find that $c_2=2$ and $c_3=0$. Substituting the values $c_2=2$ and $c_1=c_3=0$ in (3.5) and the obtained values in (3.15), we see that equality is attained, which shows that our result is sharp. For these values, from (2.1), we can derive

$$p(z) = 1 + 2\sum_{n=1}^{\infty} z^{2n} = \frac{1+z^2}{1-z^2}. \tag{3.16}$$

Therefore, in this case the extremal function is

$$f'(z) = 1 + 2\sum_{n=1}^{\infty} z^{2n}.$$

This completes the proof of our Theorem. □

Remark 5. By choosing $k = 1$ in (3.15), the result coincides with that of Janteng et al. [5].

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References

[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, The Fekete-Szegö coefficient functional for transforms of analytic functions, *Bull. Iranian Math. Soc.,* **35**(2), 119–142 (2009).

[2] L. de Branges de Bourcia, A proof of Bieberbach conjecture, *Acta Mathematica*, **154**(1-2), 137–152 (1985).

[3] P. L. Duren, "Univalent functions", vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 1983.

[4] U. Grenander and G. Szegö, "Toeplitz forms and their applications", 2nd ed., Chelsea Publishing Co., New York (NY), 1984.

[5] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, *J. Inequal. Pure Appl. Math.*,**7**(2), 1–5 (2006).

[6] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, *Proc. Amer. Math. Soc.*, **87**(2) 251–257 (1983).

[7] T. H. MacGregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.*, **104**(3) , 532 – 537 (1962).

[8] Ch. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.

[9] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, *J. Lond. Math. Soc.*, **41**(1) , 111–122 (1966).

[10] B. Simon, "Orthogonal polynomials on the unit circle", part 1. Classical theory, vol. 54, American mathematical society colloquium publications. Providence (RI): American Mathematical Society; 2005.

[11] D. Vamshee Krishna and T. RamReddy, Coefficient inequality for multivalent bounded turning functions of order alpha, *Probl. Anal. Issues Anal.*, **23**(1), 45–54 (2016).

[12] D. Vamshee Krishna and T. RamReddy, Coefficient inequality for parabolic star like functions of order alpha, *Afr. Mat.*, **27**(1-2), 121–132 (2016).

[13] D. Vamshee Krishna and T. RamReddy, An upper bound to the second Hankel determinant for functions in Mocanu class, *Viet. J. Math.*, **43**(3), 541–549 (2015).

[14] D. Vamshee Krishna and T. RamReddy,Coefficient inequality for certain p- valent analytic functions, *Rocky Mountain J. Math.*, **44**(6), 1941–1959 (2014).

[15] D. Vamshee Krishna and T. RamReddy, An upper bound to the second Hankel determinant for certain subclass of analytic functions, *Proc. Janjgeon Math. Soc.*, **16**(4), 559–568 (2013).

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New sufficient conditions for starlikeness of certain integral operators involving Bessel functions

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Abstract

The purpose of the present paper is to investigate a new integral operator associated with Bessel function. Some sufficient conditions are derived for this integral operator belonging to various subclasses of starlike functions under certain conditions.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by S the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are also univalent in U. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $\delta(0 \leq \delta < 1)$ if the following condition is satisfied

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \qquad (z \in \mathbb{U}), \tag{1.2}$$

we denote by $S^*(\delta)$ the class of starlike functions of order δ. Clearly $S^*(\delta) \subset S^*(0) = S^*(0 \leq \delta < 1)$ and $S^* \subset S$.

The Bessel function of the first kind of order ν is defined by the infinite series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)}, \tag{1.3}$$

where Γ stands for the Euler-Gamma function $z \in \mathbb{C}$ and $\nu \in \mathbb{R}$. Recently, Szasz and Kupan [17] investigated the univalence of the normalized Bessel function of first kind $g_\nu : U \longrightarrow \mathbb{C}$ defined by

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}). \tag{1.4}$$

Baricz and Frasin [3] have obtained the sufficient condition for the univalence of the various integral operators involving Bessel functions of the first kind of order ν.

Recently, Frasin [5] introduced the following integral operator involving the normalized Bessel function of the first kind

$$F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_{\nu_i}(t)}{t}\right)^{\alpha_i} dt, \tag{1.5}$$

and obtain several sufficient condition for this operator to be convex and strongly convex of given order in the open disc $\mathbb{U}$. Recently analogous to these result Porwal and Breaz [15] studied the sufficient condition for the operator defined by (1.5) for certain class of univalent functions. Very recently, Mishra and Panigrahi [9] obtain some sufficient conditions for starlikeness of certain integral operator. In this paper, motivated with the above mentioned work and work of Kumar [6] we obtain some sufficient condition for the operator defined by (1.5) is in the class S^*.

To prove our main results we shall require the following lemmas:

Lemma 1. *([17]) Let $\nu > \frac{-5+\sqrt{5}}{4}$ and consider the normalized Bessel function of the first kind $g_\nu : \mathbb{U} \longrightarrow \mathbb{C}$ defined by*

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}),$$

where J_ν stands for Bessel function of the first kind, then the following inequality hold for all $z \in \mathbb{U}$.

$$\left|\frac{z g_\nu'(z)}{g_\nu(z)} - 1\right| \leq \frac{\nu+2}{4\nu^2+10\nu+5}.$$

Lemma 2. *([16]) If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{\delta+1}{2(\delta-1)}, \qquad (z \in \mathbb{U}),$$

for some $2 \leq \delta < 3$,
or

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{5\delta-1}{2\delta+1}, \qquad (z \in \mathbb{U}),$$

for some $1 \leq \delta \leq 2$, then $f \in S^*$.

Lemma 3. *([16]) If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{\delta+1}{2\delta(\delta-1)}, \qquad (z \in \mathbb{U}),$$

for some $\delta \leq -1$,
or

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{3\delta+1}{2\delta(\delta+1)}, \qquad (z\in\mathbb{U}),$$

for some $\delta > 1$, *then* $f\in S^*(\frac{\delta+1}{2\delta})$.

Throughout this paper, we frequently use the notation $F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z) = F_{\nu_i,\alpha_i}(z)$.

2 Main Results

Theorem 4. *Let n be a natural number such that $\nu_1, \nu_2, ..., \nu_n > (\frac{-5+\sqrt{5}}{4})$. Consider the function $g_{\nu_i} : \mathbb{U}\to\mathbb{C}$ defined by*

$$g_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i+1)z^{1-\frac{\nu_i}{2}}J_{\nu_i}(z^{\frac{1}{2}}).$$

Let $\nu = min\{\nu_1, \nu_2, ..., \nu_n\}$ and suppose that the inequality

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i \leq \frac{3-\delta}{2(\delta-1)} \tag{2.1}$$

is satisfied. Then the function $F_{\nu_i,\alpha_i}(z)$ defined by (1.5) is in the class S^* for some $2 \leq \delta \leq 3$.

Proof. First we observe that, since for all $i\in\{1, 2, \ldots, n\}$, we have $g_{\nu_i}\in\mathcal{A}$ i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i,\alpha_i}(z) = \prod_{i=1}^{n}\left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}.$$

Taking logarithmic differentiation

$$\frac{F''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{g'_{\nu_i}(z)}{g_{\nu_i}(z)} - \frac{1}{z}\right)$$

$$1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)} = 1+\sum_{i=1}^{n}\alpha_i\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right)$$

$$\begin{aligned}\Re\left\{1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} &= 1+\sum_{i=1}^{n}\alpha_i\Re\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right)\\ &\leq 1+\sum_{i=1}^{n}\alpha_i\left|\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right|\\ &\leq 1+\sum_{i=1}^{n}\alpha_i\left(\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}\right).\end{aligned} \tag{2.2}$$

For all $z\in\mathbb{U}$ and $\nu_1, \nu_2, ..., \nu_n > \frac{-5+\sqrt{5}}{4}$. Since the function $\phi : \left(\frac{-5+\sqrt{5}}{4}, \infty\right) \to \mathbb{R}$, defined by,

$$\phi(x) = \frac{x+2}{4x^2+10x+5}$$

is decreasing and consequently for all $i\in\{1,2,.......,n\}$. We have

$$\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+2}\leq\frac{\nu+2}{4\nu^2+10\nu+5}.$$

Using this result, inequality (2.7) can be written as

$$\Re\left\{1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\}\leq 1+\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i.$$

Since

$$1+\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i<\frac{\delta+1}{2(\delta-1)}.$$

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i<\frac{\delta+1}{2(\delta-1)}-1$$

$$=\frac{\delta+1-2\delta+2}{2(\delta-1)}$$

$$=\frac{3-\delta}{2(\delta-1)}$$

Hence from Lemma 2 $F_{\nu_i,\alpha_i}(z)\in S^*$ for some $2\leq\delta\leq 3$.

Thus the proof of Theorem 4 is established. □

Theorem 5. *Let n be a natural number and $\nu_1,\nu_2,......,\nu_n>\left(\frac{-5+\sqrt{5}}{4}\right)$. Consider the function $g_{\nu_i}:\mathbb{U}\to\mathbb{C}$ defined by*

$$g_{\nu_i}(z)=2^{\nu_i}\Gamma(\nu_i+1)z^{1-\frac{\nu_i}{2}}J_{\nu_i}(z^{\frac{1}{2}}).$$

Let $\nu=max\{\nu_1,\nu_2,.....,\nu_n\}$ be a positive real numbers and suppose that the inequality

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i\leq\frac{2\delta^2-\delta+1}{2\delta(\delta-1)},\tag{2.3}$$

is satisfied then $F_{\nu_i,\alpha_i}(z)$ defined by (1.5) is in the class $S^*(\frac{\delta+1}{2\delta})$ for some $\delta\leq -1$.

Proof. First we observe that since for all $i\in\{1,2,3,\ldots,n\}$, we have $g_{\nu_i}\in\mathcal{A}$ i.e.

$$g_{\nu_i}(0)=g'_{\nu_i}(0)-1=0.$$

From (1.5) we have

$$F'_{\nu_i,\alpha_i}(z)=\prod_{i=1}^{n}\left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}.$$

Taking logarithmic differentiation,

$$\Re\left\{1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\}=\sum_{i=1}^{n}\alpha_i\Re\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}-1\right)+1.$$

From Lemma 1, we have

$$\left|\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}-1\right| \le \frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}.$$

Since $\Re(z) \le |z|$

$$\Re\left\{1-\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right\} \le \frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}$$

$$\Re\left\{\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right\} \ge 1-\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}$$

$$\Re\left\{1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} \ge \sum_{i=1}^{n}\alpha_i\left(1-\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}\right)+1-\sum_{i=1}^{n}\alpha_i$$

$$= -\left(\frac{\nu+2}{4\nu^2+10\nu+5}\right)\sum_{i=1}^{n}\alpha_i+1$$

$$\ge -\frac{\delta+1}{2\delta(\delta-1)}$$

$$-\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i \ge -1-\frac{\delta+1}{2\delta(\delta-1)}$$

$$\frac{\nu+2}{4\nu^2+10\nu+2}\sum_{i=1}^{n}\alpha_i \le 1+\frac{\delta+1}{2\delta(\delta-1)}$$

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i \le \frac{2\delta^2-2\delta+\delta+1}{2\delta(\delta-1)}$$

$$= \frac{2\delta^2-\delta+1}{2\delta(\delta-1)}$$

Hence from Lemma 3, $F_{\nu_i,\alpha_i}(z) \in S^*(\frac{\delta+1}{2\delta})$ for some $\delta \le -1$.
Thus the proof of Theorem 5 is established.

□

Theorem 6. *Let n be a natural number such that $\nu_1, \nu_2, \ldots, \nu_n > \left(\frac{-5+\sqrt{5}}{4}\right)$. Consider the function $g_{\nu_i} : \mathbb{U} \to \mathbb{C}$ defined by*

$$g_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i+1)z^{1-\frac{\nu_i}{2}}J_{\nu_i}(z^{\frac{1}{2}}).$$

Let $\nu = min\{\nu_1, \nu_2, \ldots\ldots, \nu_n\}$ and suppose that the inequality

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i \le \frac{5\delta-1}{2(\delta+1)} \tag{2.4}$$

is satisfied. Then the function $F_{\nu_i,\alpha_i}(z)$ defined by (1.5) is in the class S^* for some $1 \le \delta \le 2$.

Proof. First we observe that, since for all $i\in\{1,2,\ldots,n\}$. We have $g_{\nu_i}(z)\in\mathcal{A}$ i.e.

$$g_{\nu_i}(0)=g'_{\nu_i}(0)-1=0.$$

From (1.5) we have

$$F'_{\nu_i,\alpha_i}(z)=\prod_{i=1}^{n}\left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}$$

$$\left(\frac{F''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right)=\sum_{i=1}^{n}\alpha_i\left(\frac{g'_{\nu_i}(z)}{g_{\nu_i}(z)}-\frac{1}{z}\right)$$

$$\Re\left\{1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\}=1+\sum_{i=1}^{n}\alpha_i\Re\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}-1\right)$$

$$\leq 1+\sum_{i=1}^{n}\alpha_i\left|\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}-1\right|$$

$$\leq 1+\sum_{i=1}^{n}\alpha_i\left(\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}\right)$$

$$\leq 1+\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i.$$

For all $z\in\mathbb{U}$ and $\nu_1,\nu_2,........,\nu_n>\frac{-5+\sqrt{5}}{4}$.
Since the function,
$\phi:\left(\frac{-5+\sqrt{5}}{4},\infty\right)\to\mathbb{R}$ defined by

$$\phi(x)=\frac{x+2}{4x^2+10x+5}$$

is decreasing and consequently for all $i\in\{1,2,......,n\}$, we have

$$\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}\leq\frac{\nu+2}{4\nu^2+10\nu+5}$$

Since,

$$1+\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i\leq\frac{5\delta-1}{2(\delta+1)}$$

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i\leq\frac{5\delta-1}{2(\delta+1)}-1$$

$$=\frac{5\delta-1-2(\delta+1)}{2\delta+1}$$

$$=\frac{5\delta-1-2\delta-2}{2(\delta+1)}$$

$$=\frac{3(\delta-1)}{2(\delta+1)}.$$

Hence from Lemma 2 , $F_{\nu_i,\alpha_i}(z)\in S^*$ for some $1\leq\delta\leq 2$
Thus the proof of Theorem 6 is established. □

Theorem 7. *Let n be a natural number and $\nu_1, \nu_2......, \nu_n > \left(\frac{-5+\sqrt{5}}{4}\right)$. Consider the function $g_{\nu_i} : \mathbb{U} \to \mathbb{C}$ defined by*

$$g_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i + 1)z^{1-\frac{\nu_i}{2}} J_{\nu_i}(z^{\frac{1}{2}}).$$

Let $\nu = max\{\nu_1, \nu_2, ..., \nu_n\}$ be positive real numbers and suppose that the inequality

$$\frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^{n} \alpha_i \leq \frac{\delta + 1 - 2\delta^2}{2\delta(\delta + 1)}$$

is satisfied then $F_{\nu_i,\alpha_i}(z)$ defined by (1.5) is in the class $S^*(\frac{\delta+1}{2\delta})$ for some $\delta > 1$.

Proof. First we observe that since for all $i \in \{1, 2, \dots, n\}$. We have $g_{\nu_i} \in \mathcal{A}$ i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i,\alpha_i}(z) = \prod_{i=1}^{n} \left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}.$$

Taking logarithmic differentiation

$$\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} = \sum_{i=1}^{n} \alpha_i \Re\left\{\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right\} + 1$$

$$\left|\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right| \leq \frac{\nu_i + 2}{4{\nu_i}^2 + 10\nu_i + 5}$$

$$\Re\left\{1 - \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right\} \leq \frac{\nu_i + 2}{4{\nu_i}^2 + 10\nu_i + 5}$$

$$\Re\left\{\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right\} \geq 1 - \frac{\nu_i + 2}{4{\nu_i}^2 + 10\nu_i + 5}$$

$$\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} \geq \sum_{i=1}^{n} \alpha_i \left(1 - \frac{\nu_i + 2}{4{\nu_i}^2 + 10\nu_i + 5}\right) + 1 - \sum_{i=1}^{n} \alpha_i$$

$$\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} \geq \left(1 - \frac{\nu + 2}{4\nu^2 + 10\nu + 5}\right) \sum_{i=1}^{n} \alpha_i + 1 - \sum_{i=1}^{n} \alpha_i.$$

Here $\phi(x) = \frac{x+2}{4x^2+10x+5}$ is decreasing. So for all $i \in \{1, 2, \dots, n\}$.

$$\left(1 - \frac{\nu + 2}{4\nu^2 + 10\nu + 5}\right) \sum_{i=1}^{n} \alpha_i + 1 - \sum_{i=1}^{n} \alpha_i \geq \frac{3\delta + 1}{2\delta(\delta + 1)}$$

$$-\frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^{n} \alpha_i \geq \frac{3\delta + 1}{2\delta(\delta + 1)} - 1$$

$$\geq \frac{3\delta + 1 - 2\delta(\delta + 1)}{2\delta(\delta + 1)}$$

$$= \frac{3\delta + 1 - 2\delta^2 - 2\delta}{2\delta(\delta + 1)}$$

$$= \frac{\delta + 1 - 2\delta^2}{2\delta(\delta + 1)}.$$

Hence from Lemma (2), $F_{\nu_i,\alpha_i} \in S^*(\frac{\delta+1}{2\delta})$ for some $\delta > 1$.
Thus the proof of Theorem 7 is established. □

Remark 8.
1. One may also obtain the analogukes of these results for various subclasses of analytic functions e.g. Convex functions, Strongly convex functions and Strongly starlike functions, Spiral-like functions etc.
2. The results of this paper can also be extended by using the integral operator studied by Breaz *et al.* [4], Merkes and Wright [7], Miller *et al.* [8], Pesker [10], Porwal and Kumar [13] and Porwal and Singh [14].
3. Recently Baricz [1] introduced generalized Bessel functions of first kind which is a natural generalization of Bessel function, Modified Bessel function, Spherical Bessel function and Modified spherical Bessel function and give wide applications in Geometric Function Theory. For detailed study one may refer [2], (see also [11], [12]).
4. It is interesting to find the analogues of these results for harmonic starlike and convex functions.

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References

[1] A. Baricz, "Generalized Bessel functions of the first kind", Ph.D. Thesis, Babes-Bolyai University, Cluj-Napoca, (2008).

[2] A. Baricz, "Generalized Bessel functions of the first kind", Lecture Notes in Mathematics, vol. 1994, Springer-Verlag, Berlin, (2010).

[3] A. Baricz and B.A. Frasin, Univalence of integral operators involving Bessel functions, *Appl. Math. Lett.*, **23**(4) (2010), 371-376.

[4] D. Breaz, N. Breaz and H.M. Srivastava, An extension of the univalent condition for a family of integral operators, *Appl. Math. Lett.*, **22** (2009), 41-44.

[5] B.A. Frasin, Sufficient condition for integral operator defined by Bessel functions, *J. Math. Inequal.*, **4** (3), (2010), 301-306.

[6] Manish Kumar, "A study of Univalent functions", M.Phil. Dissertation, C.S.J.M. University, Kanpur, India., (2014).

[7] E.P. Merkes and D.J. Wright, On the univalence of a certain integral, *Proc. Amer. Math. Soc.*, **27** (1) (1971), 97-100.

[8] S.S. Miller, P.T. Mocanu and M.O. Reade, Starlike integral operators, *Pacific J. Math.*, **79** (19) (1978), 157-168.

[9] Akshaya Kumar Mishra and Trailokya Panigrahi, New sufficient conditions for Starlikeness of certain integral operator, *Kyungpook Math. J.*, **55**(1) (2015), 109-118.

[10] V. Pescar, A new generalization of Ahlfors's and Becker's criterion of univalence, *Bull. Malaysian Math. Soc. (Second Series)*, **19** (1996), 53-54.

[11] Saurabh Porwal, Harmonic starlikeness and convexity of integral operators generated by generalized Bessel functions, *Acta Math. Vietnam.*, **39**(2014), 337-346

[12] Saurabh Porwal and Moin Ahmad, Some sufficient condition for generalized Bessel functions associated with conic regions, *Vietnam J. Math.*, **43**(2015), 163-172.

[13] Saurabh Porwal and Manish Kumar, Mapping properties of an integral operator involving Bessel functions, *Afrika Matematika*, **28**(1)(2017), 165-170.

[14] Saurabh Porwal and M.K. Singh, Mapping properties a general integral operator defined by the Hadamard product, *Le Matematiche*, **69** (1) (2014), 179-184.

[15] Saurabh Porwal and D. Breaz, *Mapping properties of an integral operator involving Bessel functions*, "Analytic Number Theory and Special Functions", Springer, New York, (2014), 821-826.

[16] H. Shiraishi and S. Owa, Starlikeness and convexity for analytic functions concerned with Jack's Lemma, *International J. Open Problem Compute. Math.*, **2(1)**(2009), 37-47.

[17] R. Szasz and P. Kupan, About the univalence of the Bessel functions, *Stud. University Babes-Bolyoi Math.*, **54(1)**(2009), 127-132.

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Contents