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On stability, boundedness and square integrability conditions for a third-order nonlinear system of differential equations

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Abstract

This paper extends some known results on the stability, boundedness and square integrability of solutions of certain nonlinear vector differential equations of third-order. The Lyapunov's second method is used as basic tool in obtaining the criteria for the stability and boundedness of solutions. Example is included to illustrate the results.

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1 Introduction

The study of the stability and boundedness of ordinary scalar and vector nonlinear differential equations of third order have received tremendous attention. Many works have been done by notable authors, for a comprehensive treatment of this subject see Afuwape [1, 2, 3, 4], Ezeilo [8, 9, 10], Graef [12, 13], Remili [19, 20, 21, 22, 23, 24, 25, 26, 27], Tunç [32, 33, 34, 35, 36, 37, 38, 39], and the references cited therein.

In 1966, 1983,1993 and 2007 respectively, Ezeilo and Tejumola [8], Afuwape [1], Meng [16] and Omeike [17] investigated the ultimately boundedness and existence of periodic solutions of the nonlinear vector differential equation of the form

$$X''' + AX'' + BX' + H(X) = P(t, X, X', X''). \tag{1.1}$$

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Later in 1985, Afuwape [2] demonstrated a result associated with the existence of unique periodic solution of the vector differential equation

$$X''' + AX'' + G(X') + H(X) = P(t, X, X', X'').$$

In 1995, Feng [11] obtained a result associated with the existence of unique periodic solution of the similar type equation

$$X''' + A(t)X'' + B(t)X' + H(X) = P(t, X, X', X''). \tag{1.2}$$

In this paper therefore, using Lyapunov's direct method we obtain criteria for asymptotic stability, boundedness and square integrability of solutions of the equation

$$(H(X(t))X'(t))'' + A(t)X''(t) + B(t)X'(t) + C(t)F(X(t)) = P(t), \tag{1.3}$$

in which $t \in \mathbb{R}^+$ and $X(t), P(t) \in \mathbb{R}^n; A, B,$ and C are continuous $n \times n$ symetric matrices. $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F(0) = 0$, and H is a $n \times n$ symetric differentiable and inversible matrix function. Let $J_F(X)$, $A'(t), B'(t), C'(t)$ and $H'(X)$, denote the jacobian matrices corresponding to $F(X), A(t), B(t), C(t)$ and $H(X)$ respectively, that is, $J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right), A'(t) = \frac{d}{dt}(a_{ij}(t)), B'(t) = \frac{d}{dt}(b_{ij}(t)), C'(t) = \frac{d}{dt}(c_{ij}(t)), H'(X(t)) = \frac{d}{dt}(h_{ij}(X(t)), (i, j = 1, 2, \dots, n)$, where $(x_1, x_2, \dots, x_n), (f_1, f_2, \dots, f_n)$, $(a_{ij}(t)), (b_{ij}(t))$, $(c_{ij}(t))$ and $h_{ij}(X(t))$ are components of X, F, A, B, C and $H(X)$. On the other hand $X(t), Y(t)$ and $Z(t)$ are, respectively, abbreviated as X, Y and Z throughout the paper. Additionally, the symbol $\langle X, Y\rangle$ corresponding to any pair X and Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum\limits_{i=1}^{n} x_i y_i$, that is, $\langle X, Y\rangle = \sum\limits_{i=1}^{n} x_i y_i$.

2 Preliminaries

In this section, we present some lemmas that will be used to establish our main results.

Lemma 1. *[1, 3, 8, 9, 10, 31] Let D be a real symmetric positive definite $n \times n$ matrix. Then for any X in $\mathbb{R}^n$, we have*

$$\delta_d \|XVert^2 \le \langle DX, X\rangle \le \Delta_d \|XVert^2,$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D respectively.

Lemma 2. *[1, 3, 8, 9, 10, 31] Let Q, D be any two real $n \times n$ commuting matrices. Then,*

1) The eigenvalues $\lambda_i(QD)\,(i = 1, 2..., n)$ of the product matrix QD are all real and satisfy

$$\min_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D) \le \lambda_i(QD) \le \max_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D).$$

2) The eigenvalues $\lambda_i(Q + D)\,(i = 1, 2..., n)$ of the sum of matrices Q and D are all real and satisfy.

$$\left\{\min_{1\le j\le n} \lambda_j(Q) + \min_{1\le k\le n} \lambda_k(D)\right\} \le \lambda_i(Q + D) \le \left\{\max_{1\le j\le n} \lambda_j(Q) + \max_{1\le k\le n} \lambda_k(D)\right\}.$$

Lemma 3. *[1, 3, 8, 9, 10, 31, 33] Let $H(X)$ be a continuous vector function with $H(0) = 0$. Then,*

$$1)\quad \frac{d}{dt}\left(\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma\right) = \left\langle H(X), \frac{dX}{dt}\right\rangle.$$

$$2)\quad \int_0^1 \langle C(t)H(\sigma X), X\rangle d\sigma = \int_0^1\int_0^1 \sigma[\langle C(t)J_H(\sigma\tau X)X, X\rangle] d\sigma d\tau.$$

Lemma 4. *[30] Let $H(X)$ be a continuous vector function with $H(0) = 0$. Then,*

$$\begin{aligned} 1) \quad \langle H(X), H(X)\rangle &= \int_0^1\int_0^1 \sigma\langle J_H(\sigma X)J_H(\sigma\tau X)X, X\rangle d\sigma d\tau. \\ 2) \quad \langle C(t)H(X), X\rangle &= \int_0^1 \langle J_H(\sigma X)C(t)X, X\rangle d\sigma. \end{aligned}$$

3 Stability

We shall state here some assumptions which will be used on the functions that appeared in equation (1.3). Suppose that there are positive constants $\delta_A, \delta_B, \delta_C, \delta_F, \delta_H, \delta_{H^{-1}}, \Delta_A, \Delta_B, \Delta_C, \Delta_F, \Delta_H$, and $\Delta_{H^{-1}}$, such that the matrices $A(t), B(t), C(t), H(X), H^{-1}(X)$ and $J_F(X)$ (Jacobian matrix of $F(X)$) are symmetric and positive definite, and furthermore the eigenvalues $\lambda_i(A(t)), \lambda_i(B(t)), \lambda_i(C(t)), \lambda_i(H(X)), \lambda_i(H^{-1}(X))$ and $\lambda_i(J_F(X))(i = 1, 2, \ldots, n)$ of $A(t), B(t), C(t), H(X), H^{-1}(X)$ and $J_F(X)$, respectively satisfy

$$\begin{aligned} &0 < \delta_A \le \lambda_i(A(t)) \le \Delta_A, \qquad 0 < \delta_H \le \lambda_i(H(X)) \le \Delta_H, \\ &0 < \delta_B \le \lambda_i(B(t)) \le \Delta_B, \qquad 0 < \delta_{H^{-1}} \le \lambda_i\left(H^{-1}(X)\right) \le \Delta_{H^{-1}}, \\ &0 < \delta_C \le \lambda_i(C(t)) \le \Delta_C, \qquad 0 < \delta_F \le \lambda_i(J_F(X)) \le \Delta_F. \end{aligned}$$

Before stating the major theorem, we introduce the following notations

$$\begin{aligned} H_t &= H(X(t)), \\ \theta(t) &= (H_t^{-1})' = -H_t^{-1}H_t'H_t^{-1}. \end{aligned} \tag{3.1}$$

We note that equation (1.3) is equivalent to the following system

$$\begin{cases} X' = H_t^{-1}Y, \\ Y' = Z, \\ Z' = -A(t)H_t^{-1}Z - \left(A(t)\theta(t) + B(t)H_t^{-1}\right)Y - C(t)F(X) + P(t), \end{cases} \tag{3.2}$$

which was obtained by setting

$$X'' = \theta(t)Y + H_t^{-1}Z. \tag{3.3}$$

In this section, we establish some conditions for the asymptotic stability of all solutions of (1.3) in the case $P(t) = 0$. We begin with the following Theorem.

Theorem 5. *In addition to the basic assumptions imposed on the matrices A, B, C, H, H^{-1} and J_F witch commute pairwise, assume that :*

i) $\lambda_i(C') \le 0$.

ii) $\dfrac{(\Delta_C)^2(\Delta_F)^2\Delta_H}{\delta_C\delta_F\delta_B} < d < \delta_A$.

iii) $\dfrac{d}{2}\Delta_{A'} + \dfrac{1}{2}\Delta_{B'}\Delta_H + \Delta_{C'}\Delta_{H^2} < \dfrac{d\delta_B - \Delta_C\Delta_F\Delta_H}{2}$.

iv) $\displaystyle\int_0^{+\infty}\left\|\frac{d}{ds}H_s\right\| ds < +\infty$.

Then any solution of (3.2) is asymptotically stable.

Proof. Let η a positive constant which will be specified later. We define the Lyapunov functional $W = W(t, X, Y, Z)$ as

$$W = V \exp\big(-\frac{1}{\eta}\int_0^t \|\theta(s)\|\, ds\big), \tag{3.4}$$

where

$$\begin{aligned} V \ &= \ d\int_0^1 \langle C(t)F(\sigma X), X\rangle d\sigma + \langle C(t)Y, F(X)\rangle + \frac{1}{2}\langle B(t)H_t^{-1}Y, Y\rangle \\ &\quad + d\langle H_t^{-1}Y, Z\rangle + \frac{d}{2}\langle A(t)H_t^{-2}Y, Y\rangle + \frac{1}{2}\langle Z, Z\rangle. \end{aligned} \tag{3.5}$$

It is clear from (3.5) that $V(t, 0, 0, 0) = 0$. By Lemma 1, we have the following inequality

$$\begin{aligned} \langle C(t)Y, F(X)\rangle + \frac{1}{2}\langle B(t)H_t^{-1}Y, Y\rangle \ &\geq \ \langle C(t)Y, F(X)\rangle + \frac{\delta_B\delta_{H^{-1}}}{2}\langle Y, Y\rangle \\ &= \ \frac{\delta_B\delta_{H^{-1}}}{2}\|Y + \frac{1}{\delta_B\delta_{H^{-1}}}C(t)F(X)\|^2 \\ &\quad -\frac{1}{2\delta_B\delta_{H^{-1}}}\langle C^2(t)F(X), F(X)\rangle. \end{aligned}$$

Observe that

$$d\langle H_t^{-1}Y, Z\rangle + \frac{1}{2}\langle Z, Z\rangle = \frac{1}{2}\|Z + dH_t^{-1}Y\|^2 - \frac{d^2}{2}\langle H_t^{-2}Y, Y\rangle.$$

Hence

$$\begin{aligned} V \ &\geq \ d\int_0^1 \langle C(t)F(\sigma X), X\rangle d\sigma - \frac{1}{2\delta_B\delta_{H^{-1}}}\langle C^2(t)F(X), F(X)\rangle \\ &\quad + \frac{d}{2}\langle A(t)H_t^{-2}Y, Y\rangle - \frac{d^2}{2}\langle H_t^{-2}Y, Y\rangle + \frac{1}{2}\|Z + dH_t^{-1}Y\|^2. \end{aligned}$$

In view of Lemma 3 and Lemma 4, it follows that

$$\begin{aligned} \langle C^2(t)F(X), F(X)\rangle \ &\leq \ (\Delta_C)^2(\Delta_F)^2 \|X\|^2 \int_0^1\int_0^1 \sigma d\sigma d\tau = \frac{1}{2}(\Delta_C)^2(\Delta_F)^2 \|X\|^2, \\ \int_0^1 \langle dC(t)F(\sigma X), X\rangle d\sigma \ &\geq \ d\delta_C\delta_F \|X\|^2 \int_0^1\int_0^1 \sigma d\sigma d\tau = \frac{1}{2}d\delta_C\delta_F \|X\|^2. \end{aligned}$$

Therefore, since $\delta_{H^{-1}} = \Delta_H^{-1}$ we have

$$\begin{aligned} V \ &\geq \ \frac{1}{2}\big(d\delta_C\delta_F - \frac{(\Delta_C)^2(\Delta_F)^2}{\delta_B\Delta_H^{-1}}\big)\|X\|^2 \\ &\quad + \frac{d}{2}\langle\Big(A(t) - dI\Big)H_t^{-2}Y, Y\rangle + \frac{1}{2}\|Z + dH_t^{-1}Y\|^2. \end{aligned}$$

Again, in view of Lemma 1, easily, we obtain that

$$\frac{d}{2}\langle\Big(A(t) - dI\Big)H_t^{-2}Y, Y\rangle \geq \frac{d}{2}\big(\delta_A - d\big)\delta_{H^{-2}}\langle Y, Y\rangle.$$

Hence, according to the last estimates, we obtain

$$\begin{aligned} V &\geq \frac{1}{2}\big(d\delta_C\delta_F - \frac{(\Delta_C)^2(\Delta_F)^2\Delta_H}{\delta_B}\big)\,\|X\|^2 + \frac{d}{2}\big(\delta_A - d\big)\delta_{H^{-2}}\|Y\|^2 \\ &\quad + \frac{1}{2}\|Z + dH_t^{-1}Y\|^2, \end{aligned}$$

with the coefficients $\big(d\delta_C\delta_F - \frac{(\Delta_C)^2(\Delta_F)^2\Delta_H}{\delta_B}\big) > 0$ and $\big(\delta_A - d\big) > 0$ in view of condition (ii).

Thus, there exists a constant $k > 0$ small enough such that

$$V \geq k\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right). \tag{3.6}$$

On applying (3.1) and condition (iii) there exists positive constant N such that

$$\begin{aligned} \int_0^t \|\theta(s)\|\,ds &\leq \int_0^t \|H_s^{-1}\|^2 \left\|\frac{d}{ds}H_s\right\| ds \\ &\leq (\Delta_{H^{-1}})^2 \int_0^t \left\|\frac{d}{ds}H_s\right\| ds \leq N, \qquad \text{for all } t \geq 0. \end{aligned} \tag{3.7}$$

On combining this last estimate with (3.4) and (3.6) we get

$$W \geq V\exp(-\frac{N}{\eta}) \geq K_0\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right), \tag{3.8}$$

with $K_0 = k\exp(-\frac{N}{\eta})$.

Now, let $V'_{(3.2)}(t,X,Y,Z) = V'_{(3.2)}$ denote the time derivative of the functional $V(t,X,Y,Z)$ along the trajectories of the system (3.2). An easy computation shows that

$$V'_{(3.2)} = G_1 + G_2 + G_3 + G_4, \tag{3.9}$$

where

$$\begin{aligned} G_1 &= d\int_0^1 \langle C'(t)F(\sigma X), X\rangle d\sigma + \langle C'(t)Y, F(X)\rangle - \langle C'(t)Y, Y\rangle, \\ G_2 &= \left\langle \Big(\frac{d}{2}A'(t) + \frac{1}{2}B'(t)H_t + C'(t)H_t^2 - dB(t) + C(t)J_F H_t\Big)H_t^{-2}Y, Y\right\rangle, \\ G_3 &= \big\langle \big(dI - A(t)\big)H_t^{-1}Z, Z\big\rangle, \\ G_4 &= \frac{d}{2}\big\langle A(t)\theta(t)Y, H_t^{-1}Y\big\rangle + \frac{d}{2}\big\langle A(t)H_t^{-1}Y, \theta(t)Y\big\rangle + \frac{1}{2}\langle B(t)\theta(t)Y, Y\rangle \\ &\quad - d\langle H_t^{-1}Y, A(t)\theta(t)Y\rangle - \langle A(t)\theta(t)Y, Z\rangle + d\langle \theta(t)Y, Z\rangle. \end{aligned}$$

Under the assumption (i) of Theorem 5, we have

$$\begin{aligned} G_1 &\leq d\int_0^1 \langle C'(t)F(\sigma X), X\rangle d\sigma - \left\|C'^{\frac{1}{2}}(t)Y - \frac{1}{2}C'^{\frac{1}{2}}(t)F(X)\right\|^2 + \frac{\Delta_{C'}}{4}\|F(X)\|^2 \\ &\leq d\int_0^1 \langle C'(t)F(\sigma X), X\rangle d\sigma. \end{aligned}$$

By Lemma 3 and Lemma 1 we get

$$\begin{aligned} G_1 &\leq d\int_0^1 \langle C'(t)F(\sigma X), X\rangle d\sigma = \int_0^1\int_0^1 \sigma[\langle dC'(t)J_F(\sigma\tau X)X, X\rangle] d\sigma d\tau \\ &\leq \int_0^1\int_0^1 \sigma[\langle d\Delta_{C'}\Delta_F X, X\rangle] d\sigma d\tau \\ &= d\Delta_{C'}\Delta_F \|XVert^2 \leq 0. \end{aligned} \tag{3.10}$$

If we take into consideration condition (ii) of Theorem 5, we have that

$$\delta_A > d > \frac{(\Delta_C)^2(\Delta_F)^2}{\delta_C\delta_F\delta_B\Delta_H^{-1}} > \frac{\Delta_C\Delta_F\Delta_H}{\delta_B}.$$

Witch implies

$$\begin{aligned} M_1 &= \frac{d\delta_B - \Delta_C\Delta_F\Delta_H}{2} > 0, \\ M_2 &= -(d-\delta_A)\Delta_{H^{-1}} > 0. \end{aligned}$$

Clearly, as a result of the assumption (iii) we have

$$\begin{aligned} G_2 &\leq \Big(\frac{d}{2}\Delta_{A'} + \frac{1}{2}\Delta_{B'}\Delta_H + \Delta_{C'}\Delta_{H^2} - d\delta_B + \Delta_C\Delta_F\Delta_H\Big)\Delta_{H^{-2}}\|Y\|^2 \\ &\leq -M_1\|Y\|^2 \leq 0, \end{aligned}$$

and

$$G_3 \leq (d-\delta_A)\Delta_{H^{-1}}\|Z\|^2 = -M_2\|Z\|^2 \leq 0.$$

By applying the inequality $2VertuvVert \leq \|uVert^2 + \|vVert^2$ we estimate G_4 as follows

$$\begin{aligned} G_4 &= \frac{1}{2}\langle B(t)\theta(t)Y, Y\rangle - \langle A(t)\theta(t)Y, Z\rangle + d\langle\theta(t)Y, Z\rangle \\ &\leq K\|\theta(t)\|V, \end{aligned}$$

where $K = \frac{1}{k}\big(\frac{1}{2}\Delta_B + \frac{1}{2}\Delta_A + \frac{d}{2}\big)$.
Bringing together the estimates just obtained for $G_i (i = 1, 2...4)$ in (3.9) we get

$$V'_{(3.2)} \leq -M_1\|Y\|^2 - M_2\|Z\|^2 + K\|\theta(t)\| V. \tag{3.11}$$

From (3.4), we have

$$W'_{(3.2)} = \left(V'_{(3.2)} - \frac{1}{\eta}\|\theta(t)\| V\right) e^{-\frac{1}{\eta}\int_0^t \|\theta(s)\| ds}.$$

Using (3.11), (3.6) and choosing $\eta = \frac{1}{K}$, we get

$$W'_{(3.2)} \leq \left(-M_1\|Y\|^2 - M_2\|Z\|^2\right) e^{-\frac{1}{\eta}\int_0^t \|\theta(s)\| ds}.$$

Clearly from (3.7) we have $e^{-\frac{1}{\eta}\int_0^t \|\theta(s)\| ds} \geq e^{-\frac{N}{\eta}}$.
Hence

$$W'_{(3.2)} \leq -L\left(\|Y\|^2 + \|Z\|^2\right), \tag{3.12}$$

where $L = e^{-\frac{N}{\eta}} \min\{M_1, M_2\}$. In view of (3.8) and (3.12), it follow from $\big($[6, Theorem 4.1.14]$\big)$ that the solution $\big(X(t), Y(t), Z(t)\big)$ of (3.2) is stable. Now $E = \{(X, Y, Z) : W'_{(3.2)}(X, Y, Z) = 0\} = \{(X, 0, 0) : X \in \mathbb{R}^n\}$ and the largest invariant set contained in E is $F = \{(0, 0, 0)\}$. By LaSalle's invariance principe (see, for example, Haddock [14])

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} Y(t) = \lim_{t\to\infty} Z(t) = 0.$$

□

4 Boundedness

Our main theorem in this section is the following boundedness result which is stated with respect to $P(t) \neq 0$.

Theorem 6. *Let all the conditions of Theorem 5 be satisfied and in addition we assume that there exist positive constants p_1 and P_1 such that:*

I_1) $\|P(t)\| \leq p(t) < p_1, \quad \forall t \geq 0.$

I_2) $\int_0^t p(s)ds < P_1, \quad \forall t \geq 0.$

I_3) $\lim\limits_{t\to\infty} \|H'_t\|$ *exists.*

Then there exists a positive constant P_5 such that any solution $X(t)$ of (1.3) and their derivatives $X'(t)$, and $X''(t)$ satisfy

$$\|X(t)\| \leq P_5, \ \|X'(t)\| \leq P_5, \ \|X''(t)\| \leq P_5. \tag{4.1}$$

Proof. For the case $P(t) \neq 0$, on differentiating (3.5) along the system (3.2) we obtain

$$\begin{aligned} V'_{(3.2)} &\leq -U + K\|\theta(t)\|V + d\langle H_t^{-1}Y, P(t)\rangle + \langle Z, P(t)\rangle \\ &\leq -U + K\|\theta(t)\|V + p(t)\Big(d\|H_t^{-1}\| \, \|Y\| + \|Z\|\Big) \\ &\leq -U + K\|\theta(t)\|V + p(t)K_1\Big(\|Y\| + \|Z\|\Big), \end{aligned}$$

where $K_1 = \max\left\{d\delta_H^{-1}, 1\right\}$ and $U = M_1 \|Y\|^2 + M_2 \|Z\|^2$.

By using $VertuVert \ \leq \ \|uVert^2 + 1$, it is clear that

$$V'_{(3.2)} \leq -U + K\|\theta(t)\|V + p(t)K_1\Big(\|Y\|^2 + \|Z\|^2 + 2\Big). \tag{4.2}$$

From (3.4) we have

$$W'_{(3.2)} = \Big[V' - \frac{1}{\eta}\|\theta(t)\|V\Big] \exp\big(-\frac{1}{\eta}\int_0^t \|\theta(s)\| \, ds\big). \tag{4.3}$$

Since $K - \dfrac{1}{\eta} = 0$, it follows that

$$W'_{(3.2)} \leq \Big[-U + p(t)K_1\Big(\|Y\|^2 + \|Z\|^2 + 2\Big)\Big] \exp\big(-\frac{1}{\eta}\int_0^t \|\theta(s)\| \, ds\big).$$

In view of (3.12) and the fact that

$$\exp\big(-\frac{1}{\eta}\int_0^t \|\theta(s)\|\,ds\big) \le 1,$$

we have

$$W'_{(3.2)} \le -L(\|Y\|^2 + \|Z\|^2) + \frac{K_1}{K_0}p(t)\ W + K_2 p(t), \tag{4.4}$$

with $K_2 = 2K_1$. Integrating both sides (4.4) from 0 to t, one can easily obtain

$$W(t) - W(0) \le K_2\int_0^t p(s)ds + \frac{K_1}{K_0}\int_0^t W(s)p(s)ds.$$

Let

$$P_2 = W(0) + K_2 P_1. \tag{4.5}$$

Thus

$$W(t) \le P_2 + \frac{K_1}{K_0}\int_0^t W(s)p(s)ds.$$

On applying Gronwall inequality we have

$$W(t) \le P_2 \exp\Big(\frac{K_1}{K_0}\int_0^t p(s)ds\Big) \le P_3, \tag{4.6}$$

where $P_3 = P_2\exp\Big(\frac{K_1}{K_0}P_1\Big)$. Combining (4.6) and (3.8), we have

$$\|X(t)\| \le P_4,\ \ \|Y(t)\| \le P_4, \|Z(t)\| \le P_4, \tag{4.7}$$

where $P_4 = \sqrt{\frac{P_3}{K_0}}$.
Now, by (3.2) we get

$$\begin{aligned} \|X'(t)\| &= \|H_t^{-1}Y(t)\| \\ &\le \big\|H_t^{-1}\big\|\,\|Y(t)\| \\ &\le P_4\delta_H^{-1}. \end{aligned}$$

According to condition (I_3) of Theorem 6, there exists positive constant h_1 such that

$$\|H_t'\| < h_1. \tag{4.8}$$

So, by (3.1) we have

$$\|\theta(t)\| \le \|H_t^{-2}\|\ \|H_t'\| \le h_1(\delta_H^{-1})^2 = \beta. \tag{4.9}$$

In view of (3.2) and (3.3) we have

$$\begin{aligned} \|X^{''}(t)\| &\le \|\theta(t)Y(t)\| + \|H_t^{-1}Z(t)\| \\ &\le \big(h_1(\delta_H^{-1})^2 + \delta_H^{-1}\big)P_4. \end{aligned}$$

Therefore, there exists positive constant P_5 such that

$$\|X(t)\| \le P_5,\ \|X'(t)\| \le P_5,\ \|X''(t)\| \le P_5, \quad \text{for all}\ \ t \ge 0, \tag{4.10}$$

where $P_5 = \max\big\{\big(h_1(\delta_H^{-1})^2 + \delta_H^{-1}\big)P_4,\ P_4\big\}$. This completes the proof of Theorem 6. $\square$

5 Square integrability of solutions

Our next result concerns the square integrability of solutions of equation (1.3).

Theorem 7. *Let all the conditions of Theorem 5 and Theorem 6 satisfied and in addition we assume that*

I$_4$) $2\delta_C\delta_{J_F} - \Delta_A - \Delta_B > 0$.

Then, every solution X of equation (1.3) and their derivatives are elements of $L^2[0,+\infty)$.

Proof. Let $X(t)$ be a solution of (1.3) and define $Q(t) = Q(t, X(t), Y(t), Z(t))$ by

$$Q(t) = W(t) + \lambda \int_0^t \big(\|Y(s)\|^2 + \|Z(s)\|^2\big)ds, \tag{5.1}$$

where $\lambda > 0$ is a constant to be specified later and W(t) is given in (3.4). By differentiating Q(t) and using (4.4) we obtain

$$Q'(t) \leq (\lambda - L)(\|Z(t)\|^2 + \|Y(t)\|^2) + \Big(K_1W + K_2\Big)p(t).$$

Taking $\lambda - L = 0$ and using (4.6) we get

$$Q'(t) \leq K_3 p(t), \tag{5.2}$$

where $K_3 = K_1P_3 + K_2$. Integrating (5.2) from 0 to t, $t \geq 0$, and using condition (I_2) of Theorem 6 we obtain

$$Q(t) - Q(0) = \int_0^t Q'(s)ds \leq K_3P_1.$$

With (4.5) and equality $Q(0) = W(0)$ we get

$$Q(t) \leq K_3P_1 + P_2 - K_2P_1.$$

We can conclude by (5.1) that

$$\int_0^t (\|Z(s)\|^2 + \|Y(s)\|^2)ds < \frac{K_3P_1 + P_2 - K_2P_1}{\lambda},$$

which imply the existence of positive constants μ_1 and μ_2 such that

$$\int_0^t \|Y(s)\|^2 ds \leq \mu_1 \quad \text{and} \quad \int_0^t \|Z(s)\|^2 ds \leq \mu_2.$$

Observe that from (3.2)

$$\begin{aligned} \int_0^\infty \|X'(s)\|^2 ds &= \int_0^\infty \left\|H_s^{-1}Y(s)\right\|^2 ds \\ &\leq \int_0^\infty \left\|H_s^{-1}\right\|^2 \|Y(s)\|^2 ds \\ &\leq (\Delta_{H^{-1}})^2\mu_1 = l_1 < \infty. \end{aligned} \tag{5.3}$$

On the other hand, using (4.9) and (3.3) we obtain

$$\begin{aligned}\int_0^t \|X''(s)\|^2 ds &= \int_0^t \|H_s^{-1}\|^2 \|Z(s)\|^2 ds + \int_0^t \|\theta(s)\|^2 \|Y(s)\|^2 ds \\ &\quad +2\int_0^t \langle \theta(s)Y(s), H_s^{-1}Z(s)\rangle ds \\ &\leq \big(\Delta_{H^{-1}}^2 + \beta\Delta_{H^{-1}}\big)\int_0^t \|Z(s)\|^2 ds \\ &\quad +\big(\beta^2 + \beta\Delta_{H^{-1}}\big)\int_0^t \|Y(s)\|^2 ds \\ &\leq M(\mu_1+\mu_2) = l_2 < \infty, \end{aligned} \tag{5.4}$$

where $M = \max\big\{\Delta_{H^{-1}}^2 + \beta\Delta_{H^{-1}}, \beta^2 + \beta\Delta_{H^{-1}}\big\}$.
Next, multiply (1.3) by $X(t)$ and integrate by parts from 0 to t all the terms on the LHS of (1.3) obtaining

$$\int_0^t \langle C(s)F(X(s)), X(s)\rangle ds = I(t) + J(t), \tag{5.5}$$

where

$$\begin{aligned} I(t) &= -\langle H_t' X'(t) + H_t X''(t), X(t)\rangle + \langle H_t X'(t), X'(t)\rangle - \int_0^t \langle H_s X'(s), X''(s)\rangle ds + k_1, \\ J(t) &= \int_0^t \langle\big(-A(s)X''(s) - B(s)X'(s) + P(s)\big), X(s)\rangle\rangle ds, \end{aligned}$$

and

$$k_1 = \langle H_0' X'(0), X(0)\rangle + \langle H_0 X''(0), X(0)\rangle - \langle H_0 X'(0), X'(0)\rangle.$$

Using (4.8) and (4.10) we get

$$|-\langle H_t' X'(t) + H_t X''(t), X(t)\rangle + \langle H_t X'(t), X'(t)\rangle| \leq P_5^2\Big(h_1 + 2\Delta_H\Big).$$

It is clear that

$$\begin{aligned}\int_0^t \langle H_s X'(s), X''(s)\rangle ds &\leq \frac{\Delta_H}{2}\int_0^t \Big(\|X'(s)\|^2 + \|X''(s)\|^2\Big) ds. \\ &\leq \frac{\Delta_H}{2}(l_1 + l_2) = l_3. \end{aligned}$$

Hence

$$I(t) \leq l_3 + P_5^2\Big(h_1 + 2\Delta_H\Big) + |\, k_1 \,| = l_4. \tag{5.6}$$

By using assumption (I_2) of Theorem 6, Lemma 1, and inequality $2uv \leq u^2 + v^2$, we get

$$\begin{aligned} J(t) &\leq \frac{\Delta_A}{2}\int_0^t \Big(\|X''^2 + \|X(s)\|^2\Big) ds + \frac{\Delta_B}{2}\int_0^t \Big(\|X'^2 + \|X(s)\|^2\Big) ds \\ &\quad + P_5 \int_0^t \|P(s)\| ds \\ &\leq l_5 + \frac{\Delta_A + \Delta_B}{2}\int_0^t \|X(s)\|^2 ds, \end{aligned} \tag{5.7}$$

where $l_5 = \dfrac{\Delta_A}{2} l_2 + \dfrac{\Delta_B}{2} l_1 + P_1 P_5$. By Lemma 4 we have

$$\langle C(t)F(X(t)), X(t)\rangle \geq \delta_C \delta_{J_F} \|X(t)\|^2 .$$

Thus, (5.6), (5.7) and condition (I_4) imply that

$$\int_0^t \|X(s)\|^2 \, ds \leq l_0,$$

where $l_0 = \dfrac{2\big(l_4 + l_5\big)}{2\delta_C \delta_{J_F} - \big(\Delta_A + \Delta_B\big)}$. This fact completes the proof of Theorem. □

Example 8. As a special case of the following equation

$$(H(X(t))X'(t))'' + A(t)X''(t) + B(t)X'(t) + C(t)F(X(t)) = P(t) \tag{5.8}$$

where

$$\begin{aligned}
X(t) &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad F(X) = \begin{pmatrix} 0.2 \arctan x \\ 0.16 y \end{pmatrix}, \\
J_F(X) &= \begin{pmatrix} \frac{0.2}{1+x^2} & 0 \\ 0 & 0.16 \end{pmatrix}, H_t = \begin{pmatrix} h_{11}(x(t)) & 0 \\ 0 & h_{22}(y(t)) \end{pmatrix}, \\
P(t) &= \begin{pmatrix} \frac{1}{1+t^2} \\ \frac{\sin t}{3+\cos^2 t} \end{pmatrix}, \quad A(t) = \begin{pmatrix} \frac{e^{\sin t}}{10} + \frac{1}{4} & 0 \\ 0 & \frac{9\cos t}{100} + \frac{1}{3} \end{pmatrix}, \\
B(t) &= \begin{pmatrix} \frac{e^{-t^2}+1}{2} & 0 \\ 0 & \frac{\sin t}{4} + \frac{1}{2} \end{pmatrix}, \quad C(t) = \begin{pmatrix} e^{-2t} + 5 & 0 \\ 0 & e^{-t} + 5 \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
h_{11}(x(t)) &= \frac{0.02}{6}\Big(\frac{\sin(x(t))}{(1 + x^2(t))} + 3\Big), \\
h_{22}(y(t)) &= \frac{0.02}{6}\Big(\frac{\cos(y(t))}{(1 + y^2(t))} + 5\Big).
\end{aligned}$$

Clearly, $H(X), A, B$ and $J_F(X)$ are diagonal matrices, hence they are symmetric and commute pairwise. Then, by an easy calculation, we obtain eigenvalues of the matrices H, A, B, C and $J_F(X)$ as follows:

$$\begin{aligned}
\delta_h &= \frac{0.02}{3} \leq \lambda_1(H) = \frac{0.02}{6}\Big(\frac{\sin x}{(1+x^2)} + 3\Big), \\
\lambda_2(H) &= \frac{0.02}{6}\Big(\frac{\cos x}{(1+x^2)} + 5\Big) \leq 0.02 = \Delta_h, \\
\delta_A &= 0.24333 \leq \lambda_1(A(t)) = \frac{9}{100}\cos t + \frac{1}{3}, \quad \lambda_2(A(t)) = \frac{e^{\sin t}}{10} + \frac{1}{4} \leq 0.52183 = \Delta_A, \\
\delta_B &= 0.5 \leq \lambda_1(B(t)) = \frac{\sin t}{4} + \frac{3}{4}, \quad \lambda_2(B(t)) = \frac{e^{-t^2}}{2} + \frac{1}{2} \leq 1 = \Delta_B, \\
\delta_C &= 5 \leq \lambda_1(C(t)) = e^{-2t} + 5, \quad \lambda_2(C(t)) = e^{-3t} + 5 \leq 6 = \Delta_C, \\
\delta_F &= \frac{1.6}{10} = \lambda_1(J_F(X)), \quad \lambda_2(J_F(X)) = \frac{0.2}{1+x^2} \leq \frac{2}{10} = \Delta_F.
\end{aligned}$$

A simple computation gives

$$\begin{aligned}
\lambda_1\left(A'(t)\right) &= -\frac{9}{100}\sin t, \quad \lambda_2\left(A'(t)\right) = \frac{\cos t}{10}e^{\sin t} \le \frac{e}{10} = \Delta_{A'},\\
\lambda_1\left(B'(t)\right) &= -te^{-t^2}, \qquad \lambda_2\left(B'(t)\right) = -\frac{\cos t}{4} \le \frac{1}{4} = \Delta_{B'},\\
\lambda_1\left(C'(t)\right) &= -2e^{-2t}, \qquad \lambda_2\left(C'(t)\right) = -e^{-t} \le 0 = \Delta_{C'}.
\end{aligned}$$

A trivial verification shows that H is nonsingular matrix and we have

$$\frac{d}{dt}H_t = \begin{pmatrix} \frac{d}{dt}h_{11}(x(t)) & 0 \\ 0 & \frac{d}{dt}h_{22}(y(t)) \end{pmatrix},$$

where

$$\begin{aligned}
\frac{d}{dt}h_{11}(x(t)) &= \frac{0.02}{6}\Big(\frac{\cos(x(t))}{(1+x^2(t))} - \frac{2x(t)\sin(x(t))}{(1+x^2(t))^2}\Big)x'(t),\\
\frac{d}{dt}h_{22}(y(t)) &= \frac{0.02}{6}\Big(\frac{-\sin(y(t))}{(1+y^2(t))} - \frac{2y\cos(y(t))}{(1+y^2(t))^2}\Big)y'(t).
\end{aligned}$$

Thus

$$\|\frac{d}{dt}H_t\| = \max\left\{\left|\frac{d}{dt}h_{11}(x(t))\right|, \left|\frac{d}{dt}h_{22}(y(t))\right|\right\} = D(t),$$

and

$$\|\theta(t)\| \le \frac{1}{\delta_h^2}D(t), \quad \text{for all } t \ge 0.$$

A straightforward calculation give

$$\begin{aligned}
\int_0^t \|\theta(s)\|ds &\le 2.25\times 10^4 \int_0^t D(s)ds\\
&= 2.25\times 10^4 \int_0^t \max\left\{\left|\frac{d}{ds}h_{11}(x(s))\right|, \left|\frac{d}{ds}h_{22}(y(s))\right|\right\}ds\\
&\le 2.25\times 10^4 \int_0^t \frac{0.02}{6}\left|\Big(\frac{\cos x}{1+x^2} - \frac{2x\sin x}{(1+x^2)^2}\Big)x'(s)\right|ds\\
&\quad + \int_0^t \frac{0.02}{6}\left|\Big(\frac{-\sin y}{1+y^2} - \frac{2y\cos y}{(1+y^2)^2}\Big)y'(s)\right|ds\\
&\le 300\Bigg(\int_{\omega_1(t)}^{\omega_2(t)}\left|\Big(\frac{\cos u}{1+u^2} - \frac{2u\sin u}{(1+u^2)^2}\Big)du\right|\\
&\quad + \int_{\varphi_1(t)}^{\varphi_2(t)}\left|\Big(\frac{-\sin v}{1+v^2} - \frac{2v\cos v}{(1+v^2)^2}\Big)dv\right|\Bigg)\\
&< 300\Bigg(\int_{-\infty}^{+\infty}\left|\frac{1+u^2+2u}{(1+u^2)^2}\right|du + \int_{-\infty}^{+\infty}\left|\frac{1+u^2+2u}{(1+u^2)^2}\right|du\Bigg)\\
&= 150(\pi+2),
\end{aligned}$$

where

$$\omega_1(t) = \min\{x(0), x(t)\}, \qquad \omega_2(t) = \max\{x(0), x(t)\},$$

and

$$\varphi_1(t) = \min\{y(0), y(t)\}, \qquad \varphi_2(t) = \max\{y(0), y(t)\}.$$

Now, it is easy see that

$$\|P(t)\| = \sqrt{P_1^2(t) + P_2^2(t)} \leq P_1(t) + P_2(t) = p(t) < \frac{4}{3} = p_1,$$

where $P_1(t) = \frac{1}{1+t^2}$, $P_2(t) = \frac{\sin t}{3+\cos^2 t}$. So, we have for $t \in [0, +\infty)$

$$\int_0^t \|p(s)\| ds = \int_0^t \|P_1(s)\| ds + \int_0^t \|P_2(s)\| ds < \infty.$$

By taking $d = 0.23$, it follows easily that

$$\begin{aligned}
&\frac{(\Delta_C)^2(\Delta_F)^2\Delta_H}{\delta_B\delta_C\delta_F} = 0.072 < d < \delta_A = 0.24333.\\
&\frac{d}{2}\Delta_{A'} + \frac{1}{2}\Delta_{B'}\Delta_H + \Delta_{C'}\Delta_{H^2} < \frac{d}{2}\Delta_{A'} + \frac{1}{2}\Delta_{B'}\Delta_H\\
&= 3.376 \times 10^2 < \frac{d\delta_B - \Delta_C\Delta_F\Delta_H}{2} = 0.0455.
\end{aligned}$$

We have also

$$\delta_C\delta_F - \frac{\Delta_A + \Delta_B}{2} = 3.9086 \times 10^2 > 0.$$

Thus, all the conditions of Theorem 7 are satisfied.

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A note on finite lattices with many congruences

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Abstract

By a twenty year old result of Ralph Freese, an n-element lattice L has at most 2^{n-1} congruences. We prove that if L has less than 2^{n-1} congruences, then it has at most 2^{n-2} congruences. Also, we describe the n-element lattices with exactly 2^{n-2} congruences. Finally, we point out that if the congruence lattice of an n-element algebra A is distributive, then A has at most 2^{n-1} congruences; furthermore, if this maximum number is reached, then the congruence lattice of A is boolean.

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1 Introduction and motivation

It follows from Lagrange's Theorem that the size $|S|$ of an arbitrary subgroup S of a finite group G is either $|G|$, or it is at most the half of the maximum possible value, $|G|/2$. Furthermore, if the size of S is the half of its maximum possible value, then S has some special property since it is normal. Our goal is to prove something similar on the size of the congruence lattice $\mathrm{Con}(L)$ of an n-element lattice L.

For a *finite* lattice L, the relation between $|L|$ and $|\mathrm{Con}(L)|$ has been studied in some earlier papers, including Freese [5], Grätzer and Knapp [11], Grätzer, Lakser, and Schmidt [12], Grätzer, Rival, and Zaguia [13]. In particular, part (i) of Theorem 1 below is due to Freese [5]. Although Czédli and Mureşan [4] and Mureşan [14] deal only with infinite lattices, they are also among the papers motivating the present one. We will conclude the paper with some remarks on finite algebras distinct from lattices.

2 Our result on lattices and its proof

Mostly, we follow the terminology and notation of Grätzer [8]. In particular, the *glued sum* $L_0\dot{+}L_1$ of finite lattices L_0 and L_1 is their Hall–Dilworth gluing along $L_0 \cap L_1 = \{1_{L_0}\} = \{0_{L_1}\}$; see, for example, Grätzer [8, Section IV.2]. Note that $\dot{+}$ is an associative operation. Our result is the following.

Theorem 1. *If L is a finite lattice of size $n = |L|$, then the following hold.*

(i) *L has at most 2^{n-1} many congruences. Furthermore, $|\mathrm{Con}(L)| = 2^{n-1}$ if and only if L is a chain.*

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(ii) *If L has* less than 2^{n-1} *congruences, then it has at most* $2^{n-1}/2 = 2^{n-2}$ *congruences.*

(iii) $|\mathrm{Con}(L)| = 2^{n-2}$ *if and only if L is of the form $C_1 \dot{+} B_2 \dot{+} C_2$ such that C_1 and C_2 are chains and B_2 is the four-element Boolean lattice.*

For $n = 8$, part (iii) of this theorem is illustrated in Figure 1. Note that part (i) of the theorem is due to Freese [5, page 3458]; however, as a by-product of our approach leading to parts (ii) and (iii) of Theorem 1, this paper also includes a proof of part (i).

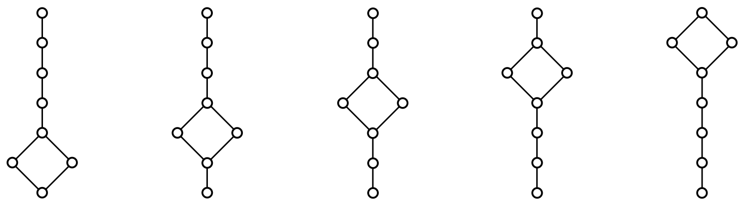

Figure 1. The full list of 8-element lattices with exactly $64 = 2^{8-2}$ many congruences

Proof of Theorem 1. We prove the theorem by induction on $n = |L|$. Since the case $n = 1$ is clear, assume as an induction hypothesis that $n > 1$ is a natural number and all the three parts of the theorem hold for every lattice with size less than n. Let L be a lattice with $|L| = n$. For $\langle a, b\rangle \in L^2$, the least congruence collapsing a and b will be denoted by $\mathrm{con}(a, b)$. A *prime interval* or an *edge* of L is an interval $[a, b]$ with $a \prec b$. For later reference, note that

$$\begin{array}{l}\mathrm{Con}(L) \text{ has an atom, and every of its atoms is of} \\ \text{the form } \mathrm{con}(a, b) \text{ for some prime interval } [a, b];\end{array} \tag{2.1}$$

this follows from the finiteness of $\mathrm{Con}(L)$ and from the fact that every congruence on L is the join of congruences generated by *covering pairs* of elements; see also Grätzer [10, page 39] for this folkloric fact.

Based on (2.1), pick a prime interval $[a, b]$ of L such that $\Theta = \mathrm{con}(a, b)$ is an atom in $\mathrm{Con}(L)$. Consider the map $f\colon \mathrm{Con}(L) \to \mathrm{Con}(L)$ defined by $\Psi \mapsto \Theta \vee \Psi$. We claim that, with respect to f,

$$\text{every element of } f(\mathrm{Con}(L)) \text{ has at most two preimages.} \tag{2.2}$$

Suppose to the contrary that there are pairwise distinct $\Psi_1, \Psi_2, \Psi_3 \in \mathrm{Con}(L)$ with the same f-image. Since the $\Theta \wedge \Psi_i$ belong to the two-element principal ideal $\downarrow\Theta := \{\Gamma \in \mathrm{Con}(L) : \Gamma \le \Theta\}$ of $\mathrm{Con}(L)$, at least two of these meets coincide. So we can assume that $\Theta \wedge \Psi_1 = \Theta \wedge \Psi_2$ and, of course, we have that $\Theta \vee \Psi_1 = f(\Psi_1) = f(\Psi_2) = \Theta \vee \Psi_2$. This means that both Ψ_1 and Ψ_2 are relative complements of Θ in the interval $[\Theta\wedge\Psi_1, \Theta\vee\Psi_1]$. According to a classical result of Funayama and Nakayama [7], $\mathrm{Con}(L)$ is distributive. Since relative complements in distributive lattices are well-known to be unique, see, for example, Grätzer [8, Corollary 103], it follows that $\Psi_1 = \Psi_2$. This is a contradiction proving (2.2).

Clearly, f is a retraction map onto the filter $\uparrow\Theta$. It follows from (2.2) that $|\uparrow\Theta| \ge |\mathrm{Con}(L)|/2$. Also, by the well-known Correspondence Theorem, see Burris and Sankappanawar [2, Theorem 6.20], or see Theorem 5.4 (under the name Second Isomorphism Theorem) in Nation [15], $|\uparrow\Theta| = |\mathrm{Con}(L/\Theta)|$ holds. Hence, it follows that

$$|\mathrm{Con}(L/\Theta)| \ge \frac{1}{2} \cdot |\mathrm{Con}(L)|. \tag{2.3}$$

Since Θ collapses at least one pair of distinct elements, $\langle a,b\rangle$, we conclude that $|L/\Theta| \le n-1$. Thus, it follows from part (i) of the induction hypothesis that $|\mathrm{Con}(L/\Theta)| \le 2^{(n-1)-1} = 2^{n-2}$. Combining this inequality with (2.3), we obtain that $|\mathrm{Con}(L)| \le 2\cdot|\mathrm{Con}(L/\Theta)| \le 2^{n-1}$. This shows the first half of part (i).

For later reference, note that we have not used that $[a,b]$ is a prime interval; this will be used only later. We only needed that $\mathrm{con}(a,b) \in \mathrm{Con}(L)$ was an atom and $\mathrm{Con}(L)$ was distributive. Hence, the same proof as above gives that

$$\text{if } A \text{ is an } n\text{-element algebra such that } \mathrm{Con}(A) \text{ is a distributive lattice, then } |\mathrm{Con}(A)| \le 2^{n-1}. \tag{2.4}$$

If L is a chain, then $\mathrm{Con}(L)$ is known to be the 2^{n-1}-element boolean lattice; see, for example, Grätzer [10, Corollaries 3.11 and 3.12]. Hence, we have that $|\mathrm{Con}(L)| = 2^{n-1}$ if L is a chain. Conversely, assume the validity of $|\mathrm{Con}(L)| = 2^{n-1}$, and let $k = |L/\Theta|$. By the induction hypothesis, $|\mathrm{Con}(L/\Theta)| \le 2^{k-1}$. On the other hand, $|\mathrm{Con}(L/\Theta)| \ge |\mathrm{Con}(L)|/2 = 2^{n-2}$ holds by (2.3). These two inequalities and $k<n$ yield that $k=n-1$ and also that $|\mathrm{Con}(L/\Theta)| = 2^{n-2} = 2^{k-1}$. Hence, the induction hypothesis implies that L/Θ is a chain. For the sake of contradiction, suppose that L is not a chain, and pick a pair $\langle u,v\rangle$ of incomparable elements of L. The Θ-blocks u/Θ and v/Θ are comparable elements of the chain L/Θ, whence we can assume that $u/\Theta \le v/\Theta$. It follows that $u/\Theta = u/\Theta \wedge v/\Theta = (u\wedge v)/\Theta$ and, by duality, $v/\Theta = (u\vee v)/\Theta$. Thus, since u, v, $u\wedge v$ and $u\vee v$ are pairwise distinct elements of L and Θ collapses both of the pairs $\langle u\wedge v, u\rangle$ and $\langle v, u\vee v\rangle$, we have that $k = |L/\Theta| \le n-2$, which is a contradiction. This proves part (i) of the theorem.

As usual, for a lattice K, let $J(K)$ and $M(K)$ denote the set of nonzero *join-irreducible elements* and the set of *meet-irreducible elements* distinct from 1, respectively. By a *narrows* we will mean a prime interval $[a,b]$ such that $a\in M(L)$ and $b \in J(L)$. Using Grätzer [9], it follows in a straightforward way that

$$\text{if } [a,b] \text{ is a narrows, then } \{a,b\} \text{ is the only non-singleton block of } \mathrm{con}(a,b). \tag{2.5}$$

Now, in order to prove part (ii) of the theorem, assume that $|\mathrm{Con}(L)| < 2^{n-1}$. By (1), we can pick a prime interval $[a,b]$ such that $\Theta := \mathrm{con}(a,b)$ is an atom in $\mathrm{Con}(L)$. There are two cases to consider depending on whether $[a,b]$ is a narrows or not; for later reference, some parts of the arguments for these two cases will be summarized in (2.6) and (2.7) redundantly. First, we deal with the case where $[a,b]$ is a narrows. We claim that

$$\text{if } |\mathrm{Con}(L)| < 2^{n-1},\ [a,b] \text{ is a narrows, and } \Theta = \mathrm{con}(a,b) \text{ is an atom in } \mathrm{Con}(L), \text{ then } L/\Theta \text{ is not a chain.} \tag{2.6}$$

By (2.5), $|L/\Theta| = n-1$. By the already proved part (i), L is not a chain, whence there are $u,v\in L$ such that $u \parallel v$. We claim that u/Θ and v/Θ are incomparable elements of L/Θ. Suppose the contrary. Since u and v play a symmetric role, we can assume that $u/\Theta \vee v/\Theta = v/\Theta$, i.e., $(u\vee v)/\Theta = v/\Theta$. But $u \vee v \ne v$ since $u\parallel v$, whereby (2.5) gives that $\{v, u\vee v\} = \{a,b\}$. Since $a<b$, this means that $v=a$ and $u\vee v = b$. Thus, $u\vee v\in J(L)$ since $[a,b]$ is a narrows. The membership $u\vee v \in J(L)$ gives that $u \vee v \in \{u,v\}$, contradicting $u\parallel v$. This shows that $u/\Theta \parallel v/\Theta$, whence L/Θ is not a chain. We have shown the validity of (2.6). Using part (i) and $|L/\Theta| = n-1$, it follows that $|\mathrm{Con}(L/\Theta)| < 2^{(n-1)-1}$. By the induction hypothesis, we can apply (ii) to L/Θ to conclude that $|\mathrm{Con}(L/\Theta)| \le 2^{(n-1)-2}$. This inequality and (2.3) yield that $|\mathrm{Con}(L)| \le 2\cdot |\mathrm{Con}(L/\Theta)| \le 2^{n-2}$, as required.

Second, assume that $[a, b]$ is not a narrows. Our immediate plan is to show that

$$\begin{aligned}&\text{if a prime interval } [a, b] \text{ of } L \text{ is not a narrows}\\ &\text{and } \Theta = \mathrm{con}(a, b), \text{ then } |L/\Theta| \le n-2.\end{aligned} \tag{2.7}$$

By duality, we can assume that a is meet-reducible. Hence, we can pick an element $c \in L$ such that $a \prec c$ and $c \neq b$. Clearly, $c \neq b \vee c$ and $\Theta = \mathrm{con}(a, b)$ collapses both $\langle a, b\rangle$ and $\langle c, b \vee c\rangle$, which are distinct pairs. Thus, we obtain that $|L/\Theta| \le n - 2$, proving (2.7). Hence, $\mathrm{Con}(L/\Theta) \le 2^{n-3}$ by part (i) of the induction hypothesis. Combining this inequality with (2.3), we obtain the validity of the required inequality $\mathrm{Con}(L) \le 2^{n-2}$. This completes the induction step for part (ii).

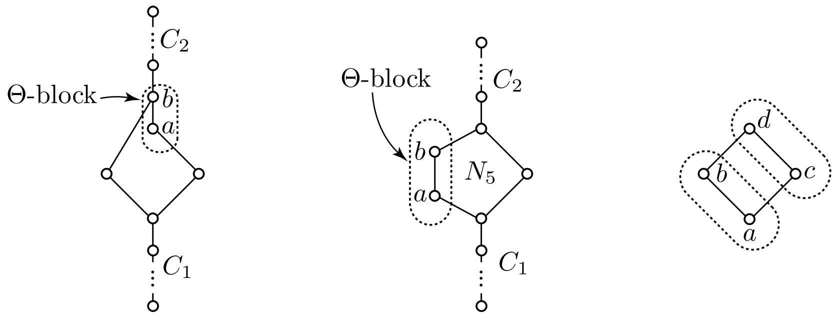

Figure 2. Illustrations for the proof

Next, in order to perform the induction step for part (iii), we assume that $|\mathrm{Con}(L)| = 2^{n-2}$. Again, there are two cases to consider. First, we assume that there exists a narrows $[a, b]$ in L such that $\Theta := \mathrm{con}(a, b)$ is an atom in $\mathrm{Con}(L)$. Then $|L/\Theta| = n-1$ by (2.5) and L/Θ is not a chain by (2.6). By the induction hypothesis, parts (i) and (ii) hold for L/Θ, whereby we have that $|\mathrm{Con}(L/\Theta)| \le 2^{(n-1)-2} = 2^{n-3}$. On the other hand, it follows from (2.3) that $|\mathrm{Con}(L/\Theta)| \ge |\mathrm{Con}(L)|/2 = 2^{n-3}$. Hence, $|\mathrm{Con}(L/\Theta)| = 2^{n-3} = 2^{|L/\Theta|-2}$. By the induction hypothesis, L/Θ is of the form $C_1 \dotplus B_2 \dotplus C_2$. We know from (2.5) that $\{a, b\} = [a, b]$ is the unique non-singleton Θ-block. If this Θ-block is outside B_2, then L is obviously of the required form. If the Θ-block $\{a, b\}$ is in $C_2 \cap B_2$, then L is of the required form simply because the situation on the left of Figure 2 would contradict the fact that $[a, b]$ is a narrows. A dual treatment applies for the case $\{a, b\} \in C_1 \cap B_2$. If the Θ-block $\{a, b\}$ is in $B_2 \setminus (C_1 \cup C_2)$, then L is of the form $C_1 \dotplus N_5 \dotplus C_2$, where N_5 is the "pentagon"; see the middle part of Figure 2. For an arbitrary bounded lattice K and the two-element chain $\mathbf{2}$, it is straightforward to see that

$$\mathrm{Con}(K \dotplus \mathbf{2}) \cong \mathrm{Con}(\mathbf{2} \dotplus K) \cong \mathrm{Con}(K) \times \mathbf{2}. \tag{2.8}$$

A trivial induction based on (2.8) yields that $|\mathrm{Con}(C_1 \dotplus N_5 \dotplus C_2)|$ is divisible by $5 = |\mathrm{Con}(N_5)|$. But 5 does not divide $|\mathrm{Con}(L)| = 2^{n-2}$, ruling out the case that the Θ-block $\{a, b\}$ is in $B_2 \setminus (C_1 \cup C_2)$. Hence, L is of the required form.

Second, we assume that no narrows in L generates an atom of $\mathrm{Con}(L)$. By (2.1), we can pick a prime interval $[a, b]$ such that $\Theta := \mathrm{con}(a, b)$ is an atom of $\mathrm{Con}(L)$. Since $[a, b]$ is not a narrows, (2.7) gives that $|L/\Theta| \le n - 2$. We claim that we have equality here, that is, $|L/\Theta| = n - 2$. Suppose to the contrary that $|L/\Theta| \le n - 3$. Then part (i) and (2.3) yield that

$$2^{n-2} = |\mathrm{Con}(L)| \le 2 \cdot |\mathrm{Con}(L/\Theta)| \le 2 \cdot 2^{(n-3)-1} = 2^{n-3},$$

which is a contradiction. Hence, $|L/\Theta| = n-2$. Thus, we obtain from part (i) that $|\mathrm{Con}(L/\Theta)| \le 2^{n-3}$. On the other hand, (2.3) yields that $|\mathrm{Con}(L/\Theta)| \ge |\mathrm{Con}(L)|/2 = 2^{n-3}$, whence $|\mathrm{Con}(L/\Theta)| = 2^{n-3} = 2^{|L/\Theta|-1}$, and it follows by part (i) that L/Θ is a chain. Now, we have to look at the prime interval $[a,b]$ closely. It is not a narrows, whereby duality allows us to assume that b is not the only cover of a. So we can pick an element $c \in L \setminus \{b\}$ such that $a \prec c$, and let $d := b \vee c$; see on the right of Figure 2. Since $\langle c,d\rangle = \langle c \vee a, c \vee b\rangle \in \mathrm{con}(a,b) = \Theta$, any two elements of $[c,d]$ are collapsed by Θ. Using $\langle a,b\rangle \in \Theta$, $\langle c,d\rangle \in \Theta$, and $|L/\Theta| = n-2 = |L|-2$, it follows that there is no "internal element" in the interval $[c,d]$, that is, $c \prec d$. Furthermore, $[a,b] = \{a,b\}$ and $[c,d] = \{c,d\}$ are the only non-singleton blocks of Θ. In order to show that $b \prec d$, suppose to the contrary that $b < e < d$ holds for some $e \in L$. Since $d = b \vee c \le e \vee c \le d$, we have that $e \vee c = d$, implying $e \nleq c$. Hence, $c \wedge e < e$. Since $\langle c\wedge e, e\rangle = \langle c \wedge e, d \wedge e\rangle \in \Theta$, the Θ-block of e is not a singleton. This contradicts the fact that $\{a,b\}$ and $\{c,d\}$ are the only non-singleton Θ-blocks, whereby we conclude that $b \prec d$. The covering relations established so far show that $S := \{a = b \wedge c, b, c, d = b \vee c\}$ is a covering square in L. We know that both non-singleton Θ-blocks are subsets of S and L/Θ is a chain. Consequently, $L \setminus S$ is also a chain.

Hence, to complete the analysis of the second case when $[a,b]$ is not a narrows, it suffices to show that for every $x \in L \setminus S$, we have that either $x \le a$, or $x \ge d$. So, assume that $x \in L \setminus S$. Since L/Θ is a chain, $\{a,b\}$ and $\{x\}$ are comparable in L/Θ. If $\{x\} < \{a,b\}$, then $\{x\} \vee \{a,b\} = \{a,b\}$ gives that $x \vee a \in \{a,b\}$. If $x \vee a$ happens to equal b, then $x \nleq a$ leads to $x \wedge a < x$ and $\langle x \wedge a, x\rangle = \langle x \wedge a, x \wedge b\rangle \in \Theta$, contradicting the fact the $\{a,b\}$ and $\{c,d\}$ are the only non-singleton Θ-blocks. So if $\{x\} < \{a,b\}$, then $x \vee a = a$ and $x < a$, as required. Thus, we can assume that $\{x\} > \{a,b\}$. If $\{x\} > \{c,d\}$, then the dual of the easy argument just completed shows that $x \ge d$. So, we are left with the case $\{a,b\} < \{x\} < \{c,d\}$. Then the equalities $\{a,b\} \vee \{x\} = \{x\}$ and $\{x\} = \{x\} \wedge \{c,d\}$ give that $b \vee x = x = d \wedge x$, that is, $b \le x \le d$. But $x \notin S$, so $b < x < d$, contradicting $b \prec d$. This completes the second case of the induction step for part (iii) and the proof of Theorem 1. □

3 Remarks and problems on other finite algebras

We conclude the paper with some remarks and problems on finite algebras that are not necessarily lattices. Part (a) below has been pointed out in (2.4).

Remark 2. If A is an n-element algebra such that $\mathrm{Con}(A)$ is distributive, then the following two statements hold.

(a) $|\mathrm{Con}(A)| \le 2^{n-1}$.

(b) If $|\mathrm{Con}(A)| = 2^{n-1}$, then $\mathrm{Con}(A)$ is a boolean lattice.

First proof of Remark 2. As mentioned above, part (a) follows from (2.4). With straightforward changes, the same argument is appropriate to prove part (b); we outline this possibility as follows. Again, we use induction on n. If $|\mathrm{Con}(A)| = 2^{n-1}$, then the induction hypothesis together with (2.3) yield that $\uparrow\Theta$ is a boolean lattice and $|\uparrow\Theta| = 2^{n-2}$, whence the "at most" in (2.2) turns into "exactly". Hence, for each $\Psi \in \uparrow\Theta$, there is exactly one $g(\Psi) \in \mathrm{Con}(A)$ such that $g(\Psi) \ne \Psi = f(g(\Psi))$. Using that $|\mathrm{Con}(A)| = 2^{n-1} = 2 \cdot |\uparrow\Theta|$, it follows that $g(\uparrow\Theta) = \{g(\Psi) : \Psi \in \uparrow\Theta\}$ is disjoint from $\uparrow\Theta$ and so $\mathrm{Con}(A)$ is the disjoint union of $\uparrow\Theta$ and $g(\uparrow\Theta)$. Furthermore, g and the restriction $f\rceil_{g(\uparrow\Theta)}$ of f to the subset $g(\uparrow\Theta) = \mathrm{Con}(A) \setminus \uparrow\Theta$ are reciprocal bijections. For $\Psi \in g(\uparrow\Theta)$, we have that

$f\rceil_{g(\uparrow\Theta)}(\Psi) = \Theta \vee \Psi$, whereby it follows from distributivity that $f\rceil_{g(\uparrow\Theta)}$ is a lattice homomorphism from $g(\uparrow\Theta)$ onto $\uparrow\Theta$, so it is an isomorphism. Since $\Psi < f\rceil_{g(\uparrow\Theta)}(\Psi)$ for every $\Psi \in g(\Theta)$, we conclude that $\mathrm{Con}(A)$ is the direct product of the two-element chain and the boolean lattice $\uparrow\Theta$. Consequently, $\mathrm{Con}(A)$ is also boolean, proving part (b). $\square$

As a preparation for another remark, we also give an alternative proof.

Second proof of Remark 2. The equivalence lattice $\mathrm{Equ}(A)$ is semimodular by Ore [16]; see also Grätzer [8, Theorem 404]. Let $\emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_n = A$ be a maximal chain of subsets of A and denote by Δ_A the equality relation on A. Then $\{\Delta_A \cup (X_i \times X_i) : 1 \le i \le n\}$ is a maximal chain of length $n-1$ in $\mathrm{Equ}(A)$. By semimodularity, $\mathrm{Equ}(A)$ has no longer chain, and neither has $\mathrm{Con}(A)$ since it is a sublattice of $\mathrm{Equ}(A)$. Finally, we know, say, from Grätzer [8, Lemma 170 and Corollaries 169 and 171] that a distributive lattice of length $n-1$ has at most 2^{n-1} elements and we have equality only in the boolean case. $\square$

It follows easily from Freese and Nation [6] that parts (a) and (b) above hold even if A is an n-element semilattice, where $\mathrm{Con}(A)$ is not distributive in general; see also Czédli [3]. This fact and the second proof above raise the problem how to relax the assumption that $\mathrm{Con}(A)$ is distributive if we want to ensure the validity of parts (a) and (b) of Remark 2.

Denote by $B(n) = |\mathrm{Equ}(\{1, 2, \dots, n\})|$ the n-th *Bell number*; see Bell [1] and Rota [17]. For example, $B(5) = 52$ and $B(6) = 203$; see [1, page 540]; these equalities show that $B(n)$ is much larger than 2^{n-1}. Hence, any meaningful generalization of Remark 2 must exclude that $\mathrm{Con}(A) = \mathrm{Equ}(A)$. Since, for n large enough, every element of $\mathrm{Equ}(A)$ is meet reducible or join-reducible with high multiplicity, we cannot leave only few elements from $\mathrm{Equ}(A)$ to get a proper sublattice. This means that the difference $B(n) - |\mathrm{Con}(A)|$ cannot be too small. This difference can be even larger than what the lattice theoretical analysis of $\mathrm{Equ}(A)$ gives, because many sublattices of $\mathrm{Equ}(A)$ cannot be congruence lattices of A; this follows easily from Zádori [18]. As a second problem, we are far from finding the largest number $m(n)$ in the set $\{|\mathrm{Con}(A)| : A \text{ is an } n\text{-element algebra and } \mathrm{Con}(A) \ne \mathrm{Equ}(A)\}$. All we know is a lower bound given in the following remark; this remark and the inequality $1 + B(6-1) = 53 > 2^{6-1}$ will show that $m(n)$ is much larger than 2^{n-1} in general.

Remark 3. For every integer $n \ge 2$, there exists an n-element algebra $\langle A; F\rangle$ such that $\mathrm{Con}(\langle A; F\rangle) \ne \mathrm{Equ}(A)$ and $|\mathrm{Con}(\langle A; F\rangle)| = 1 + B(n-1)$.

Proof. Fix an element $u \in A$ and let $H := A \setminus \{u\}$. A pair $\langle a, b\rangle$ is *nontrivial* if $a \ne b$. For each nontrivial pair $\langle a, b\rangle \in H^2$, define the following unary operation:

$$f_{a,b} \colon A^2 \to A, \quad , x \mapsto \begin{cases} a & \text{if } x \ne u \\ b & \text{if } x = u. \end{cases}$$

Let $F := \{f_{a,b} : \langle a, b\rangle \in H^2 \text{ is a nontrivial pair}\}$. We claim that

$$\text{for each } \Psi \in \mathrm{Equ}(H),\ \ \{\langle u, u\rangle\} \cup \Psi \in \mathrm{Con}(\langle A; F\rangle), \text{ and} \tag{3.1}$$

$$\text{if } \Theta \in \mathrm{Con}(\langle A; F\rangle) \text{ and the } \Theta\text{-block of } u \text{ is not a singleton, then } \Theta = A^2. \tag{3.2}$$

Every operation $f_{a,b}$ is constant on H and every nontrivial pair from $\langle u, u\rangle \cup \Psi$ belongs to H^2, whence (3.1) follows trivially. Assuming the premise of (3.2), pick an element

$x \neq u$ in the Θ-block of u. Then $\langle a, b\rangle = \langle f_{a,b}(x), f_{a,b}(u)\rangle \in \Theta$ for every nontrivial pair $\langle a, b\rangle \in H^2$, implying $\Theta = A^2$ and (3.2). Finally, Remark 3 follows from (3.1) and (3.2). $\square$

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On finitely generated free orthomodular lattices

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Abstract

This is a survey on a series of four papers devoted to study of finitely generated free orthomodular lattices. It aims to recall this research conducted about two decades ago and point out to two particular varieties of orthomodular lattices where similar results would be desirable but are still missing.
Firstly, in this paper an abstract description is presented of the n-generated free algebras $F_{\mathcal{MO}_k}(n)$ in the varieties $\mathcal{MO}_k$ $(k \geq 2, n \geq 1)$ of modular ortholattices generated by the ortholattices $\mathbf{MO}_k$ of height 2 with $2k$ atoms. Also formulas for the cardinalities of these algebras are given. We notice that before our research was conducted, even the cardinality of the free algebra with three generators in the variety $\mathcal{MO}_2$ covering the variety of Boolean algebras was not known. Full abstract descriptions of the free algebras with $n > 2$ generators in the varieties of modular ortholattices were only known in the variety of Boolean algebras.
Secondly, an abstract description of the finitely generated free algebras $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ in the varieties $\mathbf{V}(\mathbf{L}_k)$ $(k \geq 2, n \geq 3)$ of orthomodular lattices generated by the ortholattices $\mathbf{L}_k$ which are horizontal sums of one block $\mathbf{2}^3$ and $k-1$ blocks $\mathbf{2}^2$ is given. This is the simplest case stepping outside the varieties of modular ortholattices and shows how even such a small step increases the complexity of the descriptions. Finally, the finitely generated free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ with n generators in the varieties $\mathbf{V}(\mathbf{O}_k)$ $(2 \leq k \leq n)$ of non-modular ortholattices generated by the orthomodular lattices $\mathbf{O}_k$ which are horizontal sums of k Boolean blocks $\mathbf{2}^3$ are described.
Algebraic methods of the theory of orthomodular lattices are combined with natural duality theory for varieties of algebras. The free algebras are decomposed by central elements into products of canonical intervals of different types. The structures of the intervals are obtained from natural dualities for the varieties of the considered orthomodular lattices. Then Stirling numbers of the second kind are used to count the number of intervals and to give the full abstract descriptions of the free algebras as well as (recursive) formulas for their cardinalities. The structures of the free algebras are illustrated and their cardinalities are for small values of the parameters explicitly displayed in tables.

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1 Introduction

We recall that the origin of the study of orthomodular lattices is due to G. Birkhoff and J. von Neumann [3] who in 1936 suggested taking the lattice of closed subspaces of a Hilbert space as a suitable model for 'the logic of quantum mechanics'. They were interested in discovering, we cite, *"what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces [of a Hilbert space] with respect to set products (i.e. intersections), linear*

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sums, and orthogonal complements — and resembles the usual calculus of propositions with respect to and, or and not."

It is well-known that if the Hilbert space H is finite-dimensional, then the lattice $\langle \mathcal{L}(H); \cap, + \rangle$ of its closed subspaces satisfies the *modular law*

$$P \subseteq Q \implies P + (S \cap Q) = (P + S) \cap Q \quad (P, Q, S \in \mathcal{L}(H)).$$

This law is known to fail in case H is infinite-dimensional, but then a weaker *orthomodular law*

$$P \subseteq Q \ \Rightarrow \ P + (Q \cap P^{\perp}) = Q \quad (P, Q \in \mathcal{L}(H))$$

is satisfied in $\langle \mathcal{L}(H); \cap, + \rangle$. This was discovered by K. Husimi in [11]. The name *orthomodular* is due to I. Kaplansky (1955).

The first systematic treatment of the theory of orthomodular lattices was given by G. Kalmbach [12]. Her monograph together with the monographs by L. Beran [2], by P. Pták and S. Pulmanová [14] and by A. Dvurečenskij and S. Pulmanová [5] are recommended for the basic knowledge about orthomodular lattices and quantum structures.

Roughly speaking, to compare orthomodular lattices with Boolean algebras we can say that a Boolean algebra is an orthomodular lattice in which every two elements are *compatible* while in a general orthomodular lattice there are also non-compatible pairs. Therefore the distributive law cannot be used in orthomodular lattices as in Boolean algebras. A certain version of distributivity however holds in orthomodular lattices, we shall give details later.

The basic information about the subvariety lattice of the variety $\mathcal{OM}$ of all orthomodular lattices can be found in [12]. There is a three-element (covering) chain

$$\mathcal{T} \subsetneq \mathcal{B} \subsetneq \mathcal{MO}_2$$

at the bottom of the subvariety lattice of $\mathcal{OM}$, where $\mathcal{T}$ and $\mathcal{B}$ are the varieties of trivial algebras and Boolean algebras, respectively, and $\mathcal{MO}_2 = \mathbf{V}(\mathbf{MO}_2)$ is the variety generated by the orthomodular lattice $\mathbf{MO}_2$ of height 2 with 4 atoms a_1, a_1', a_2, a_2' (see Figure 1).

The only finite subdirectly irreducible algebras in the variety of all modular ortholattices $\mathcal{MO}$ are $\mathbf{MO}_k$ $(k \geq 2)$ and $\mathbf{2}$. That is why the subvarieties of $\mathcal{MO}$ form the chain

$$\mathcal{T} \subsetneq \mathcal{B} \subsetneq \mathcal{MO}_2 \subsetneq \mathcal{MO}_3 \subsetneq \cdots \subsetneq \mathcal{MO}_k \subsetneq \mathcal{MO}_{k+1} \subsetneq \cdots \subsetneq \mathcal{MO}$$

of type $\omega + 1$ where $\mathcal{MO}_k = \mathbf{V}(\mathbf{MO}_k)$ is the variety generated by $\mathbf{MO}_k$. The strict inclusions $\mathcal{MO}_k \subsetneq \mathcal{MO}_{k+1}$ follow from the fact that $\mathbf{MO}_k$ satisfies the identity

$$\bigwedge_{\substack{i,j=1 \\ i<j}}^{k+1} c'(x_i, x_j) = 0 \quad \text{where } c'(x_i, x_j) = (x_i \vee x_j) \wedge (x_i' \vee x_j) \wedge (x_i \vee x_j') \wedge (x_i' \vee x_j')$$

but $\mathbf{MO}_{k+1}$ does not.

In our study $F_{\mathbf{V}}(n)$ generally denotes the free algebra with n generators in a variety $\mathbf{V}$. The free orthomodular lattice $F_{\mathcal{OM}}(1)$ with one generator is isomorphic to the four-element Boolean algebra $\{0, x, x', 1\}$. Thus

$$F_{\mathcal{OM}}(1) = F_{\mathcal{B}}(1) \cong \mathbf{2}^2$$

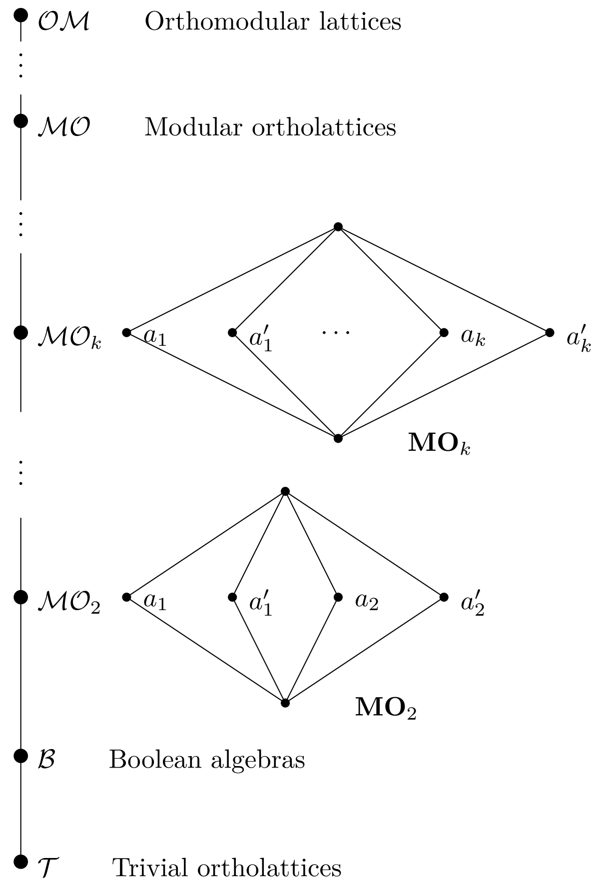

Figure 1. The lattice of subvarieties of modular ortholattices

where $\mathbf{2}$ denotes the two-element Boolean algebra $\mathbf{2} = (\{0,1\}; \vee, \wedge, ', 0, 1)$. It is also well-known that the free orthomodular lattice with two generators $F_{\mathcal{OM}}(2)$ is a direct product of the free Boolean algebra with two generators $F_{\mathcal{B}}(2)$ and the lattice $\mathbf{MO_2}$:

$$F_{\mathcal{OM}}(2) = F_{\mathcal{MO}_2}(2) \cong F_{\mathcal{B}}(2) \times \mathbf{MO_2} \cong \mathbf{2}^4 \times \mathbf{MO_2}.$$

This free algebra has 96 elements and is described in detail in [2].

The free orthomodular lattice with three generators $F_{\mathcal{OM}}(3)$ is infinite. Even the free modular ortholattice $F_{\mathcal{MO}}(3)$ is infinite since it has the orthomodular lattice of closed subspaces of $\mathbb{R}^3$ as a homomorphic image (cf. [12, p. 229]). While $F_{\mathcal{MO}}(3)$ is infinite, the free algebras $F_{\mathcal{MO}_k}(n)$ ($k \geq 2, n \geq 3$) are finite since the varieties $\mathcal{MO}_k$ are locally finite (cf. [4, chapter 1.3]).

In Section 3 we present a description, from our paper [8], of the n-generated free algebras $F_{\mathcal{MO}_k}(n)$ ($k \geq 2, n \geq 1$) in the varieties $\mathcal{MO}_k$ of modular ortholattices generated by the ortholattices $\mathbf{MO}_k$ of height 2 with $2k$ atoms. This study was a continuation of the paper [7], where the cases $k = 2$, $n > 2$ were solved with full details. We notice that in parallel to our theoretical investigations, the calculation of the cardinality of $F_{\mathcal{MO}_2}(3)$ was done by C.B. Wegener using her computer program at the fastest (at the time, in the mid-1990s) computer in Oxford.

In Section 4 we present our investigations, from the paper [9], pursued outside the variety $\mathcal{MO}$ of modular ortholattices. There we described the finitely generated free algebras in the varieties $\mathbf{V}(\mathbf{L}_k)$ generated by the orthomodular lattices $\mathbf{L}_k$ ($k \geq 2$) which are the horizontal sums of one Boolean block $\mathbf{2}^3$ and $k-1$ Boolean blocks $\mathbf{2}^2$. These varieties form an infinite chain "parallel" to the chain of varieties $\mathcal{MO}_k$ in the sense that each $\mathbf{V}(\mathbf{L}_k)$ contains the variety $\mathcal{MO}_k$ (see Figure 2 on page 40). This meant the smallest possible step outside the varieties $\mathcal{MO}_k$ of modular ortholattices — in the generator $\mathbf{MO}_k$ we only replaced one of the blocks $\mathbf{2}^2$ by a larger block $\mathbf{2}^3$.

In Section 5 we present a full description, from our paper [10], of the finitely generated free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ ($2 \leq k \leq n$) in the varieties $\mathbf{V}(\mathbf{O}_k)$ of non-modular ortholattices generated by the orthomodular lattices $\mathbf{O}_k$ which are horizontal sums of k Boolean blocks $\mathbf{2}^3$. These varieties form an another infinite chain "parallel" to the chains of varieties $\mathcal{MO}_k$ and $\mathbf{V}(\mathbf{L}_k)$ in the sense that each $\mathbf{V}(\mathbf{O}_k)$ contains the variety $\mathbf{V}(\mathbf{L}_k)$ (see Figure 3 on page 46). This more ambitious step outside the varieties $\mathcal{MO}_k$ of modular ortholattices resulted in a quite complex description.

Finally, in Section 6 we point out that similar descriptions are still missing and would be desirable in two particular varieties of orthomodular lattices. These two varieties together with the varieties $\mathcal{MO}_3$ and $\mathbf{V}(\mathbf{L}_2)$ are among the four varieties of orthomodular lattices covering the variety $\mathcal{MO}_2$. We present our desire for these missing descriptions as an open problem.

2 Preliminaries

2.1 Orthomodular lattices

By an *orthomodular lattice* is meant an algebra $(L; \vee, \wedge, ', 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $'$ is the unary operation of orthocomplementation. The following identities are satisfied:

$$(a')' = a,\ a \wedge a' = 0,\ a \vee a' = 1,\ 0' = 1,\ 1' = 0;$$
$$(a \wedge b)' = a' \vee b',\ (a \vee b)' = a' \wedge b';$$
$$b = (b \wedge a) \vee [b \wedge (b \wedge a)'].$$

The last identity is called the *orthomodular law* and its equivalent form is

$$a \leq b \ \Rightarrow\ b = a \vee (b \wedge a').$$

In an orthomodular lattice L, the *commutator* of elements $x_1, \ldots, x_n \in L$ is defined by

$$c(x_1, \ldots, x_n) = \bigvee_{(i_1, \ldots, i_n) \in \{0,1\}^n} x_1^{i_1} \wedge \cdots \wedge x_n^{i_n},$$

where $x_i^0 = x_i$ and $x_i^1 = x_i'$. By $c'(x_1, \ldots, x_n)$ is denoted the element $(c(x_1, \ldots, x_n))'$. The commutator of two elements x, y is then

$$c(x, y) = (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y').$$

A *compatibility relation* $a \leftrightarrow b$ on L is defined by

$$a \leftrightarrow b \ \text{ if } \ a = (a \wedge b) \vee (a \wedge b') \quad (a, b \in L)$$

and satisfies

$$a \leq b \ \Rightarrow\ a \leftrightarrow b,\ a \leq b' \ \Rightarrow\ a \leftrightarrow b;$$

$a \leftrightarrow b \;\Rightarrow\; a \leftrightarrow b',\; a' \leftrightarrow b,\; a' \leftrightarrow b';$
$a \leftrightarrow b \;\Leftrightarrow\; c(a,b) = 1.$

The compatibility relation is symmetric and a version of distributivity related to it holds: if $M \subseteq L$ is such that $\bigvee M$ exists in L and $a \in L$ is such that $a \leftrightarrow m$ for every $m \in M$ then

$a \leftrightarrow \bigvee M$ and $a \wedge (\bigvee M) = \bigvee_{m \in M}(a \wedge m)$.

One can show that

$c(x_1, \dots, x_n) \leftrightarrow x_i$ for every $i = 1, 2, \dots, n$ and $c(x_1, \dots, x_n) \leftrightarrow t(x_1, \dots, x_n)$

for any $x_1, \dots, x_n \in L$ and any n-ary term t.

By *central* elements $a \in L$ are meant elements which are compatible with every $x \in L$. By a *centre* of L is meant the set $Z(L)$ of all central elements of L. It forms a Boolean subalgebra of L. Moreover, $a \in Z(L)$ and $v \in L$ imply $a \wedge v \in Z([0, v])$ and

(1) $c \in Z(L) \Leftrightarrow L \cong [0, c] \times [0, c']$ (cf. [12, p. 20]).

2.2 Natural dualities for the varieties of orthomodular lattices

The fundamental facts about the theory of natural dualities can be found in [4]. We recall that a variety generated by an algebra $\underline{\mathbf{M}}$ is arithmetical if $\underline{\mathbf{M}}$ has an arithmeticity (Pixley) term function $p(x, y, z) : \underline{\mathbf{M}}^3 \to \underline{\mathbf{M}}$ satisfying

$$p(a, b, b) = p(a, b, a) = p(b, b, a) = a \quad \text{for all } a, b \in M.$$

The term function

$$p(x,y,z) = (x \vee z) \wedge (x \vee y') \wedge (z \vee y') \wedge [(c(x,y) \wedge z) \vee (c(y,z) \wedge x) \vee (c(x,z) \wedge x \wedge z)],$$

is an arithmeticity term function for the generator $\mathbf{MO}_k$. To see this, note that if x, z belong to the same block of $\mathbf{MO}_k$ then $(x \vee z) \wedge (x' \vee z) = z$ and $c(x, z) = 1$; if x, z are atoms of different blocks of $\mathbf{MO}_k$, $(x \vee z) \wedge (x' \vee z) = 1$ and $c(x, z) = 0$.

By the Arithmetic Strong Duality Theorem of the Natural duality theory (cf. [4, Theorem 3.11]), the n-generated free algebra $F_{\mathcal{MO}_k}(n)$ $(k \geq 2, n \geq 1)$ is isomorphic to the algebra of all functions from ${\mathbf{MO}_k}^n$ to $\mathbf{MO}_k$ preserving the partial endomorphisms of $\mathbf{MO}_k$.

To discuss the partial endomorphisms of $\mathbf{MO}_k$, we firstly notice that for $k \geq 2$, every endomorphism of $\mathbf{MO}_k$ is an automorphism. It is easy to see that each partial endomorphism of $\mathbf{MO}_k$ must map the top to the top, the bottom to the bottom and if it maps an atom a to $c \in \{0, 1\}$, then it must map a' to $c' \in \{0, 1\}$. Such partial endomorphisms are not extendable. Any other partial endomorphism must map all atoms in its domain to distinct atoms of $\mathbf{MO}_k$, while preserving the complementation $'$. Partial endomorphisms of this kind extend to automorphisms and their graphs can be obtained by intersection from the automorphism group, $\mathrm{Aut}(\mathbf{MO}_k)$. Let us consider a non-extendable partial endomorphism r mapping onto $\{0, 1\}$, with graph $r^{\square} = \{(0, 0), (a, 0), (a', 1), (1, 1)\}$, where a is some atom in $\mathbf{MO}_k$.

Theorem 2.1 ([8, Theorem 2.3])**.** Let a be an atom of $\mathbf{MO}_k$ and let r be the partial endomorphism with graph $r^{\square} = \{(0, 0), (a, 0), (a', 1), (1, 1)\}$. Then for $k \geq 2$, $H = \mathrm{Aut}(\mathbf{MO}_k) \cup \{r\}$ yields a duality on the variety $\mathcal{MO}_k = \mathbb{ISP}(\mathbf{MO}_k)$.

Corollary 2.2 ([8, Corollary 2.4])**.** Let $H = \mathrm{Aut}(\mathbf{MO}_k) \cup \{r\}$. Then the n-generated free algebra $F_{\mathcal{MO}_k}(n)$ in the variety $\mathcal{MO}_k$ is isomorphic to the algebra of all H-preserving functions from $(\mathbf{MO}_k)^n$ to $\mathbf{MO}_k$.

3 Finitely generated free algebras in $\mathcal{MO}_k$

The last section showed that the free orthomodular lattice $F_{\mathcal{MO}_k}(n)$ with n generators in the variety $\mathcal{MO}_k = \mathbb{ISP}(\mathbf{MO}_k)$ is isomorphic to the algebra of all those functions from $(\mathbf{MO}_k)^n$ to $\mathbf{MO}_k$ which preserve $H = \mathrm{Aut}(\mathbf{MO}_k) \cup \{r\}$. We notice that these functions are exactly the n-ary term functions on $\mathbf{MO}_k$. Our strategy is to find central elements for a decomposition of $F_{\mathcal{MO}_k}(n)$ into a product of intervals and then to describe these intervals using those H-preserving functions.

3.1 The decomposition by central elements

We firstly find central elements for the decomposition of the free orthomodular lattice $F_{\mathcal{MO}_k}(n)$. Let $t(x_1, \dots, x_n) \colon (\mathbf{MO}_k)^n \to \mathbf{MO}_k$ be a term function into $\{0, 1\}$. Then for any term function $u(x_1, \dots, x_n) \colon (\mathbf{MO}_k)^n \to \mathbf{MO}_k$,

$$t(x_1, \dots, x_n) = (t(x_1, \dots, x_n) \wedge u(x_1, \dots, x_n)) \vee (t(x_1, \dots, x_n) \wedge u'(x_1, \dots, x_n)).$$

Hence any term function $t(x_1, \dots, x_n)$ mapping into $\{0, 1\}$ is a central element of $F_{\mathcal{MO}_k}(n)$. Since the commutators are such term functions, by (1) we get the decomposition

$$F_{\mathcal{MO}_k}(n) = [0, c(x_1, \dots, x_n)] \times [0, c'(x_1, \dots, x_n)].$$

The structure of the first interval $[0, c(x_1, \dots, x_n)]$ is analysed in the next theorem.

Theorem 3.1 ([8, Theorem 3.1])**.** The interval $[0, c(x_1, \dots, x_n)]$ in $F_{\mathcal{MO}_k}(n)$ is isomorphic to the n-generated free Boolean algebra $F_{\mathcal{B}}(n)$. Hence

$$[0, c(x_1, \dots, x_n)] \cong \mathbf{2}^{2^n}.$$

The second interval $[0, c'(x_1, \dots, x_n)]$ is further decomposed. The binary commutators $c(x_i, x_j)$ are used for $i, j = 1, \dots, n$, $i < j$ and we arrive at the decomposition

$$[0, c'(x_1, \dots, x_n)] \cong \prod_{\tilde{\mathrm{w}} \in \{0,1\}^N} [0, \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \dots, x_n)].$$

Here the product is taken over all N-tuples

$$\tilde{\mathrm{w}} = (w_{1,2}, \dots, w_{1,n}, w_{2,3}, \dots, w_{n-1,n}) \in \{0, 1\}^N$$

where $N = \binom{n}{2}$ and

$$c^{w_{i,j}}(x_i, x_j) = \begin{cases} c(x_i, x_j), & \text{if } w_{i,j} = 0, \\ c'(x_i, x_j), & \text{if } w_{i,j} = 1. \end{cases}$$

A labelled unoriented graph $G_{\tilde{\mathrm{w}}}$ (without multiple edges and loops) can now be constructed for every term function

$$t_{\tilde{\mathrm{w}}}(x_1, \dots, x_n) = \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \dots, x_n)$$

on vertex set $\{x_1, \dots, x_n\}$ with edges $x_i x_j$ whenever $w_{i,j} = 1$ for $i < j$. From this graph G we are able to reconstruct the term function $t_{\tilde{\mathrm{w}}}$, which is also denoted by C_G. We notice that any one of $\tilde{\mathrm{w}}$, $t_{\tilde{\mathrm{w}}}$ ($=C_G$) and G determines the other two. For analysing the structure of the interval $[0, c'(x_1, \dots, x_n)]$ we investigate the intervals $[0, t_{\tilde{\mathrm{w}}}(x_1, \dots, x_n)]$ for every N-tuple $\tilde{\mathrm{w}}$. Since some of these intervals can be trivial, it is useful to give a necessary and sufficient condition on the structure of the corresponding graph G for the interval $[0, t_{\tilde{\mathrm{w}}}(x_1, \dots, x_n)] = [0, C_G(x_1, \dots, x_n)]$ to be non-trivial:

Proposition 3.2 ([8, Proposition 3.2])**.** Consider the term function $C_G(x_1,\dots,x_n):(\mathbf{MO}_k)^n\to\mathbf{MO}_k$ given by

$$\bigwedge_{\substack{i,j=1,\\ i<j}}^{n} c^{w_{i,j}}(x_i,x_j)\wedge c'(x_1,\dots,x_n)$$

and its associated graph G. The following conditions are equivalent:

(a) The term function $C_G(x_1,\dots,x_n)$ is not identically equal to zero.

(b) There exist elements $a_1,\dots,a_n\in\mathbf{MO}_k$ such that

  (i) $C_G(a_1,\dots,a_n)=1$ and

  (ii) the elements $a_1,\dots,a_n$ are not all from the same block of $\mathbf{MO}_k$ and

  (iii) a_i,a_j are atoms of different blocks in $\mathbf{MO}_k$ if and only if x_ix_j is an edge of G.

(c) The graph $G_p:=G$ consists of l isolated vertices $(0\le l\le n-p)$ and one connected component which is a complete p-partite graph $(2\le p\le n)$.

Moreover, if the graph $G=G_p$ is as in (c), then there are exactly $2^n\binom{k}{p}p!$ n-tuples $(a_1,\dots,a_n)$ in $(\mathbf{MO}_k)^n$ with the value $C_G(a_1,\dots,a_n)$ non-zero.

Proof. If (a) holds then there exist $a_1,\dots,a_n\in\mathbf{MO}_k$ with $C_G(a_1,\dots,a_n)\neq 0$. It follows that $c^{w_{i,j}}(a_i,a_j)$ and $c'(a_1,\dots,a_n)$ are non-zero for all i,j. Then necessarily $C_G(a_1,\dots,a_n)=1$. We notice that $c'(a_1,\dots,a_n)=1$ if and only if there exist i,j $(i<j)$ such that a_i,a_j are atoms of different blocks of $\mathbf{MO}_k$. Then we have $c^{w_{i,j}}(a_i,a_j)=1$ if and only if $w_{i,j}=1$ if and only if x_ix_j is an edge in G. This proves (b).

Now let (b) hold and $a_1,\dots,a_n\in\mathbf{MO}_k$ be as in (b). By the condition (b)(iii), x_i must be an isolated vertex in G for $a_i\in\{0,1\}$. In case a_i is an atom in $\mathbf{MO}_k$ there exists j such that a_j is an atom in a different block by (b)(ii), and for all such i,j there is an edge x_ix_j in G by (b)(iii). Hence the graph G has isolated vertices corresponding to those $a_i\in\{a_1,\dots,a_n\}$ that are from $\{0,1\}$ while the other vertices can be partitioned according to which block the corresponding a_i comes from. This results in a complete p-partite graph by (b)(iii) such that $p\ge 2$ by (b)(ii). We have proven (c).

Let us now assume that (c) hold. We have already shown that, given a labelled graph $G=G_p$ as in (c), one can choose $a_1,\dots,a_n\in\mathbf{MO}_k$ with $C_G(a_1,\dots,a_n)$ non-zero. We notice that $C_G(a_1,\dots,a_n)\neq 0$ if and only if all the expressions $c^{w_{i,j}}(a_i,a_j)$ and $c'(a_1,\dots,a_n)$ have values 1. We firstly consider the connected component of G that is partitioned into $p\ge 2$ parts and a vertex x_i in this connected component. For every j such that x_ix_j is an edge in G, the term C_G contains the subterm $c'(x_i,x_j)$ in case $i<j$ and the subterm $c'(x_j,x_i)$ in case $j<i$. This subterm takes value 1 at (a_i,a_j) if and only if a_i,a_j are from different blocks of $\mathbf{MO}_k$. Now for x_j in the same block of the p-partite graph as x_i, the term C_G contains the subterm $c(x_i,x_j)$ in case $i<j$ and the subterm $c(x_j,x_i)$ in case $j<i$. These subterms take value 1 at (a_i,a_j) if and only if a_i,a_j are from the same block of $\mathbf{MO}_k$. In case x_i is an isolated vertex of G, any subterm $c^{w_{i,j}}(x_i,x_j)$ in the term C_G is $c(x_i,x_j)$ (and analogously for $c^{w_{j,i}}(x_j,x_i)$), hence a_i must lie in the same block as a_j for all j. It follows that we have to choose a_i to be either 0 or 1. So in order to have $C_G(a_1,\dots,a_n)\neq 0$, we associate to each block of the p-partite component of the graph G a unique block of $\mathbf{MO}_k$ and we choose the corresponding a_i to be atoms of the associated blocks. And we choose $a_i\in\{0,1\}$ for isolated vertices x_i. We have proven (a).

To count the number of n-tuples $(a_1,\dots,a_n)$ such that $C_G(a_1,\dots,a_n)\neq 0$, we have seen that we need to associate p blocks of $\mathbf{MO}_k$, in any order, to the p blocks of the connected p-partite component of G. Clearly, once the order of the blocks has been chosen, there are two choices for any a_i: more precisely, these choices are either of the two atoms in the corresponding block for the vertex x_i in the connected component, or 0 or 1 for an isolated vertex x_i. Altogether this gives $2^n\binom{k}{p}p!$ such n-tuples $(a_1,\dots,a_n)$ and the proof is complete. $\square$

3.2 The use of natural duality

By using the natural duality for $\mathcal{MO}_k$ given by $H=\mathrm{Aut}(\mathbf{MO}_k)\cup\{r\}$ we are able to analyse the structure of the intervals $[0,C_G(x_1,\dots,x_n)]$ associated with graphs $G=G_p$ as they were described in Proposition 3.2(c). By the duality, the interval $[0,C_G(x_1,\dots,x_n)]$ can be described as the algebra of all those H-preserving functions $f\colon(\mathbf{MO}_k)^n\to\mathbf{MO}_k$ that are pointwise less than or equal to $C_G(x_1,\dots,x_n)$. This yields that any such function f must take value zero whenever the term C_G does. Now by

$$T_G:=\{(a_1,\dots,a_n)\in(\mathbf{MO}_k)^n\mid C_G(a_1,\dots,a_n)=1\}$$

we denote the set consisting of the $2^n\binom{k}{p}p!$ n-tuples $(a_1,\dots,a_n)\in(\mathbf{MO}_k)^n$ at which C_G is non-zero; this automatically means $C_G(a_1,\dots,a_n)=1$.

Now we discuss the preservation of the dualising structure $H=\mathrm{Aut}(\mathbf{MO}_k)\cup\{r\}$. We first recall a general definition saying that a function $f:(\mathbf{MO}_k)^n\to\mathbf{MO}_k$ preserves a partial endomorphism e of $\mathbf{MO}_k$ with the graph $e^{\square}$ if for $\underline{\mathbf{a}}=(a_1,\dots,a_n),\underline{\mathbf{b}}=(b_1,\dots,b_n)\in(\mathbf{MO}_k)^n$,

(2) $(a_1,b_1)\in e^{\square},\dots,(a_n,b_n)\in e^{\square}\Rightarrow(f(\underline{\mathbf{a}}),f(\underline{\mathbf{b}}))\in e^{\square}$.

We consider our partial endomorphism $r\in H$ with the graph

$$r^{\square}=\{(0,0),(a,0),(a',1),(1,1)\}$$

where a is an atom of $\mathbf{MO}_k$. We notice that for the left-hand side of (2) to hold, the elements a_i must lie in $\{0,a,a',1\}$ and the elements b_i in $\{0,1\}$, yielding that neither $\underline{\mathbf{a}}$ nor $\underline{\mathbf{b}}$ can lie in the set T_G. Hence $(f(\underline{\mathbf{a}}),f(\underline{\mathbf{b}}))=(0,0)\in r^{\square}$ for any $f\le C_G$ and this means that the function f is r-preserving.

To investigate the preservation of the automorphisms of $\mathbf{MO}_k$, we firstly consider the action of the automorphism group $\mathrm{Aut}(\mathbf{MO}_k)$ on $(\mathbf{MO}_k)^n$ (we refer here e.g. to [13] for the basic notions). Naturally, the automorphism group $\mathrm{Aut}(\mathbf{MO}_k)$ acts on $\mathbf{MO}_k$ by permuting its atoms. We denote the action of an automorphism α on $a\in\mathbf{MO}_k$ by a^{α} (other common notations are $\alpha(a)$ or $a\alpha$ depending on whether α is treated as a function or as a permutation). One can extend the action of $\mathrm{Aut}(\mathbf{MO}_k)$ on $\mathbf{MO}_k$ pointwise to $(\mathbf{MO}_k)^n$, thus $\underline{\mathbf{a}}^{\alpha}=(a_1^{\alpha},\dots,a_n^{\alpha})\in(\mathbf{MO}_k)^n$ for $\underline{\mathbf{a}}=(a_1,\dots,a_n)\in(\mathbf{MO}_k)^n$ and $\alpha\in\mathrm{Aut}(\mathbf{MO}_k)$. We denote the orbit of $\underline{\mathbf{a}}$ for such $\underline{\mathbf{a}}$ and α by

$$\mathrm{Orb}\,\underline{\mathbf{a}}=\{\underline{\mathbf{a}}^{\beta}\mid\beta\in\mathrm{Aut}(\mathbf{MO}_k)\},$$

and the stabiliser of $\underline{\mathbf{a}}$ by

$$\mathrm{Stab}\,\underline{\mathbf{a}}=\{\beta\in\mathrm{Aut}(\mathbf{MO}_k)\mid\underline{\mathbf{a}}^{\beta}=\underline{\mathbf{a}}\}.$$

Moreover, the set of elements fixed by α under the action on $\mathbf{MO}_k$ is denoted by

$$\mathrm{fix}_{\mathbf{MO}_k}\alpha=\{b\in\mathbf{MO}_k\mid b^{\alpha}=b\}.$$

Now the well-known Stabiliser-Orbit Theorem (cf. also [13, Corollary 6.2]) gives us that for all $\underline{\mathbf{a}} \in (\mathbf{MO}_k)^n$,

(3) $|\mathrm{Aut}(\mathbf{MO}_k)| = |\mathrm{Stab}\ \underline{\mathbf{a}}| \cdot |\mathrm{Orb}\ \underline{\mathbf{a}}|$.

In order to determine the size $|\mathrm{Aut}(\mathbf{MO}_k)|$ of the automorphism group $\mathrm{Aut}(\mathbf{MO}_k)$ of $\mathbf{MO}_k$ for $k \geq 2$, we point out that every automorphism is determined by the images of k atoms, one from each block. These atoms have to be mapped to atoms of distinct blocks of $\mathbf{MO}_k$ and this gives us two choices per such atom as soon as the order of blocks has been determined. This leads to the size $|\mathrm{Aut}(\mathbf{MO}_k)| = 2^k k!$.

We are ready to rewrite (2) for an automorphism $\alpha \in \mathrm{Aut}(\mathbf{MO}_k)$. A function $f : (\mathbf{MO}_k)^n \to \mathbf{MO}_k$ is α-preserving if for all $\underline{\mathbf{a}} = (a_1, \dots, a_n) \in (\mathbf{MO}_k)^n$,

(4) $f(\underline{\mathbf{a}}^\alpha) = f(\underline{\mathbf{a}})^\alpha$.

We return to the investigation of the interval $[0, C_G(x_1, \dots, x_n)]$ associated with a graph $G = G_p$ $(2 \leq p \leq k)$. We first notice that

$$\underline{\mathbf{a}} \in T_G \text{ if and only if } \underline{\mathbf{a}}^\alpha \in T_G$$

for any $\alpha \in \mathrm{Aut}(\mathbf{MO}_k)$. The equality (4) is satisfied on the set $(\mathbf{MO}_k)^n \setminus T_G$ for any $\alpha \in \mathrm{Aut}(\mathbf{MO}_k)$, since $f(\underline{\mathbf{b}}) = 0$ for all $\underline{\mathbf{b}} \in (\mathbf{MO}_k)^n \setminus T_G$. Now we consider $\underline{\mathbf{a}} \in T_G$. We notice that the coordinates of $\underline{\mathbf{a}}$ lie exactly in p blocks of $\mathbf{MO}_k$ and that any such $\underline{\mathbf{a}}$ is fixed by exactly those $\alpha \in \mathrm{Aut}(\mathbf{MO}_k)$ which only permute atoms in the remaining $k - p$ blocks of $\mathbf{MO}_k$. This means that

$$|\mathrm{Stab}\ \underline{\mathbf{a}}| = |\mathrm{Aut}(\mathbf{MO}_{k-p})| = 2^{k-p}(k-p)!$$

which does not depend on $\underline{\mathbf{a}}$. Hence by (3) the set T_G is partitioned by the action of $\mathrm{Aut}(\mathbf{MO}_k)$ into orbits of size

(5) $|\mathrm{Orb}\ \underline{\mathbf{a}}| = \frac{|\mathrm{Aut}(\mathbf{MO}_k)|}{|\mathrm{Stab}\ \underline{\mathbf{a}}|} = \frac{2^k k!}{2^{k-p}(k-p)!} = 2^p \binom{k}{p} p!$

We point out that to define an $\mathrm{Aut}(\mathbf{MO}_k)$preserving map $f \leq C_G$, we cannot freely choose images from $\mathbf{MO}_k$ for representatives of the orbits within T_G and then use (4) in order to define the images of the other members of T_G (we did this in the first paper [7] in the case of the variety $\mathcal{MO}_2$). The reason is that when $p < k$, there exist elements $\alpha \neq \beta$ of $\mathrm{Aut}(\mathbf{MO}_k)$ such that $\underline{\mathbf{a}}^\alpha = \underline{\mathbf{a}}^\beta$ for any representative $\underline{\mathbf{a}}$ of orbit Orb $\underline{\mathbf{a}}$, and this restricts the choices for $f(\underline{\mathbf{a}})$ only to those which satisfy $f(\underline{\mathbf{a}})^\alpha = f(\underline{\mathbf{a}})^\beta$.

For any element $b \in \mathbf{MO}_k$,

$$b^\alpha = b^\beta \iff b^{\alpha\beta^{-1}} = b \iff \alpha\beta^{-1} \in \mathrm{Stab}\ b \iff b \in \mathrm{fix}_{\mathbf{MO}_k}(\alpha\beta^{-1}).$$

So an $\mathrm{Aut}(\mathbf{MO}_k)$-preserving function f is restricted to values $f(\underline{\mathbf{a}}) \in \mathrm{fix}_{\mathbf{MO}_k}(\gamma)$, for $\gamma \in \mathrm{Stab}\ \underline{\mathbf{a}}$, and thus

(6) $f(\underline{\mathbf{a}}) \in \bigcap_{\gamma \in \mathrm{Stab}\ \underline{\mathbf{a}}} \mathrm{fix}_{\mathbf{MO}_k}(\gamma)$.

Since the stabiliser of $\underline{\mathbf{a}}$ consists of exactly those automorphisms which only permute the $k - p$ blocks not covered by the coordinates of $\underline{\mathbf{a}}$, we obtain that $\bigcap_{\gamma \in \mathrm{Stab}\ \underline{\mathbf{a}}} \mathrm{fix}_{\mathbf{MO}_k}(\gamma)$ is the set of atoms of the p blocks covered by $\underline{\mathbf{a}}$ together with 0 and 1 that are always fixed. Hence ordered by the usual order relation $\leq$ on $\mathbf{MO}_k$, we have

(7) $\bigcap_{\gamma \in \mathrm{Stab}\ \underline{\mathbf{a}}} \mathrm{fix}_{\mathbf{MO}_k}(\gamma) \cong \mathbf{MO}_p$.

In order to construct the $\mathrm{Aut}(\mathbf{MO}_k)$-preserving functions $f\colon (\mathbf{MO}_k)^n \to \mathbf{MO}_k$ which are pointwise less than or equal to $C_G(x_1,\ldots,x_n)$, one has to define f to be zero whenever C_G is and to partition the set T_G on which C_G is non-zero into orbits under the automorphism action. According to (6) one can freely choose the image $f(\underline{\mathbf{a}})$ for each orbit-representative $\underline{\mathbf{a}}$ within $\bigcap_{\gamma\in \mathrm{Stab}\underline{\mathbf{a}}} \mathrm{fix}_{\mathbf{MO}_k}(\gamma)$. This forces the values of the other points in Orb $\underline{\mathbf{a}}$ to be $f(\underline{\mathbf{a}}^\alpha) = f(\underline{\mathbf{a}})^\alpha$. Hence by (7) each orbit within T_G contributes a factor $\mathbf{MO}_p$ to the algebra of $\mathrm{Aut}(\mathbf{MO}_k)$-preserving functions $f\colon (\mathbf{MO}_k)^n \to \mathbf{MO}_k$. The orbits are all of the same size and the number of them within T_G is

$$\frac{|T_G|}{|\mathrm{Orb}\ \underline{\mathbf{a}}|} = \frac{2^n\binom{k}{p}p!}{2^p\binom{k}{p}p!} = 2^{n-p}.$$

Consequently,

(8) $[0, C_G(x_1,\ldots,x_n)] \cong (\mathbf{MO}_p)^{2^{n-p}}$.

3.3 Counting the intervals

We know that the interval $[0, c'(x_1,\ldots,x_n)]$ is the product of intervals $[0, C_G(x_1,\ldots,x_n)]$ over all graphs $G = G_p$ $(2 \le p \le k)$ that satisfy the condition (c) from Proposition 3.2. The number of labelled complete p-partite graphs on m vertices is obviously the same as the number of partitions of a labelled m-element set into p parts. This number is given by the Stirling numbers $S(m,p)$ of the second kind (we refer to [1, 2.66, 3.29, 3.39]):

$$S(m,p) = pS(m-1,p) + S(m-1,p-1) = \frac{1}{p!}\sum_{s=1}^{p}(-1)^{p-s}\binom{p}{s}s^m.$$

As p ranges from 2 to k and the number of isolated vertices l from 0 to $n-p$, the number of the graphs $G = G_p$ on n vertices is given by

$$\phi'(n,p) = \sum_{l=0}^{n-p}\binom{n}{l}S(n-l,p).$$

We notice that $\phi'(1,p) = 0$ since $p \ge 2$. We define

$$\phi(n,p) = 2^{n-p}\phi'(n,p).$$

We remark that when $p = 2$, which is satisfied whenever $k = 2$, $\phi(n,2)$ corresponds to the function $\phi(n)$ in [7, Theorem 1.1].

We give a table of values of $\phi(n,p)$ for $1 \le n,p \le 10$. To do this we need to compute, for $0 \le l \le p$, the binomial coefficients $\binom{n}{l}$ and the Stirling numbers of the second kind, $S(n-l,p)$. Our first table gives part of Pascal's triangle computed by the recursive definition

$$\binom{n}{0} = 1,$$
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We notice that $\binom{n}{k} = 0$ for $k > n$, and this is represented by empty cells in the table.

Binomial coefficients (Pascal's triangle)

$\binom{n}{k}$	k=0	1	2	3	4	5	6	7	8	9	10
n=1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Our second table displays the demanded Stirling numbers. These can be defined recursively by

$$\begin{aligned} &S(0,0) = 1,\ S(n,0) = 0 \quad \text{for } n > 0,\\ &S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k). \end{aligned}$$

As before, the empty cells are to be filled with the values 0.

Stirling numbers of the second kind

$S(n,k)$	k=1	2	3	4	5	6	7	8	9	10
n=1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

Finally, the table of values of $\phi(n,k)$ can be established by the following procedure:
(1) The first column's entries are $\phi(n,1) = 2^n$.

Other entries are calculated by $\phi(n,k) = 2^{n-k} \sum_{l=0}^{n-k} \binom{n}{l} S(n-l,k)$.
(2) For $n < k$, $\phi(n,k)$ takes value 0.
(3) For $n \geq k$, the sum in the expression is taken over the products of row n entries of the Pascal triangle with column k entries of the Stirling table. At last the result is multiplied by 2^{n-k} to give $\phi(n,k)$.

Values of $\phi(n,k)$

$\phi(n,k)$	k=1	2	3	4	5	6	7	8	9	10
n=1	2									
2	4	1								
3	8	12	1							
4	16	100	20	1						
5	32	720	260	30	1					
6	64	4816	2800	560	42	1				
7	128	30912	27216	8400	1064	56	1			
8	256	193600	248640	111216	21168	1848	72	1		
9	512	1194240	2182720	1360800	365232	47040	3000	90	1	
10	1024	7296256	18656000	15790720	5743584	1023792	95040	4620	110	1

3.4 The results and their illustration

As stated above, each $2 \le p \le k$ contributes a factor of $(\mathbf{MO}_p)^{\phi(n,p)}$ to the structure of the interval $[0, c'(x_1,\dots,x_n)]$. Consequently,

$$[0, c'(x_1,\dots,x_n)] \cong \prod_G [0, C_G(x_1,\dots,x_n)] \cong \prod_{p=2}^{k} (\mathbf{MO}_p)^{\phi(n,p)}.$$

Now we can present our results. Firstly, an abstract description of $F_{\mathcal{MO}_k}(n)$ for all $n \ge 1$, $k \ge 2$.

Theorem 3.3 ([8, Theorem 3.3])**.** For all $n \ge 1$, $k \ge 2$,

$$F_{\mathcal{MO}_k}(n) \cong F_{\mathcal{B}}(n) \times \prod_{p=2}^{k} (\mathbf{MO}_p)^{\phi(n,p)}$$

where $F_{\mathcal{B}}(n)$ is the n-generated free Boolean algebra $\mathbf{2}^{2^n}$,

$$\phi(n,p) = 2^{n-p}\phi'(n,p) = 2^{n-p}\sum_{l=0}^{n-p}\binom{n}{l} S(n-l,p),$$

and the Stirling numbers of the second kind are given by

$$S(m,p) = \frac{1}{p!}\sum_{s=1}^{p}(-1)^{p-s}\binom{p}{s}s^m.$$

Secondly, we can give a formula for the cardinality of $F_{\mathcal{MO}_k}(n)$ for all $n \ge 1$, $k \ge 2$.

Corollary 3.4 ([8, Corollary 3.4])**.** For all $n \ge 1$, $k \ge 2$,

$$|F_{\mathcal{MO}_k}(n)| = 2^{2^n} \cdot \prod_{p=2}^{k} (2(p+1))^{2^{n-p}\sum_{l=0}^{n-p}\binom{n}{l}S(n-l,p)},$$

where the Stirling numbers of the second kind are defined by

$$S(m,p) = \frac{1}{p!}\sum_{s=1}^{p}(-1)^{p-s}\binom{p}{s}s^m.$$

From the table of values of $\phi(n,k)$ one can read off the structure of all free algebras $F_{\mathcal{MO}_k}(n)$ for k, $n \leq 10$. Firstly, we define $\mathbf{MO_1} := \mathbf{2}$ and secondly, we extend the formula $\phi(n,p)$ to include values at $p = 1$ by defining $\phi(n,1) = 2^n$. This enables us to write

$$F_{\mathcal{MO}_k}(n) \cong \prod_{p=1}^{k} (\mathbf{MO}_p)^{\phi(n,p)}.$$

To determine, for example, the structure of the free algebra $F_{\mathcal{MO}_3}(7)$, we consider the first three entries in the 7th row of the above table. The first entry gives the power of $\mathbf{MO}_1 = \mathbf{2}$ in $F_{\mathcal{MO}_3}(7)$, the next one gives the power of $\mathbf{MO}_2$, etc. Thus

$$F_{\mathcal{MO}_3}(7) \cong \mathbf{2}^{128} \times (\mathbf{MO}_2)^{30912} \times (\mathbf{MO}_3)^{27216} \quad \text{and}$$
$$|F_{\mathcal{MO}_3}(7)| = 2^{128} \cdot (2(2+1))^{30912} \cdot (2(3+1))^{27216}.$$

We notice that, for $k > n$, $F_{\mathcal{MO}_k}(n) = F_{\mathcal{MO}_n}(n)$ and that, for $k < n$, the free algebra $F_{\mathcal{MO}_{k+1}}(n)$ has an additional non-trivial factor $(\mathbf{MO}_{k+1})^{\phi(n,k+1)}$ when compared to the structure of $F_{\mathcal{MO}_k}(n)$.

4 Finitely generated free algebras in $\mathbf{V}(\mathbf{L}_k)$

In this section we consider the chain of varieties $\mathbf{V}(\mathbf{L}_k)$ $(k \geq 2)$ of orthomodular lattices where $\mathbf{L}_k$ is the ortholattice which is the horizontal sum of one block $\mathbf{2}^3$ and $k-1$ blocks $\mathbf{2}^2$. More precisely, this chain of varieties is such that for every $k \geq 2$, $\mathbf{V}(\mathbf{L}_k)$ contains the variety $\mathbf{V}(\mathbf{MO}_k)$ (see Figure 2).

We present an abstract description of the finitely generated free algebras $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ $(k \geq 2, n \geq 3)$ with n generators in the varieties $\mathbf{V}(\mathbf{L}_k)$ from our paper [9]. We recall that these free algebras are finite because the varieties $\mathbf{V}(\mathbf{L}_k) = \mathbb{ISP}(\mathbf{L}_k)$ are locally finite (see [4, Chapter 1.3]).

4.1 Similarities with the modular ortholattices

The arithmeticity term function for the ortholattices $\mathbf{L}_k$ is the same as for the modular ortholattices $\mathbf{MO}_k$. From the Arithmetic Strong Duality Theorem (cf. [4, Theorem 3.11]) it follows again that the n-generated free algebra $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ $(k \geq 2, n \geq 3)$ is isomorphic to the algebra of all functions from ${L_k}^n$ to L_k preserving the partial endomorphisms of $\mathbf{L}_k$.

The decomposition process is analogous to the one in the previous section. In the first step the n-generated free algebra $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ is expressed as the product

$$F_{\mathbf{V}(\mathbf{L}_k)}(n) = [0, c(x_1,\ldots,x_n)] \times [0, c'(x_1,\ldots,x_n)]$$

where $c(x_1,\ldots,x_n) = \bigvee_{(i_1,\ldots,i_n)\in\{0,1\}^n} (x_1^{i_1} \wedge \cdots \wedge x_n^{i_n})$ denotes the commutator of the generators $x_1,\ldots,x_n$ of $F_{\mathbf{V}(\mathbf{L}_k)}(n)$, $x_i^0 = x_i$, $x_i^1 = x_i'$ and $c'(x_1,\ldots,x_n)$ is $(c(x_1,\ldots,x_n))')$. The interval $[0, c(x_1,\ldots,x_n)]$ again represents the n-generated free Boolean algebra $F_{\mathcal{B}}(n) \cong \mathbf{2}^{2^n}$. In the second step the interval $[0, c'(x_1,\ldots,x_n)]$ is decomposed by the commutators $c(x_i,x_j)$ $(i,j = 1,\ldots,n,\ i<j)$ into the form

$$[0, c'(x_1,\ldots,x_n)] \cong \prod_{\tilde{\mathrm{w}}\in\{0,1\}^N} [0, \bigwedge_{\substack{i,j=1\\ i<j}}^{n} c^{w_{i,j}}(x_i,x_j) \wedge c'(x_1,\ldots,x_n)],$$

where the product is taken over all N-tuples $\tilde{\mathrm{w}} = (w_{1,2},\ldots,w_{n-1,n}) \in \{0,1\}^N$, $N = \binom{n}{2}$ and

$$c^{w_{i,j}}(x_i,x_j) = \begin{cases} c(x_i,x_j), & \text{if } w_{i,j} = 0,\\ c'(x_i,x_j), & \text{if } w_{i,j} = 1.\end{cases}$$

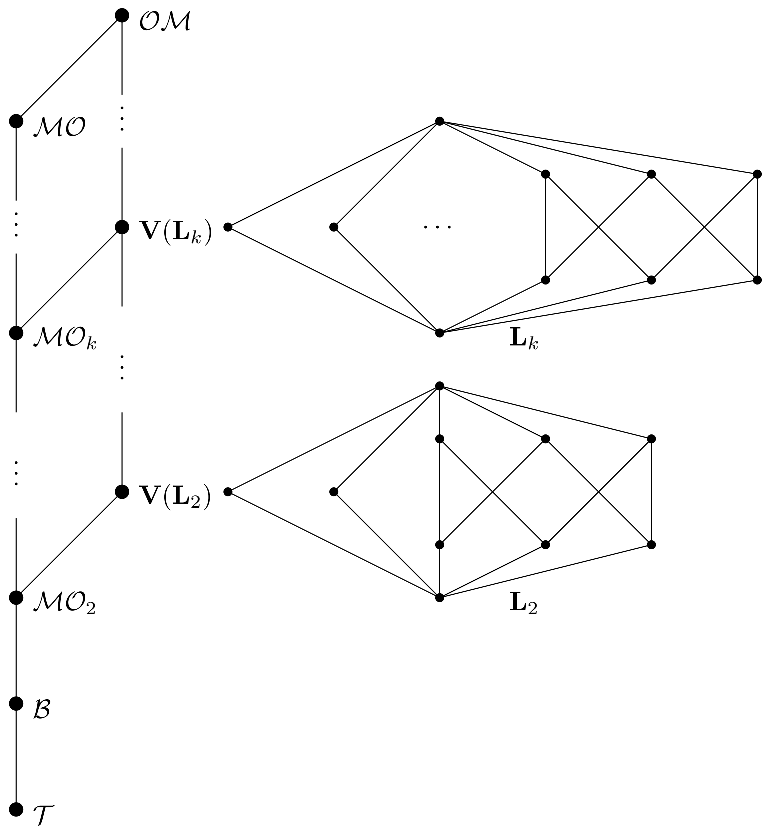

Figure 2. The subvarieties $\mathbf{V}(\mathbf{L}_k)$ and their generators $\mathbf{L}_k$

As in the previous section, the term function

$$t_{\tilde{\mathrm{w}}}(x_1,\dots,x_n)=\bigwedge_{\substack{i,j=1\\ i<j}}^{n} c^{w_{i,j}}(x_i,x_j)\wedge c'(x_1,\dots,x_n)=C_G(x_1,\dots,x_n)$$

corresponds to a labelled unoriented graph $G=G_{\tilde{\mathrm{w}}}$ on the vertex set $\{x_1,\dots,x_n\}$ with edges x_ix_j whenever $w_{i,j}=1$ for $i<j$. The next proposition gives a necessary and sufficient condition on the structure of the graph G for the interval $[0,C_G(x_1,\dots,x_n)]$ in $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ to be non-trivial (its proof is analogous to that of Proposition 3.2).

Proposition 4.1 ([9, Proposition 1]). Consider the term function $C_G(x_1,\dots,x_n)$: $(\mathbf{L}_k)^n\to\mathbf{L}_k$ given by

$$\bigwedge_{\substack{i,j=1,\\ i<j}}^{n} c^{w_{i,j}}(x_i,x_j)\wedge c'(x_1,\dots,x_n)$$

and its associated graph G. The following conditions are equivalent:

(a) The function $C_G(x_1,\dots,x_n)$ is not identically equal to zero.

(b) There exist elements $a_1,\dots,a_n\in\mathbf{L}_k$ such that

(i) $C_G(a_1, \dots, a_n) = 1$ and
(ii) the elements $a_1, \dots, a_n$ are not all from the same block of $\mathbf{L}_k$ and
(iii) a_i, a_j are elements of different blocks in $\mathbf{L}_k$ if and only if $x_i x_j$ is an edge of G.

(c) The graph $G_p := G$ consists of l isolated vertices ($0 \le l \le n-p$) and one connected component which is a complete p-partite graph ($2 \le p \le k$).

As in the case of the modular ortholattices, the interval $[0, C_G(x_1, \dots, x_n)]$ is isomorphic to the algebra of all functions from ${L_k}^n$ to L_k which are pointwise less than or equal to $C_G(x_1, \dots, x_n)$ and preserve all partial endomorphisms of $\mathbf{L}_k$. Such functions take values zero whenever the term C_G does.

4.2 Orbits of three types

Let

$$T_G := \{(a_1, \dots, a_n) \in (L_k)^n \mid C_G(a_1, \dots, a_n) = 1\}$$

be the set of all n-tuples $\underline{\mathbf{a}} = (a_1, \dots, a_n)$ from $(L_k)^n$ where the function C_G is non-zero, thus $C_G(a_1, \dots, a_n) = 1$. We call the coordinates $a_i \in \{0,1\}$ corresponding to isolated vertices of G *trivial*, otherwise they are *non-trivial*.

By Proposition 4.1, the non-trivial coordinates of $\underline{\mathbf{a}} \in T_G$ lie in exactly p of the k Boolean blocks $B_1, \dots, B_k$ of $\mathbf{L}_k$ corresponding to the blocks of the p-partite component of the graph $G = G_p$ ($2 \le p \le k$). Let us denote the cardinalities of these blocks by $k_1, \dots, k_p$, where $k_1 \ge k_2 \ge \dots \ge k_p \ge 1$ and $\sum_{i=1}^{p} k_i \le n$. Assume that $(B_1, \dots, B_p)(\underline{\mathbf{a}})$ is a sequence of the p Boolean blocks of $\mathbf{L}_k$ which contain the non-trivial coordinates of $\underline{\mathbf{a}}$ and the number of the non-trivial coordinates of $\underline{\mathbf{a}}$ from the block B_i is k_i, $i = 1, \dots, p$.

4.2.1 First step

In the first step of our procedure we consider the partition of T_G into orbits under the action of the automorphism group $\mathrm{Aut}(\mathbf{L}_k)$. We also count the number of orbits of $\mathrm{Aut}(\mathbf{L}_k)$ in T_G. In case the block $\mathbf{2}^3$ of $\mathbf{L}_k$ is a member of the sequence $(B_1, \dots, B_p)(\underline{\mathbf{a}})$, i.e. $B_i \cong \mathbf{2}^3$ for a unique $i \in \{1, \dots, p\}$, we distinguish types I and II of the n-tuples $\underline{\mathbf{a}} = (a_1, \dots, a_n) \in T_G$ (and the corresponding orbits $\mathrm{Orb}(\underline{\mathbf{a}})$). We say that of *type I* are the n-tuples $\underline{\mathbf{a}}$ (and orbits $\mathrm{Orb}(\underline{\mathbf{a}})$) such that the k_i coordinates of $\underline{\mathbf{a}}$ belonging to the block $B_i = \{0, b, b', c, c', d, d', 1\}$ are only from the set $\{b, b'\}$ of atoms of B_i. Of *type II* are the n-tuples $\underline{\mathbf{a}}$ (orbits $\mathrm{Orb}(\underline{\mathbf{a}})$) such that the k_i non-trivial coordinates of $\underline{\mathbf{a}}$ belonging to the block B_i contain distinct elements b, c where b, c are not an atom and its complement in B_i.

We now assume for simplicity that $i = 1$ and the first k_1 coordinates of $\underline{\mathbf{a}}$ are from $B_1 \cong \mathbf{2}^3$. The considered types of the n-tuples $\underline{\mathbf{a}}$ (orbits $\mathrm{Orb}(\underline{\mathbf{a}})$) in this case are specified as I.1 and II.1. We notice that there are automorphisms of $\mathbf{L}_k$ which permute any two of the three atoms b, c, d of the block B_1 and which permute the atoms a_j, a'_j of other blocks $B_2, \dots, B_p$. Hence to pick up a representative of an orbit $\mathrm{Orb}(\underline{\mathbf{a}})$ of type I.1 we obviously have 2^{k_1} choices for the k_1 coordinates from B_1, $2^{k_i - 1}$ choices for the k_i coordinates from B_i ($i \in \{2, \dots, p\}$) and $2^{n-(k_1 + \dots + k_p)}$ choices for the coordinates of $\underline{\mathbf{a}}$ from $\{0,1\}$. Altogether this gives

$$2^{k_1} \cdot 2^{k_2 - 1} \cdot \dots \cdot 2^{k_p - 1} \cdot 2^{n-(k_1 + \dots + k_p)} = 2^{n-p+1}$$

different orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of $\mathrm{Aut}(\mathbf{L}_k)$ of type I.1 on T_G. (We remark that we later showed in [9] that among all orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type I it is sufficient to consider only the orbits of type I.1.) For the number of orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type II.1 in T_G under the action of the automorphism group, the following lemma can be used. (We give its proof to make our presentation as much self-contained as possible.)

Lemma 4.2 ([9, Lemma 2])**.** There are (up to the automorphism action)

$$P(k) = 2^{k-1} + 6^{k-1}$$

choices for the $k := k_1$ coordinates of the n-tuples $\underline{\mathbf{a}} = (a_1, \dots, a_n)$ of type I.1 or II.1 in T_G to be taken from the block $B_1 \cong \mathbf{2}^3$.

Proof. Notice that if the pair of the first two coordinates of $\underline{\mathbf{a}}$ from B_1 is one of the four pairs $(b, c), (b, c'), (b', c), (b', c')$, where the distinct elements $b, c \notin \{0, 1\}$ are not an atom of B_1 and its complement, then arbitrary of the remaining $k-2$ coordinates from B_1 might be chosen freely from the six elements $\{b, b', c, c', d, d'\}$ of B_1. This gives $4 \cdot 6^{k-2}$ choices for the k coordinates from B_1 starting with such first two coordinates. In the remaining case the pair of the first two coordinates is one of $(b, b), (b, b'), (b', b), (b', b')$ for an atom b of B_1. This gives us 2 choices, namely b and b', for the first coordinate (up to the automorphism action) and, it gives us, recursively, $P(k-1)$ choices for the remaining $k-1$ coordinates. Therefore we obtain the recursive formula

$$P(k) = 4 \cdot 6^{k-2} + 2 \cdot P(k-1).$$

A standard method of solving such formulas gradually gives us

$$\begin{aligned}
\frac{P(k) - 2P(k-1)}{P(k-1) - 2P(k-2)} &= 6\\
P(k) - 8P(k-1) + 12P(k-2) &= 0\\
u^2 - 8u + 12 &= 0\\
u_1 = 2,\ u_2 &= 6,\\
P(k) = \alpha \cdot 2^k + \beta \cdot 6^k,\ \alpha, \beta &\in R.
\end{aligned}$$

We arrive at $P(2) = 8$ and $P(3) = 40$, and this gives us

$$\alpha = \frac{1}{2},\ \beta = \frac{1}{6}.$$

Consequently $P(k) = 2^{k-1} + 6^{k-1}$ as claimed. □

To continue, again there are automorphisms permuting the atoms a_j, a_j' of other blocks $B_2, \dots, B_p$. Hence to pick up a representative $\underline{\mathbf{a}}$ of an orbit $\mathrm{Orb}(\underline{\mathbf{a}})$ of one of the types I.1, II.1, we notice we have $2^{k_1-1} + 6^{k_1-1}$ choices for the coordinates from the block B_1, we have 2^{k_i-1} choices for the coordinates from B_i for $i = 2, \dots, p$ and we finally have $2^{n-(k_1+\dots+k_p)}$ choices for the coordinates of $\underline{\mathbf{a}}$ from $\{0, 1\}$. Altogether this gives

$$(2^{k_1-1} + 6^{k_1-1}) \cdot 2^{k_2-1} \cdot \ldots 2^{k_p-1} \cdot 2^{n-(k_1+\dots+k_p)} = 2^{n-p}(3^{k_1-1} + 1)$$

orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of $\mathrm{Aut}(\mathbf{L}_k)$ in T_G of types I.1 or II.1. Consequently the number of orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type II.1 is $2^{n-p}(3^{k_1-1}+1) - 2^{n-p+1} = 2^{n-p}(3^{k_1-1} - 1)$ while the total number of orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type II is

$$N(k_1, \dots, k_p) = 2^{n-p}[(\sum_{i=1}^{p} 3^{k_i-1}) - p].$$

To describe the third type of orbits, we assume now that for the n-tuple $\underline{\mathbf{a}} = (a_1, \dots, a_n) \in T_G$, the block $\mathbf{2}^3$ of $\mathbf{L}_k$ is not a member of the sequence $(B_1, \dots, B_p)(\underline{\mathbf{a}})$.

This means that $B_i \cong \mathbf{2}^2$ for all $i = 1, \dots, p$. We say such n-tuples $\underline{\mathbf{a}} \in T_G$ (the corresponding orbits $\mathrm{Orb}(\underline{\mathbf{a}})$) are of *type III.* Since there exist automorphisms permuting the atoms a_j, a'_j of any of the blocks $B_1, \dots, B_p$, there are

$$2^{k_1-1} \cdot 2^{k_2-1} \cdot \dots \cdot 2^{k_p-1} \cdot 2^{n-(k_1+\dots+k_p)} = 2^{n-p}$$

orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of $\mathrm{Aut}(\mathbf{L}_k)$ of type III.

4.2.2 Second step

In the second step we determine the structure of the algebra of $\mathrm{Aut}(\mathbf{L}_k)$-preserving functions from ${L_k}^n$ to L_k which are pointwise less than or equal to $C_G(x_1, \dots, x_n)$. As in the previous section, we extend the action of the automorphism group $\mathrm{Aut}(\mathbf{L}_k)$ on $\mathbf{L}_k$ pointwise to $(\mathbf{L}_k)^n$: this means that for $\underline{\mathbf{a}} = (a_1, \dots, a_n) \in (\mathbf{L}_k)^n$ and $\alpha \in \mathrm{Aut}(\mathbf{L}_k)$ we have $\underline{\mathbf{a}}^\alpha = (a_1^\alpha, \dots, a_n^\alpha) \in (\mathbf{L}_k)^n$ and a function $f : (\mathbf{L}_k)^n \to \mathbf{L}_k$ is α-preserving if for all $\underline{\mathbf{a}} \in (\mathbf{L}_k)^n$, $f(\underline{\mathbf{a}}^\alpha) = f(\underline{\mathbf{a}})^\alpha$. We notice that in order to define an $\mathrm{Aut}(\mathbf{L}_k)$-preserving function $f \leq C_G$, one cannot freely choose images from $\mathbf{L}_k$ for representatives of the orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ within T_G since in case $p < k$ there are automorphisms $\alpha \neq \beta$ in $\mathrm{Aut}(\mathbf{L}_k)$ such that $\underline{\mathbf{a}}^\alpha = \underline{\mathbf{a}}^\beta$ for any representative $\underline{\mathbf{a}}$ of orbit $\mathrm{Orb}\ \underline{\mathbf{a}}$. This restricts the choices for $f(\underline{\mathbf{a}})$ to those satisfying $f(\underline{\mathbf{a}})^\alpha = f(\underline{\mathbf{a}})^\beta$. Consequently, one can freely choose the image $f(\underline{\mathbf{a}})$ for each orbit-representative $\underline{\mathbf{a}}$ within $\bigcap_{\gamma \in \mathrm{Stab}\underline{\mathbf{a}}} \mathrm{fix}_{\mathbf{L}_k}(\gamma)$ and this forces the values of the other elements $\underline{\mathbf{a}}^\alpha$ in $\mathrm{Orb}(\underline{\mathbf{a}})$ to be $f(\underline{\mathbf{a}}^\alpha) = f(\underline{\mathbf{a}})^\alpha$. To compare it with the previous section, now only the orbits of types I and III within T_G contribute a factor $\mathbf{MO}_p$ to the algebra of $\mathrm{Aut}(\mathbf{L}_k)$-preserving functions $f \colon (\mathbf{L}_k)^n \to \mathbf{L}_k$. On the other hand, the orbits of type II within T_G contribute a factor $\mathbf{L}_p$. To see this, we notice that the stabiliser of n-tuples $\underline{\mathbf{a}} \in T_G$ of type II with associated sequences of blocks $(B_1, \dots, B_p)(\underline{\mathbf{a}})$ with $B_i \cong \mathbf{2}^3$ for a unique $i \in \{1, \dots, p\}$, consists exactly of those automorphisms in $\mathrm{Aut}(\mathbf{L}_k)$ that fix all elements of the blocks $B_1, \dots, B_p$ in $\mathbf{L}_k$ and permute only atoms in the remaining $k - p$ blocks $\mathbf{2}^2$ of $\mathbf{L}_k$.

4.2.3 Third step

In the third step of our procedure we determine which of the orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of types I, II and III can be "glued together" by the action of the partial endomorphisms of $\mathbf{L}_k$. We note at the beginning that for $\underline{\mathbf{a}} = (a_1, \dots, a_n)$, $\underline{\mathbf{b}} = (b_1, \dots, b_n)$ in T_G, the action $e(a_1) = b_1, \dots, e(a_n) = b_n$ by a partial endomorphism e of $\mathbf{L}_k$ is impossible if the domain $\mathrm{dom}(e)$ is a subalgebra of one of the blocks of $\mathbf{L}_k$. To see this, notice that the non-trivial coordinates of $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$ from T_G always lie in at least two different blocks of $\mathbf{L}_k$. We also note that for any partial endomorphism e of $\mathbf{L}_k$ with the action $e(a_1) = b_1, \dots, e(a_n) = b_n$ for some $(a_1, \dots, a_n), (b_1, \dots, b_n) \in T_G$, there is always a partial endomorphism e' of $\mathbf{L}_k$ with the "reverse action" $e'(b_1) = a_1, \dots, e'(b_n) = a_n$. This means that the binary relation E on the set of orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of types I, II and III given by $(\mathrm{Orb}(\underline{\mathbf{a}}), \mathrm{Orb}(\underline{\mathbf{b}})) \in E$ if there is a partial endomorphism e with the action $e(a_1) = b_1, \dots, e(a_n) = b_n$ is an equivalence.

We remark that in the first step we dealt with the n-tuples $\underline{\mathbf{a}} \in T_G$ of type I.1 which were determined by a sequence $(B_1, B_2 \dots, B_p)(\underline{\mathbf{a}})$ of blocks where $B_1 \cong \mathbf{2}^3$. Now we can see that for any orbit $\mathrm{Orb}(\underline{\mathbf{b}})$ of type I with an associated sequence $(B'_1, B'_2 \dots, B'_p)(\underline{\mathbf{b}})$ with $B'_i \cong \mathbf{2}^3$ for a unique $i \in \{2, \dots, p\}$ we have

$$(\mathrm{Orb}(\underline{\mathbf{a}}), \mathrm{Orb}(\underline{\mathbf{b}})) \in E$$

for the n-tuple $\underline{\mathbf{a}} = (a_1, \dots, a_n) \in T_G$ of type I.1 obtained from $\underline{\mathbf{b}}$ by mutually replacing in $\underline{\mathbf{b}}$ the coordinates from the first block $B'_1 \cong \mathbf{2}^2$ with the coordinates from the i-th block $B'_i \cong \mathbf{2}^3$. This is witnessed by the action $e(a_1) = b_1, \dots, e(a_n) = b_n$ where the partial endomorphism e of $\mathbf{L}_k$ is as follows:

(i) It mutually replaces the atoms of $\mathbf{L}_k$ that are the coordinates of $\underline{\mathbf{b}}$ coming from the block $B'_1 \cong \mathbf{2}^2$ with the elements b, b' of $\mathbf{L}_k$ that are the coordinates of $\underline{\mathbf{b}}$ coming from the block $B'_i \cong \mathbf{2}^3$.

(ii) It fixes all elements of the blocks $B'_2, \ldots, B'_{i-1}, B'_{i+1}, \ldots, B'_p$ of $\mathbf{L}_k$.

From this it follows that a function $f : {L_k}^n \to L_k$ preserving all partial endomorphisms of $\mathbf{L}_k$ can map the representative $\underline{\mathbf{a}}$ of $\mathrm{Orb}(\underline{\mathbf{a}})$ of type I.1 freely into the subalgebra $\mathbf{MO}_p$ of $\mathbf{L}_k$ while the image of the representative $\underline{\mathbf{b}}$ of $\mathrm{Orb}(\underline{\mathbf{b}})$ of a given type I (different from I.1) is determined by

$$f(b_1, \ldots, b_n) = f(e(a_1), \ldots, e(a_n)) = e(f(a_1, \ldots, a_n)).$$

That is why the factors $\mathbf{MO}_p$ contributed by the orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type I different from I.1 will not be considered.

In a similar way each orbit $\mathrm{Orb}(\underline{\mathbf{a}})$, such that $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$ is of type III and the associated sequence $(B_1, B_2 \ldots, B_p)(\underline{\mathbf{a}})$ has all blocks $B_i \cong \mathbf{2}^2,\ i = 1, \ldots, p$, can be "glued together" by the equivalence E with an orbit $\mathrm{Orb}(\underline{\mathbf{b}})$ such that $\underline{\mathbf{b}} = (b_1, \ldots, b_n)$ is of type I.1 and the associated sequence $(B'_1, B_2 \ldots, B_p)(\underline{\mathbf{b}})$ has $B'_1 = \mathbf{2}^3$. This is witnessed by the action $e(a_1) = b_1, \ldots, e(a_n) = b_n$ where the partial endomorphism e of $\mathbf{L}_k$ is as follows:

(i) It mutually replaces the atoms of $\mathbf{L}_k$ which are the coordinates of $\underline{\mathbf{a}}$ coming from the block $B_1 \cong \mathbf{2}^2$ with the elements b, b' of $\mathbf{L}_k$ which are the coordinates of $\underline{\mathbf{b}}$ coming from the block $B'_1 \cong \mathbf{2}^3$.

(ii) It fixes all elements of the blocks $B_2 \ldots, B_p$ of $\mathbf{L}_k$.

That is why the factors $\mathbf{MO}_p$ contributed by the orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type III will not be considered, too.

Finally we notice that each orbit $\mathrm{Orb}(\underline{\mathbf{a}})$ of type I.1 with a sequence $(B_1, B_2 \ldots, B_p)(\underline{\mathbf{a}})$ such that the k_1 coordinates of $\underline{\mathbf{a}}$ are coming from the set $\{b, b'\}$ of the block $B_1 \cong \mathbf{2}^3$ for some atom b of B_1 can be "glued together" by the equivalence E with an orbit $\mathrm{Orb}(\underline{\mathbf{b}})$ where $\underline{\mathbf{b}} = (b_1, \ldots, b_n)$ can be obtained from $\underline{\mathbf{a}} = (a_1, \ldots, a_n)$ by only mutually replacing the atom b with its complement b'. This is witnessed by the partial endomorphism of $\mathbf{L}_k$ mutually replacing b and b' and fixing all elements of the blocks $B_2 \ldots, B_p$ of $\mathbf{L}_k$. This will reduce the number of factors $\mathbf{MO}_p$ contributed by the orbits $\mathrm{Orb}(\underline{\mathbf{a}})$ of type I.1 to the half, that is, 2^{n-p}.

To summarise above, the structure of the interval $[0, C_G(x_1, \ldots, x_n)]$ associated to a p-partite graph $G = G_p(k_1, \ldots, k_p)$ with blocks of cardinalities $k_1, \ldots, k_p$ such that each $k_i \geq 1$ and $\sum_{i=1}^p k_i \leq n$ is

$$[0, C_G(x_1, \ldots, x_n)] \cong (\mathbf{MO}_p)^{2^{n-p}} \times (L_p)^{N(k_1, \ldots, k_p)}$$

where $N(k_1, \ldots, k_p) = 2^{n-p}[(\sum_{i=1}^p 3^{k_i - 1}) - p]$. We see that, compared to the previous section, this structure now depends on the sequence $(k_1, \ldots, k_p)$ of the cardinalities of the blocks of the p-partite graph G where one can assume $k_1 \geq \cdots \geq k_p$.

4.2.4 Counting the graphs

We calculate the number $\phi(n; k_1, \ldots, k_p)$ of the p-partite graphs $G = G_p(k_1, \ldots, k_p)$ on n-element vertex set with blocks of cardinalities $k_1, \ldots, k_p$ ($k_1 \geq \cdots \geq k_p \geq 1, \sum_{i=1}^p k_i \leq n$) and with $l = n - \sum_{i=1}^p k_i$ isolated vertices. We firstly have $\binom{n}{l}$ choices for the isolated vertices. And secondly, the number of partitions of a labelled $(n-l)$-element set $S = \{1, \ldots, n-l\}$ into exactly p blocks $S^1, \ldots, S^p$ of cardinalities $k_1, \ldots, k_p$, respectively is given by (cf. [1, 3.15])

(9) $S(n-l;k_1,\dots,k_p) = \frac{(n-l)!}{b_1!b_2!\dots b_{n-l}!(2!)^{b_2}\dots((n-l)!)^{b_{n-l}}}$

where for $i = 1,\dots,n-l$, b_i denotes the number of blocks of cardinality i among the blocks $S^1,\dots,S^p$. Consequently, we have

(10) $\phi(n;k_1,\dots,k_p) = \binom{n}{\sum_{i=1}^p k_i} S(\sum_{i=1}^p k_i;k_1,\dots,k_p)$.

4.3 The results

Theorem 4.3 ([9, Theorem 3])**.** For any $n \geq 3$, $k \geq 2$, the finitely generated free algebra $F_{\mathbf{V}(\mathbf{L}_k)}(n)$ is isomorphic to the product

$$\mathbf{2}^{2^n} \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\dots,k_p)\\ k_1\geq\dots\geq k_p\geq 1\\ \sum_{i=1}^p k_i\leq n}} [(\mathbf{MO}_p)^{2^{n-p}} \times (\mathbf{L}_p)^{N(k_1,\dots,k_p)}]^{\phi(n;k_1,\dots,k_p)}$$

where $\phi(n;k_1,\dots,k_p)$ is given by (9), (10) and $N(k_1,\dots,k_p) = 2^{n-p}[(\sum_{i=1}^p 3^{k_i-1}) - p]$.

It can be seen that

$$\sum_{\substack{(k_1,\dots,k_p)\\ k_1\geq\dots\geq k_p\geq 1\\ \sum_{i=1}^p k_i=n-l}} S(n-l;k_1,\dots,k_p) = S(n-l,p),$$

where $S(n-l,p)$ is the Stirling number of the second kind. We recall from the previous section that it gives the number of partitions of a labelled $(n-l)$-element set into exactly p parts and that it is given by the formula

$$S(n-l,p) = \frac{1}{p!}\sum_{s=1}^{p}(-1)^{p-s}\binom{p}{s}s^{n-l}$$

(cf. [1, 3.39]). This means that

$$\mathbf{2}^{2^n} \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\dots,k_p)\\ k_1\geq\dots\geq k_p\geq 1\\ \sum_{i=1}^p k_i\leq n}} [(\mathbf{MO}_p)^{2^{n-p}}]^{\phi(n;k_1,\dots,k_p)} = \mathbf{2}^{2^n} \times \prod_{p=2}^{k}(\mathbf{MO}_p)^{(2^{n-p}\phi'(n,p))}$$

with

$$\phi'(n,p) = \sum_{l=0}^{n-p}\binom{n}{l}S(n-l,p).$$

Notice that this is isomorphic to the free modular ortholattice $F_{\mathcal{MO}_k}(n)$ with n generators in the variety $\mathcal{MO}_k$ from the previous section. We arrive at our final result.

Corollary 4.4 ([9, Corollary 4])**.** For any $n \geq 3$, $k \geq 2$,

$$F_{\mathbf{V}(\mathbf{L}_k)}(n) \cong F_{\mathcal{MO}_k}(n) \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\dots,k_p)\\ k_1\geq\dots\geq k_p\geq 1\\ \sum_{i=1}^p k_i\leq n}} [(\mathbf{L}_p)^{N(k_1,\dots,k_p)}]^{\phi(n;k_1,\dots,k_p)}$$

where $F_{\mathcal{MO}_k}(n)$ is the free modular ortholattice in the variety $\mathcal{MO}_k$ with n generators, $\phi(n;k_1,\dots,k_p)$ is given by (9) and (10) and $N(k_1,\dots,k_p) = 2^{n-p}[(\sum_{i=1}^p 3^{k_i-1}) - p]$.

5 Finitely generated free algebras in $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$

In this section we present, based on our paper [10], a full description of the finitely generated free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ $(2 \leq k \leq n)$ with n generators in the varieties $\mathbf{V}(\mathbf{O}_k)$ of non-modular ortholattices. These varieties are generated by the orthomodular lattices $\mathbf{O}_k$ which are horizontal sums of k Boolean blocks $\mathbf{2}^3$ and form an another infinite chain "parallel" to the chains of varieties $\mathcal{MO}_k$ and $\mathbf{V}(\mathbf{L}_k)$ in the sense that each $\mathbf{V}(\mathbf{O}_k)$ contains the variety $\mathbf{V}(\mathbf{L}_k)$ (see Figure 3). As we shall see, this very ambitious step outside the varieties of modular ortholattices results in a very complex description.

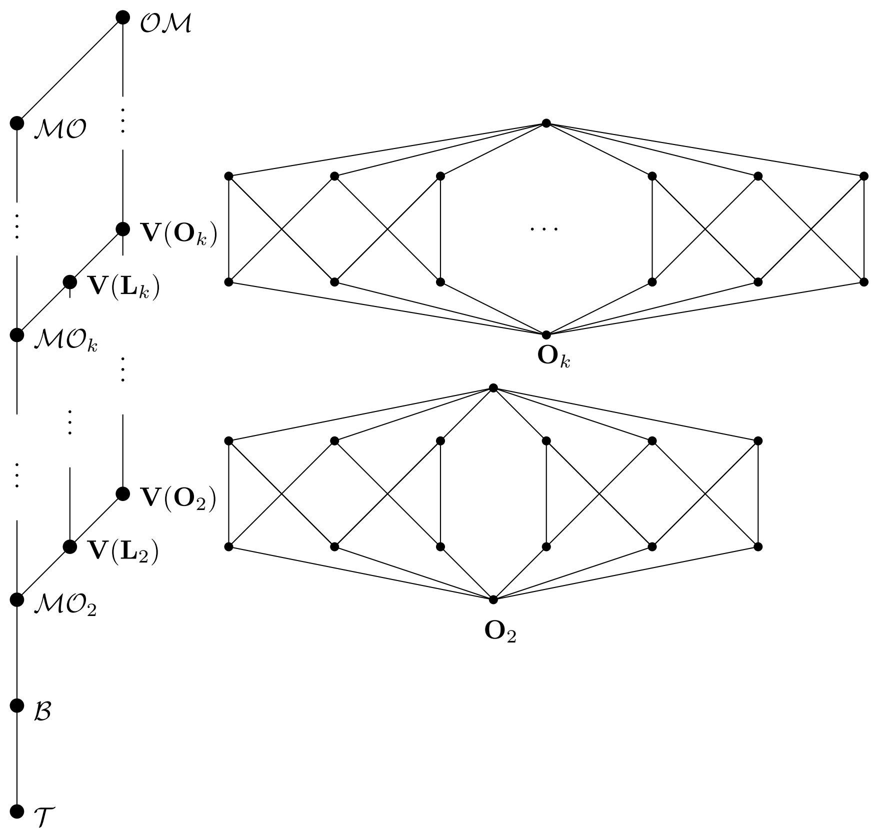

Figure 3. The subvarieties $\mathbf{V}(\mathbf{O}_k)$ and their generators $\mathbf{O}_k$

5.1 Similarities with the modular ortholattices once again

The free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ are certainly finite because the varieties $\mathbf{V}(\mathbf{O}_k)$ are locally finite (cf. [4, Chapter 1.3]). The varieties $\mathbf{V}(\mathbf{O}_k)$ are arithmetical with the same Pixley arithmeticity term as in the previous two sections. From the Arithmetic Strong Duality Theorem of Theory of natural dualities (cf. [4, Theorem 3.11]) we once again have a concrete description of the free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$.

Theorem 5.1 ([10, Theorem 3.1])**.** The free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ with n generators in the variety $\mathbf{V}(\mathbf{O}_k)$ $(2 \leq k \leq n)$ is isomorphic to the algebra of all functions from ${O_k}^n$ to O_k preserving the unary partial endomorphisms of $\mathbf{O}_k$.

Since the commutator $c(x_1, \dots, x_n)$ is a central element of $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$, the free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ can be decomposed into the product

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n) = [0, c(x_1, \dots, x_n)] \times [0, c'(x_1, \dots, x_n)].$$

Again the interval $[0, c(x_1, \dots, x_n)]$ is isomorphic to the n-generated free Boolean algebra $\mathbf{F}_{\mathcal{B}}(n) \cong \mathbf{2}^{2^n}$. The interval $[0, c'(x_1, \dots, x_n)]$ is decomposed by the binary commutators $c(x_i, x_j)$ $(i, j = 1, \dots, n,\ i < j)$ in the form

$$[0, c'(x_1, \dots, x_n)] \cong \prod_{\tilde{\mathrm{w}} \in \{0,1\}^N} [0, \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \dots, x_n)],$$

where the product is taken over all N-tuples $\tilde{\mathrm{w}} = (w_{1,2}, \dots, w_{n-1,n}) \in \{0,1\}^N$, $N = \binom{n}{2}$ and

$$c^{w_{i,j}}(x_i, x_j) = \begin{cases} c(x_i, x_j), & \text{if } w_{i,j} = 0, \\ c'(x_i, x_j), & \text{if } w_{i,j} = 1. \end{cases}$$

The term function $t_{\tilde{\mathrm{w}}}(x_1, \dots, x_n) = \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \dots, x_n) = C_G(x_1, \dots, x_n)$ can be associated with a labelled unoriented graph $G_{\tilde{\mathrm{w}}}$ on the vertex set $\{x_1, \dots, x_n\}$ with edges $x_i x_j$ whenever $w_{i,j} = 1$ for $i < j$. The next proposition, analogous to Propositions 3.2 and 4.1 (and with its proof similar to that of Proposition 3.2), gives a necessary and sufficient condition on the structure of the graph G for the interval $[0, C_G(x_1, \dots, x_n)]$ in $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ to be non-trivial.

Proposition 5.2 ([10, Proposition 3.2])**.** Let $C_G(x_1, \dots, x_n) \colon (\mathbf{MO}_k)^n \to \mathbf{MO}_k$ be a term function given by

$$\bigwedge_{\substack{i,j=1, \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \dots, x_n)$$

which is associated to a labelled unoriented graph $G = G_{\tilde{\mathrm{w}}}$. The following conditions are equivalent:

(a) The function $C_G(x_1, \dots, x_n) : O_k^n \to O_k$ is not identically equal to zero.

(b) There exist elements $a_1, \dots, a_n \in \mathbf{O}_k$ such that

   (i) $C_G(a_1, \dots, a_n) = 1$ and

   (ii) the elements $a_1, \dots, a_n$ are not all from the same block of $\mathbf{O}_k$ and

   (iii) a_i, a_j are elements of different blocks in $\mathbf{O}_k$ if and only if $x_i x_j$ is an edge of G.

(c) The graph $G_p := G$ consists of l isolated vertices $(0 \leq l \leq n-p)$ and one connected component which is a complete p-partite graph $(2 \leq p \leq k)$.

Once again, let

$$T_G := \{(a_1, \dots, a_n) \in (O_k)^n \mid C_G(a_1, \dots, a_n) = 1\}$$

be the set of all $\mathbf{a} = (a_1, \dots, a_n)$ from $(O_k)^n$ where C_G is non-zero, hence it takes value 1. As in the previous section, we call the coordinates $a_i \in \{0,1\}$ *trivial* if they correspond to isolated vertices of G. Then Proposition 5.2 means that the non-trivial coordinates of $\mathbf{a} \in T_G$ lie in exactly p of the k Boolean blocks $B_1, \dots, B_k$ of $\mathbf{O}_k$ associated to blocks of the p-partite component of $G = G_p$, $2 \le p \le k$. Henceforth we assume that the blocks of the p-partite component of the graph G have cardinalities $k_1, \dots, k_p$, where $k_1 \ge k_2 \ge \cdots \ge k_p \ge 1$ and $\sum_{i=1}^{p} k_i \le n$. Accordingly, we sometimes use the notation $G_p(k_1, \dots, k_p)$ for the graph $G = G_p$. We call the elements $\mathbf{a} \in T_G$ *standard* if their $k_1, \dots, k_p$ non-trivial coordinates are taken respectively from the first p blocks $B_1, \dots, B_p$ of $\mathbf{O}_k$.

5.2 Types of orbits

5.2.1 First step

The orbits of the automorphism group $\mathrm{Aut}(\mathbf{O}_k)$ on T_G will be counted. We denote the atoms and the coatoms of each block B_i b_i, c_i, d_i and b_i', c_i', d_i', respectively. We notice that any automorphism f_i of the Boolean algebra $\mathbf{B}_i$ fixes $\{0, 1\}$ and is determined by a permutation of the atoms of $\mathbf{B}_i$ for $i \in \{1, 2, \dots, k\}$. A description of the automorphisms of $\mathbf{O}_k$ is given in the following lemma. It guarantees that it suffices to focus on the standard elements $\mathbf{a} \in T_G$ when counting the orbits of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G.

Lemma 5.3 ([10, Lemma 3.3])**.**

(i) For each automorphism $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$ there is a permutation ν on the index set $I_k := \{1, 2, \dots, k\}$ and on the set of automorphisms $\{f_i : B_i \to B_i \mid i \in I_k\}$ of the Boolean blocks $\mathbf{B}_i$ such that

(11) $\alpha(x_i) = f_{\nu(i)}(x_{\nu(i)})$ for every $i \in I_k$ and $x_i \in B_i$.

(ii) Conversely, for each permutation ν on the index set $I_k := \{1, 2, \dots, k\}$ and on the set $\{f_i : B_i \to B_i \mid i \in I_k\}$ of automorphisms of the Boolean blocks $\mathbf{B}_i$, the unary map $\alpha : O_k \to O_k$ defined by (11) is an automorphism of $\mathbf{O}_k$.

(iii) For each $\mathbf{b} \in T_G$ such that the non-trivial coordinates of $\mathbf{b}$ lie in p Boolean blocks $B_{i_1}, \dots, B_{i_p}$ of $\mathbf{O}_k$, with $2 \le p \le k$ and $\{i_1, \dots, i_p\} \subseteq I_k$, there exists a standard element $\mathbf{a} \in T_G$ and $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$ such that

(12) $\alpha(a_1) = b_1, \dots, \alpha(a_n) = b_n$,

whence $\mathbf{b}$ belongs to the orbit $\mathrm{Orb}(\mathbf{a})$ of the automorphism group $\mathrm{Aut}(\mathbf{O}_k)$ on T_G.

Proof. While (i) and (ii) are easy, to show (iii) we define a permutation ν on the set I_k such that $\nu(j) = i_j$ for $j = 1, \dots, p$ and ν maps the set $I_k \setminus \{1, \dots, p\}$ arbitrarily onto the set $I_k \setminus \{i_1, \dots, i_p\}$. For the automorphisms $\{f_i : B_i \to B_i \mid i \in I_k\}$ of the Boolean blocks $\mathbf{B}_i$ we take the identity maps. By this we define an automorphism $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$ such that $\alpha \restriction \mathbf{B}_j : \mathbf{B}_j \to \mathbf{B}_{i_j}$ is an isomorphism for every $j \in \{1, \dots, p\}$. Now let α^{-1} denote the inverse of α and $a_i := \alpha^{-1}(b_i)$ for $i = 1, \dots, n$. Then obviously $\mathbf{a} = (a_1, \dots, a_n) \in T_G$ by Proposition 5.2. Also $\mathbf{a}$ is standard and (12) holds. Thus $\mathbf{b}$ belongs to the orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G. □

For standard element $\mathbf{a} \in T_G$ we call the corresponding orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G *standard*, too. Lemma 5.3(iii) shows that when counting the orbits $\mathrm{Orb}(\mathbf{a})$ of the automorphism group $\mathrm{Aut}(\mathbf{O}_k)$ on T_G for a p-partite graph $G = G_p(k_1, \dots, k_p)$ with $k_1 \ge k_2 \ge \cdots \ge k_p$ and $\sum_{i=1}^{p} k_i \le n$, one can count only standard orbits $\mathrm{Orb}(\mathbf{a})$. This means that one can assume that for $i = 1, \dots, p$ the k_i non-trivial coordinates of $\mathbf{a}$ are taken from the Boolean block B_i. We accordingly use the notation $\mathbf{a}(k_1, \dots, k_p)$ for $\mathbf{a}$.

Lemma 5.4. (i) There are (up to the automorphism action)

$$P(k_i) = 2^{k_i - 1} + 6^{k_i - 1}$$

choices for the non-trivial k_i coordinates of $\mathbf{a}(k_1,\ldots,k_p) = (a_1,\ldots,a_n)$ to be selected from the block B_i, $i = 1,\ldots,p$.

(ii) There are

$$N(n,p;k_1,\ldots,k_p) = 2^{n-p} \cdot \prod_{i=1}^{p}(3^{k_i-1}+1)$$

orbits $\mathrm{Orb}(\mathbf{a}(k_1,\ldots,k_p))$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G.

Proof. The part (i) comes from Lemma 4.2.

(ii) We use (i) and the fact that there are $2^{n-\sum_{i=1}^{p} k_i}$ choices for the trivial coordinates of $\mathbf{a}$ to be selected from the set $\{0,1\}$. This implies that the number of orbits $\mathrm{Orb}(\mathbf{a}(k_1,\ldots,k_p))$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G is

$$\begin{aligned} N(n,p;k_1,\ldots,k_p) &= (\prod_{i=1}^{p}(2^{k_i-1}+6^{k_i-1}))\cdot 2^{n-\sum_{i=1}^{p} k_i} \\ &= 2^{(\sum_{i=1}^{p} k_i)-p}\cdot(\prod_{i=1}^{p}(3^{k_i-1}+1))\cdot 2^{n-\sum_{i=1}^{p} k_i} \\ &= 2^{n-p}\cdot\prod_{i=1}^{p}(3^{k_i-1}+1). \end{aligned}$$

□

The k_i non-trivial coordinates of $\mathbf{a}$ belonging to the block B_i are said to be of *type I* if they all are from the set $\{b, b'\}$ for an atom b and its orthocomplement b' in $\mathbf{B}_i$. They are called of *type II* otherwise. It is easy to see that among $P(k_i) = 2^{k_i-1}+6^{k_i-1}$ choices for the non-trivial k_i coordinates of $\mathbf{a}$ from the block B_i there exist exactly 2^{k_i} choices for the k_i coordinates of type I while the rest, $2^{k_i-1}+6^{k_i-1}-2^{k_i} = 6^{k_i-1}-2^{k_i-1}$, are the choices for the k_i coordinates of type II. Hence we can express the product $N(n,p;k_1,\ldots,k_p)$ in Lemma 5.4 as

$$N(n,p;k_1,\ldots,k_p) = (\textstyle\prod_{i=1}^{p}[(6^{k_i-1}-2^{k_i-1})+2^{k_i}])\cdot 2^{n-\sum_{i=1}^{p} k_i}. \tag{13}$$

From this it follows how the coordinates of types I and II contribute to the resulting number.

We call a standard orbit $\mathrm{Orb}(\mathbf{a})$ of the automorphism group $\mathrm{Aut}(\mathbf{O}_k)$ on T_G of *type* $\{i_1,\ldots,i_s\}$ if the non-trivial coordinates of $\mathbf{a}$ of type II are exactly from the blocks $B_{i_1},\ldots,B_{i_s}$ for some subset $\{i_1,\ldots,i_s\} \subseteq \{1,\ldots,p\}$.

5.2.2 Second step

In the second step we determine the structure of the interval of $\mathrm{Aut}(\mathbf{O}_k)$-preserving functions from ${O_k}^n$ to O_k which are pointwise less than or equal to the term function $C_G(x_1,\ldots,x_n)$. Again, we extend the action of $\mathrm{Aut}(\mathbf{O}_k)$ on $\mathbf{O}_k$ pointwise to $(\mathbf{O}_k)^n$: for $\mathbf{a} = (a_1,\ldots,a_n) \in (\mathbf{O}_k)^n$ and $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$, $\alpha(\mathbf{a}) := (\alpha(a_1),\ldots,\alpha(a_n)) \in (\mathbf{O}_k)^n$. A function $f\colon (\mathbf{L}_k)^n \to \mathbf{L}_k$ is α-preserving if for all $\mathbf{a} \in (\mathbf{L}_k)^n$, $f(\alpha(\mathbf{a})) = \alpha(f(\mathbf{a}))$. We again notice that to define an $\mathrm{Aut}(\mathbf{L}_k)$-preserving function $f \leq C_G$, we cannot arbitrarily choose images from $\mathbf{O}_k$ for representatives of the orbits $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G. The reason is that for $p < k$ there exist automorphisms $\alpha \neq \beta$ in $\mathrm{Aut}(\mathbf{O}_k)$ such that for any representative $\mathbf{a}$ of orbit $\mathrm{Orb}\ \mathbf{a}$, $\alpha(\mathbf{a}) = \beta(\mathbf{a})$. This restricts the choices for $f(\mathbf{a})$ to only those that satisfy $f(\alpha(\mathbf{a})) = f(\beta(\mathbf{a}))$. Therefore we can arbitrarily choose the image $f(\mathbf{a})$ for each orbit-representative $\mathbf{a}$ within $\bigcap_{\gamma\in\mathrm{Stab}\,\mathbf{a}} \mathrm{fix}_{\mathbf{L}_k}(\gamma)$ while the values of the other elements $\alpha(\mathbf{a})$ in $\mathrm{Orb}(\mathbf{a})$ are determined by

$$f(\alpha(\mathbf{a})) = \alpha(f(\mathbf{a})).$$

It follows that the algebra $\mathbf{A}_G$ of the $\mathrm{Aut}(\mathbf{O}_k)$-preserving functions from ${O_k}^n$ to O_k that are pointwise less than or equal to C_G is isomorphic to the product of the subalgebras $\bigcap_{\gamma\in\mathrm{Stab}\,\mathbf{a}} \mathrm{fix}_{\mathbf{L}_k}(\gamma)$ of $\mathbf{O}_k$ taken over all standard orbits $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G.

We denote by $\mathbf{L}_{(s,p-s)}$ $(s \in \{0,1,\dots,p\})$ a subalgebra of $\mathbf{O}_k$ which consists of s Boolean blocks $\mathbf{2}^3$ and $p-s$ Boolean blocks $\mathbf{2}^2$.

Proposition 5.5 ([10, Proposition 3.5])**.** Let $G = G_p(k_1,\dots,k_p)$ be a p-partite graph with blocks of cardinalities $k_1,\dots,k_p$ such that $k_1 \geq \dots \geq k_p \geq 1$ and $\sum_{i=1}^{p} k_i \leq n$. The algebra $\mathbf{A}_G$ of the $\mathrm{Aut}(\mathbf{O}_k)$-preserving functions from ${O_k}^n$ to O_k which are pointwise less than or equal to $C_G(x_1,\dots,x_n)$ is

$$\mathbf{A}_G \cong (\mathbf{MO}_p)^{2^n} \times \prod_{s=1}^{p} (\mathbf{L}_{(s,p-s)})^{N_A(n,p,s;k_1,\dots,k_p)},$$

where

$$N_A(n,p,s;k_1,\dots,k_p) = 2^n \cdot \sum_{\{i_1,\dots,i_s\}\subseteq\{1,\dots,p\}} \prod_{r=1}^{s} \frac{6^{k_{i_r}-1} - 2^{k_{i_r}-1}}{2^{k_{i_r}}}.$$

Proof. We already know that every standard orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G contributes a factor $\bigcap_{\gamma\in\mathrm{Stab}\mathbf{a}} \mathrm{fix}_{\mathbf{L}_k}(\gamma)$ to the algebra $\mathbf{A}_G$. Now for every type of the orbits $\mathrm{Orb}(\mathbf{a})$ we investigate the structure of $\bigcap_{\gamma\in\mathrm{Stab}\mathbf{a}} \mathrm{fix}_{\mathbf{L}_k}(\gamma)$ and we state the number of orbits $\mathrm{Orb}(\mathbf{a})$ of a given type.

We firstly consider a standard orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G of type $I_p = \{1,\dots,p\}$, i.e. all the non-trivial coordinates of $\mathbf{a}$ are of type II. Then Stab $\mathbf{a}$ consists of the automorphisms $\gamma \in \mathrm{Aut}(\mathbf{O}_k)$ that fix all elements of the blocks $B_1,\dots,B_p$ in $\mathbf{O}_k$ and permute the atoms (together with their complementary coatoms) in the remaining $k-p$ blocks of $\mathbf{O}_k$. Thus

$$\bigcap_{\gamma\in\mathrm{Stab}\mathbf{a}} \mathrm{fix}_{\mathbf{L}_k}(\gamma) \cong \mathbf{L}_{(p,0)} = \mathbf{O}_p,$$

whence the orbit $\mathrm{Orb}(\mathbf{a})$ contributes a factor $\mathbf{O}_p$ to $\mathbf{A}_G$. Notice that the number of standard orbits $\mathrm{Orb}(\mathbf{a}(k_1,\dots,k_p))$ of type I_p is

$$N(n,p;k_1,\dots,k_p;I_p) = (\prod_{i=1}^{p}(6^{k_i-1} - 2^{k_i-1})) \cdot 2^{n-\sum_{i=1}^{p} k_i} = 2^n \cdot \prod_{i=1}^{p} \frac{6^{k_i-1} - 2^{k_i-1}}{2^{k_i}}.$$

Hence this is the number of $\mathbf{O}_p$ contributed by all standard orbits $\mathrm{Orb}(\mathbf{a}(k_1,\dots,k_p))$ of type $\{1,\dots,p\}$. We remark that one can get this number from $N(n,p;k_1,\dots,k_p)$ in (13) by removing in each of the first p factors the term 2^{k_i} expressing the number of the non-trivial coordinates of type I.

We secondly consider a standard orbit $\mathrm{Orb}(\mathbf{a})$ of type $S := \{i_1,\dots,i_s\}$ where we have $\varnothing \neq S \subsetneq I_p$, i.e. $1 \leq s \leq p-1$. For every $j \in I_p \setminus S$, the k_j non-trivial coordinates of $\mathbf{a}$ taken from the block B_j are from the subset $\{b_j, b'_j\} \subset B_j$ where $b_j \in B_j$ is an atom. Now Stab $\mathbf{a}$ consists of the automorphisms $\gamma \in \mathrm{Aut}(\mathbf{O}_k)$ that fix the elements of the $p-s$ subalgebras $\{0, b_j, b'_j, 1\}$ of B_j for $j \in I_p \setminus S$, fix the elements of the s blocks B_j for $j \in S$

and permute the atoms (together with their complements) in the remaining $k-p$ blocks of $\mathbf{O}_k$. Thus

$$\bigcap_{\gamma \in \mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{L}_k}(\gamma) \cong \mathbf{L}_{(s,p-s)}.$$

Hence each such orbit $\mathrm{Orb}(\mathbf{a})$ contributes a factor $\mathbf{L}_{(s,p-s)}$ to $\mathbf{A}_G$. Notice that the number of orbits $\mathrm{Orb}(\mathbf{a}(k_1,\dots,k_p))$ of type $S=\{i_1,\dots,i_s\}$ is

(14) $N(n,p,s;k_1,\dots,k_p;\{i_1,\dots,i_s\}) = 2^n \cdot \prod_{r=1}^{s} \frac{6^{k_{i_r}-1}-2^{k_{i_r}-1}}{2^{k_{i_r}}}.$

This is the number of factors $\mathbf{L}_{(s,p-s)}$ contributed to $\mathbf{A}_G$ by all standard orbits $\mathrm{Orb}(\mathbf{a}(k_1,\dots,k_p))$ of type $S=\{i_1,\dots,i_s\}$ where $\varnothing \neq S \subsetneq I_p$. We remark that this number can be obtained from $N(n,p;k_1,\dots,k_p)$ in (14) by removing

(i) in the factors corresponding to $i \in S$ the term 2^{k_i} which expresses the number of the non-trivial coordinates of type I and

(ii) in the factors corresponding to $i \in I_p \setminus S$ the term $6^{k_i-1}-2^{k_i-1}$ expressing the number of the non-trivial coordinates of type II.

Finally, we consider the orbits $\mathrm{Orb}(\mathbf{a})$ of type $\varnothing$, i.e. those that all the non-trivial coordinates of $\mathbf{a}$ are of type I. Every such orbit contributes a factor $\mathbf{L}_{(0,p)} \cong \mathbf{MO}_p$ to $\mathbf{A}_G$ and the number of these $\mathbf{MO}_p$ is given by

$$N(n,p;k_1,\dots,k_p;\varnothing) = 2^{\sum_{i=1}^{p} k_i} \cdot 2^{n-\sum_{i=1}^{p} k_i} = 2^n.$$

□

5.2.3 Third step

We investigate which of the different standard orbits $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G can be "glued together" by the action of the unary partial endomorphisms e of $\mathbf{O}_k$. By this we mean that

(15) $e(a_1)=b_1,\dots,e(a_n)=b_n$

for representatives $\mathbf{a}=(a_1,\dots,a_n)$, $\mathbf{b}=(b_1,\dots,b_n)$ of different standard orbits $\mathrm{Orb}(\mathbf{a})$ and $\mathrm{Orb}(\mathbf{b})$ and for a unary partial endomorphism e of $\mathbf{O}_k$. As the functions $f: O_k^n \to O_k$ which we consider preserve all unary partial endomorphisms of $\mathbf{O}_k$, we must guarantee the condition

(16) $(e(a_1)=b_1,\dots,e(a_n)=b_n) \Longrightarrow f(b_1,\dots,b_n)=e(f(a_1,\dots,a_n))$

for every unary partial endomorphism e of $\mathbf{O}_k$.

Definition 5.6 ([10, Definition 3.6])**.** A unary partial endomorphism e of $\mathbf{O}_k$ is called

(i) *straight*, if e maps all elements of $\mathrm{dom}(e) \cap B_i$ into B_i for every $i \in \{1,\dots,k\}$;

(ii) *proper*, if the domain $\mathrm{dom}(e)$ consists of the elements from at least two different blocks of $\mathbf{O}_k$;

(iii) $0,1$-*separating*, if for any $x \in O_k$

$$e(x)=0 \text{ implies } x=0 \text{ and } e(x)=1 \text{ implies } x=1.$$

The following lemma describes the partial endomorphisms of $\mathbf{O}_k$.

Lemma 5.7 ([10, Lemma 3.7])**.** Every unary partial endomorphism e of $\mathbf{O}_k$ can be expressed as

$$e = \alpha \circ e'$$

for some automorphism $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$ and a straight partial endomorphism e' on $\mathbf{O}_k$ with domain $\mathrm{dom}(e') = \mathrm{dom}(e)$.

Proof. In case the partial endomorphism e of $\mathbf{O}_k$ is straight, the claim is satisfied for $e' = e$ and α being the identity map on $\mathbf{O}_k$.

If the partial endomorphism e is not straight, there exist elements $x_i \in B_i \setminus \{0,1\}$ and $y_j \in B_j \setminus \{0,1\}$ $(i, j \in \{1,\dots,k\},\ i \neq j)$ such that $e(x_i) = y_j$, whence $e(x_i') = y_j'$. Suppose by contradiction that e maps an element z_l of a block $B_l \neq B_i$ into the block B_j. It follows that the element $e(z_l)$ must be comparable to one of the elements y_j, y_j'. One can assume, without loss of generality, that $e(z_l) \neq 0$. Since $z_l \wedge x_i = z_l \wedge x_i' = 0$, we get $e(z_l) \wedge y_j = e(z_l) \wedge y_j' = e(0) = 0$, a contradiction.

Hence every partial endomorphism e maps different blocks of $\mathbf{O}_k$ to mutually different blocks of $\mathbf{O}_k$. Thus there is a permutation ν of the index set $I_k = \{1, 2, \dots, k\}$ such that e maps each $x_i \in \mathrm{dom}(e) \cap B_i$ into $B_{\nu(i)}$. Assume that $\alpha \in \mathrm{Aut}(\mathbf{O}_k)$ is determined by this permutation and by the identity maps $\{f_i : B_i \to B_i \mid i \in I_k\}$ (see Lemma 5.3). We define a partial endomorphism e' on $\mathbf{O}_k$ with domain $\mathrm{dom}(e') = \mathrm{dom}(e)$ by $e' := \alpha^{-1} \circ e$. Now it is easy to see that e' is straight and $\alpha \circ e' = e$. □

By Lemma 5.7, a function $f : O_k^n \to O_k$ preserves all unary partial endomorphisms e of $\mathbf{O}_k$ if it preserves the automorphisms of $\mathbf{O}_k$ and the straight partial endomorphisms e' of $\mathbf{O}_k$. This means that it is sufficient to consider the condition (16) only for straight partial endomorphisms e of $\mathbf{O}_k$.

We remark that (15) is possible only if the partial endomorphism e is proper since the non-trivial coordinates of $\mathbf{a}, \mathbf{b} \in T_G$ always lie in at least two different blocks of $\mathbf{O}_k$. The next lemma shows that proper partial endomorphisms e are necessarily $0,1$-separating.

Lemma 5.8 ([10, Lemma 3.8])**.** Every proper partial endomorphisms of $\mathbf{O}_k$ is $0,1$-separating.

Proof. Assume that $x_i \in \mathrm{dom}(e) \cap B_i$ and $y_j \in \mathrm{dom}(e) \cap B_j$ for different blocks B_i, B_j of $\mathbf{O}_k$. It follows $\{x_i, y_j\} \cap \{0,1\} = \varnothing$. Suppose by contradiction that, without loss of generality, $e(x_i) = 0$. Since $y_j \vee x_i = 1 = y_j' \vee x_i$, we get

$$e(y_j) = e(y_j) \vee e(x_i) = e(y_j \vee x_i) = e(1) = 1.$$

Analogously,

$$e(y_j') = e(y_j') \vee e(x_i) = e(y_j' \vee x_i) = e(1) = 1.$$

From this it follows $e(0) = e(y_j \wedge y_j') = e(y_j) \wedge e(y_j') = 1$, a contradiction. □

We conclude that it is sufficient to consider the condition (16) only for straight and proper (i.e. $0,1$-separating) partial endomorphisms e of $\mathbf{O}_k$.

We call a unary partial endomorphism u_j of $\mathbf{O}_k$ *primitive* if its graph is

$$(u_j)^{\square} = \{(0,0), (b_j, b_j'), (b_j', b_j), (1,1)\}$$

where b_j is an atom of the block B_j, $j \in I_k$. We call it $\{j_1, \dots, j_s\}$*-primitive* if $u \restriction B_{j_r}$ is primitive for $r = 1, \dots, s$ and $u(x) = x$ for all $x \in \mathrm{dom}(u) \setminus (B_{j_1} \cup \dots \cup B_{j_s})$.

Our final lemma is now easy to see.

Lemma 5.9 ([10, Lemma 3.9])**.** Every straight and proper (i.e. $0,1$-separating) partial endomorphism e of $\mathbf{O}_k$ which is not an automorphism is of the form

$$e = \alpha \circ u$$

for some $\{j_1, \ldots, j_s\}$-primitive partial endomorphism u of $\mathbf{O}_k$ and a straight automorphism α of $\mathbf{O}_k$.

Consequently it is sufficient to consider the condition (16) only for $\{j_1, \ldots, j_s\}$-primitive partial endomorphisms e of $\mathbf{O}_k$.

We finally consider a standard orbit $\mathrm{Orb}(\mathbf{a})$ of $\mathrm{Aut}(\mathbf{O}_k)$ on T_G of a type $S \subsetneq I_p$, where $s := |S| \geq 0$. This means that for each $j \in I_p \setminus S$, the k_j non-trivial coordinates of $\mathbf{a}$ taken from the block B_j are from the subset $\{b_j, b_j'\} \subset B_j$ where b_j is an atom of B_j. Let $I_p \setminus S = \{j_1, \ldots, j_{p-s}\}$. Assume that e is an $\{j_1, \ldots, j_{p-s}\}$-primitive partial endomorphism of $\mathbf{O}_k$ and that (15) holds for e and $\mathbf{b} \in T_G \setminus \mathrm{Orb}(\mathbf{a})$. It follows that the orbit $\mathrm{Orb}(\mathbf{b})$ is also of type S. In case the image $f(a_1, \ldots, a_n)$ of $\mathbf{a}$ in f is chosen from

$$\bigcap_{\gamma \in \mathrm{Stab}\ \mathbf{a}} \mathrm{fix}_{\mathbf{L}_k}(\gamma) \cong \mathbf{L}_{(s,p-s)},$$

by (16) the image $f(b_1, \ldots, b_n)$ of $\mathbf{b}$ in f is determined by $f(b_1, \ldots, b_n) = e(f(a_1, \ldots, a_n))$. Hence only one of the orbits $\mathrm{Orb}(\mathbf{a})$ and $\mathrm{Orb}(\mathbf{b})$ contributes a factor $\mathbf{L}_{(s,p-s)}$ to the interval $[0, C_G(x_1, \ldots, x_n)]$ of functions $f : (\mathbf{L}_k)^n \to \mathbf{L}_k$ which are pointwise less than or equal to $C_G(x_1, \ldots, x_n)$ and preserve all unary partial endomorphisms of $\mathbf{O}_k$. It follows that for $0 \leq s \leq p-1$, the number of factors $\mathbf{L}_{(s,p-s)}$ in the structure of the interval $[0, C_G(x_1, \ldots, x_n)]$ can be obtained by dividing every $N(n,p,s;k_1,\ldots,k_p;S)$ in (14) by two for each $j \in I_p \setminus S = \{j_1, \ldots, j_{p-s}\}$, hence by dividing the exponents $N_A(n,p,s;k_1,\ldots,k_p)$ in Proposition 5.5 by 2^{p-s}. The number of factors $\mathbf{MO}_p$ in the interval $[0, C_G(x_1, \ldots, x_n)]$ can be obtained by dividing $N(n,p;k_1,\ldots,k_p;\varnothing) = 2^n$ by 2^p, hence it is equal to 2^{n-p}. We denote for every $1 \leq s \leq p$,

$$N(n,p;s;k_1,\ldots,k_p) := \frac{N_A(n,p,s;k_1,\ldots,k_p)}{2^{p-s}}$$

and we arrive to our final proposition.

Proposition 5.10 ([10, Proposition 3.10])**.** For the interval $[0, C_G(x_1, \ldots, x_n)]$ associated to a p-partite graph $G = G_p(k_1, \ldots, k_p)$ with blocks of cardinalities $k_1, \ldots, k_p$ such that $k_1 \geq \cdots \geq k_p \geq 1$ and $\sum_{i=1}^p k_i \leq n$ we have

$$[0, C_G(x_1, \ldots, x_n)] \cong (\mathbf{MO}_p)^{2^{n-p}} \times \prod_{s=1}^{p} (\mathbf{L}_{(s,p-s)})^{N(n,p,s;k_1,\ldots,k_p)}$$

where

$$N(n,p,s;k_1,\ldots,k_p) = 2^{n-p+s} \cdot \sum_{\{i_1,\ldots,i_s\} \subseteq \{1,\ldots,p\}} \prod_{r=1}^{s} \frac{6^{k_{i_r}-1} - 2^{k_{i_r}-1}}{2^{k_{i_r}}}.$$

5.3 The results and their illustration

The calculation of the number of the p-partite graphs $G = G_p(k_1, \ldots, k_p)$ on an n-element vertex set with blocks of cardinalities $k_1, \ldots, k_p$ ($k_1 \geq \cdots \geq k_p \geq 1, \sum_{i=1}^p k_i \leq n$) and with $l = n - \sum_{i=1}^p k_i$ isolated vertices was given by (9) and (10) in the previous section.

Also similarly to the previous sections, $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n) \cong \mathbf{F}_{\mathbf{V}(\mathbf{O}_n)}(n)$ if $n < k$. Thus it suffices to consider $k \leq n$ in the description of the finitely generated free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$. Our final note is that in the case $n = k = 2$ we have $\phi(2; 1, 1) = 1$ and so we have the known description $\mathbf{F}_{\mathbf{V}(\mathbf{O}_2)}(2) \cong \mathbf{F}_{\mathcal{B}}(2) \times \mathbf{MO}_2$.

Theorem 5.11 ([10, Theorem 3.11])**.** For any $2 \leq k \leq n$, the finitely generated free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ is isomorphic to the product of the n-generated free Boolean algebra $\mathbf{F}_{\mathcal{B}}(n)$ with

$$\prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_p) \\ k_1 \geq \cdots \geq k_p \geq 1 \\ \sum_{i=1}^{p} k_i \leq n}} [(\mathbf{MO}_p)^{2^{n-p}} \times \prod_{s=1}^{p} (\mathbf{L}_{(s,p-s)})^{N(n,p,s;k_1,\ldots,k_p)}]^{\phi(n;k_1,\ldots,k_p)}$$

where $\phi(n; k_1, \ldots, k_p)$ are given by (9) and (10) on page 44 and

$$N(n, p, s; k_1, \ldots, k_p) = 2^{n-p+s} \cdot \sum_{\{i_1,\ldots,i_s\} \subseteq \{1,\ldots,p\}} \prod_{r=1}^{s} \frac{6^{k_{i_r}-1} - 2^{k_{i_r}-1}}{2^{k_{i_r}}}.$$

We notice that (cf. [9])

$$\sum_{\substack{(k_1,\ldots,k_p) \\ k_1 \geq \cdots \geq k_p \geq 1 \\ \sum_{i=1}^{p} k_i = n-l}} S(n-l; k_1, \ldots, k_p) = S(n-l, p),$$

where the Stirling number $S(n-l, p)$ of the second kind is the number of partitions of a labelled $(n-l)$-element set into exactly p parts (cf. [1, 3.39]). It follows that

$$(17) \quad \prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_p) \\ k_1 \geq \cdots \geq k_p \geq 1 \\ \sum_{i=1}^{p} k_i \leq n}} [(\mathbf{MO}_p)^{2^{n-p}}]^{\phi(n;k_1,\ldots,k_p)} = \prod_{p=2}^{k} \mathbf{MO}_p)^{(2^{n-p}\phi'(n,p))}$$

where

$$\phi'(n, p) = \sum_{l=0}^{n-p} \binom{n}{l} S(n-l, p).$$

Notice that on the right hand side of (17) we have an isomorphic copy of the n-generated free modular ortholattice $F_{\mathcal{MO}_k}(n)$ in the variety $\mathcal{MO}_k$. We arrive at our final result.

Corollary 5.12 ([10, Corollary 3.12])**.** For any $2 \leq k \leq n$, the finitely generated free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ is isomorphic to

$$F_{\mathcal{MO}_k}(n) \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_p) \\ k_1 \geq \cdots \geq k_p \geq 1 \\ \sum_{i=1}^{p} k_i \leq n}} [\prod_{s=1}^{p} (\mathbf{L}_{(s,p-s)})^{N(n,p,s;k_1,\ldots,k_p)}]^{\phi(n;k_1,\ldots,k_p)}$$

where $F_{\mathcal{MO}_k}(n)$ is the n-generated free modular ortholattice in the variety $\mathcal{MO}_k$, $\phi(n; k_1, \ldots, k_p)$ are given by (9) and (10) on page 44 and

$$N(n, p, s; k_1, \ldots, k_p) = 2^{n-p+s} \cdot \sum_{\{i_1,\ldots,i_s\} \subseteq \{1,\ldots,p\}} \prod_{r=1}^{s} \frac{6^{k_{i_r}-1} - 2^{k_{i_r}-1}}{2^{k_{i_r}}}.$$

Remark 5.13 ([10, Remark 3.13]). We notice that for $s = 1$,

$$N(n,p,1;k_1,\ldots,k_p) = 2^{n-p+1} \cdot \sum_{\{i\}\subseteq\{1,\ldots,p\}} \frac{6^{k_i-1} - 2^{k_i-1}}{2^{k_i}} = 2^{n-p}[(\sum_{i=1}^{p} 3^{k_i-1}) - p]$$

and the factor

$$F_{\mathcal{MO}_k}(n) \times \prod_{p=2}^{k} \prod_{\substack{(k_1,\ldots,k_p)\\ k_1\geq\cdots\geq k_p\geq 1\\ \sum_{i=1}^{p} k_i\leq n}} [(\mathbf{L}_{(1,p-1)})^{N(n,p,1;k_1,\ldots,k_p)}]^{\phi(n;k_1,\ldots,k_p)}$$

of the free algebra $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ in Corollary 5.12 is isomorphic to the n-generated free algebra $F_{\mathbf{V}(\mathbf{L}_{(1,p-1)})}(n)$ in the variety $V(\mathbf{L}_{(1,p-1)})$ described in the previous section (there for the algebra $\mathbf{L}_{(1,p-1)}$ we use the notation $\mathbf{L}_p$).

We finally illustrate the obtained results by presenting the structures of the free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ for $k = 2, 3, 4$ and for $n = 3, 4, 5$.

In the first of our next tables are displayed the values of the coefficients $\phi(n;k_1,\ldots,k_p)$.

$n=3$	$n=4$	$n=5$
$\phi(3;1,1) = 3$	$\phi(4;1,1) = 6$	$\phi(5;1,1) = 10$
$\phi(3;2,1) = 3$	$\phi(4;2,1) = 12$	$\phi(5;2,1) = 30$
	$\phi(4;3,1) = 4$	$\phi(5;3,1) = 20$
	$\phi(4;2,2) = 3$	$\phi(5;2,2) = 15$
	$\phi(4;1,1,1) = 4$	$\phi(5;4,1) = 5$
	$\phi(4;2,1,1,) = 6$	$\phi(5;3,2) = 10$
	$\phi(4;1,1,1,1,) = 1$	$\phi(5;1,1,1) = 10$
		$\phi(5;2,1,1) = 30$
		$\phi(5;3,1,1) = 10$
		$\phi(5;2,2,1) = 15$
		$\phi(5;1,1,1,1) = 5$
		$\phi(5;2,1,1,1) = 10$

$n=3$	$n=4$	$n=5$
$N(3,2,1;2,1) = 4$	$N(4,2,1;2,1) = 8$	$N(5,2,1;2,1) = 16$
	$N(4,2,1;3,1) = 32$	$N(5,2,1;3,1) = 64$
	$N(4,2,1;2,2) = 16$	$N(5,2,1;2,2) = 32$
	$N(4,2,2;2,2) = 16$	$N(5,2,2;2,2) = 32$
	$N(4,3,1;2,1,1) = 4$	$N(5,3,1;2,1,1) = 8$
		$N(5,2,1;4,1) = 208$
		$N(5,2,1;3,2) = 80$
		$N(5,2,2;3,2) = 128$
		$N(5,3,1;3,1,1) = 32$
		$N(5,3,1;2,2,1) = 16$
		$N(5,3,2;2,2,1) = 16$
		$N(5,4,1;2,1,1,1) = 4$

The coefficients $N(n, p, s; k_1, \ldots, k_p)$ which are non-zero for $n = 3, 4, 5$ are displayed in our second table (all other coefficients $N(n, p, s; k_1, \ldots, k_p)$ for $n = 3, 4, 5$ take value zero).

Hence we have the following structures of the free algebras $\mathbf{F}_{\mathbf{V}(\mathbf{O}_k)}(n)$ for $k = 2, 3, 4$ and for $n = 3, 4, 5$:

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_2)}(3) \cong \mathbf{F}_{\mathcal{B}}(3) \times (\mathbf{MO}_2)^{12} \times (\mathbf{L}_{1,1})^{12}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_2)}(4) \cong \mathbf{F}_{\mathcal{B}}(4) \times (\mathbf{MO}_2)^{100} \times (\mathbf{L}_{1,1})^{272} \times (\mathbf{L}_{2,0})^{48}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_2)}(5) \cong \mathbf{F}_{\mathcal{B}}(5) \times (\mathbf{MO}_2)^{720} \times (\mathbf{L}_{1,1})^{4080} \times (\mathbf{L}_{2,0})^{1760}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_3)}(3) \cong \mathbf{F}_{\mathcal{B}}(3) \times (\mathbf{MO}_2)^{12} \times (\mathbf{L}_{1,1})^{12} \times (\mathbf{MO}_3)^{1}$$

$$\mathbf{F}_{\mathbf{V}(\mathbf{O}_3)}(4) \cong \mathbf{F}_{\mathcal{B}}(4) \times (\mathbf{MO}_2)^{100} \times (\mathbf{MO}_3)^{20} \times (\mathbf{L}_{1,1})^{272} \times (\mathbf{L}_{2,0})^{48} \times (\mathbf{L}_{1,2})^{24}$$

$$\begin{aligned}\mathbf{F}_{\mathbf{V}(\mathbf{O}_3)}(5) \cong{}& \mathbf{F}_{\mathcal{B}}(5) \times (\mathbf{MO}_2)^{720} \times (\mathbf{MO}_3)^{260} \times (\mathbf{L}_{1,1})^{4080} \times (\mathbf{L}_{2,0})^{1760} \times (\mathbf{L}_{1,2})^{800} \\ & \times (\mathbf{L}_{2,1})^{240}\end{aligned}$$

$$\begin{aligned}\mathbf{F}_{\mathbf{V}(\mathbf{O}_4)}(4) \cong{}& \mathbf{F}_{\mathcal{B}}(4) \times (\mathbf{MO}_2)^{100} \times (\mathbf{MO}_3)^{20} \times (\mathbf{MO}_4)^{1} \times (\mathbf{L}_{1,1})^{272} \times (\mathbf{L}_{2,0})^{48} \\ & \times (\mathbf{L}_{1,2})^{24}\end{aligned}$$

$$\begin{aligned}\mathbf{F}_{\mathbf{V}(\mathbf{O}_4)}(5) \cong{}& \mathbf{F}_{\mathcal{B}}(5) \times (\mathbf{MO}_2)^{720} \times (\mathbf{MO}_3)^{260} \times (\mathbf{MO}_4)^{30} \times (\mathbf{L}_{1,1})^{4080} \times (\mathbf{L}_{2,0})^{1760} \\ & \times (\mathbf{L}_{1,2})^{800} \times (\mathbf{L}_{2,1})^{240} \times (\mathbf{L}_{1,3})^{40}\end{aligned}$$

6 Problem

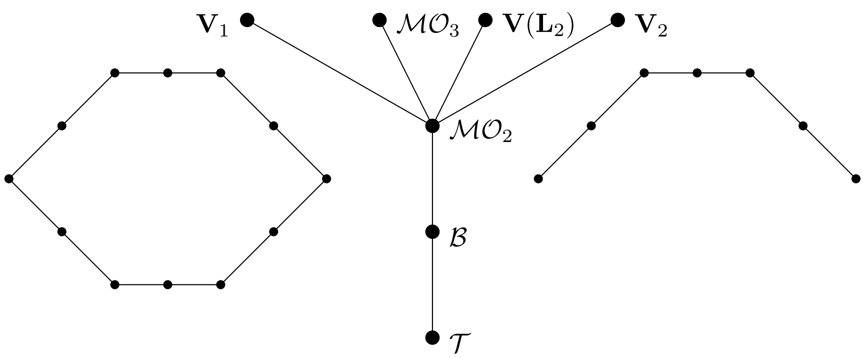

Figure 4. The four subvarieties covering the variety $\mathcal{MO}_2$

Similar descriptions as above are still missing in two particular varieties of orthomodular lattices which together with the varieties $\mathcal{MO}_3$ and $\mathbf{V}(\mathbf{L}_2)$ are among the four varieties of orthomodular lattices covering the variety $\mathcal{MO}_2$. More precisely, these are the variety $\mathbf{V}_1$ generated by the ortholattice whose Greechie diagram is 'the pentagon' (see the left part of Figure 4) and the variety $\mathbf{V}_2$ generated by the ortholattice whose Greechie diagram is 'the 6-path' (see the right part of Figure 4). (For the concept of the Greechie diagram we refer to the original Greechie's paper [6].) The question is if one can attack the problem of describing the n-generated free algebras in these two varieties using the methods analogous to those presented here. We close this paper with an open problem.

Problem 6.1. Give for any $n \geq 1$ abstract descriptions, similar to those presented here, of the n-generated free algebras in the variety $\mathbf{V}_1$ generated by 'the pentagon' and in the variety $\mathbf{V}_2$ generated by the 'the 6-path'. Also give (recursive) formulas for the cardinalities of these free algebras.

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Riemann-Stieltjes composition operators between weighted Banach spaces of holomorphic functions

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Abstract
We characterize boundedness and compactness of Riemann-Stieltjes composition operators acting between weighted Banach spaces of holomorphic functions.

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1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane and $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$. Moreover, we consider an analytic self-map φ of $\mathbb{D}$ as well as an analytic map $g : \mathbb{D} \to \mathbb{C}$. Such maps induce the following *Riemann-Stieltjes composition operator*

$$I_{g,\varphi} : H(\mathbb{D}) \to H(\mathbb{D}),\ [I_{g,\varphi} f](z) = \int_0^1 f(\varphi(tz))g'(tz)z\, dt.$$

Recently, this type of operator has been of great interest, see e.g. [1], [2], [3], [6], [9], [10].

In this article we study operators of the above type acting in the following setting: Let v be a strictly positive, bounded and continuous function (*weight*) on $\mathbb{D}$. Then the *weighted Banach space of holomorphic functions* is defined by

$$H_v^\infty := \left\{ f \in H(\mathbb{D});\ \|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\}.$$

Endowed with the weighted sup-norm $\|.\|_v$ this is a Banach space. Such spaces occur naturally in a variety of problems. For more information on that topic we refer the reader to the articles [4] and [7] and the references therein.

In [6] Li characterized boundedness and compactness of operators $I_{g,\varphi}$ acting between weighted Bergman spaces and weighed Bloch spaces, both generated by standard weights. In [8] we generalized his results to a more general setting. In this article we continue this branch of research by considering operators $I_{g,\varphi}$ acting between different weighted Banach spaces of holomorphic functions. We give a characterization of the boundedness and compactness of such operators that only involve the given weights as well as the holomorphic map g as well as the symbol φ.

2 Basics

Let ν be a holomorphic function on $\mathbb{D}$ that is additionally non-vanishing and strictly positive on $[0,1[$ and satisfies $\lim_{r\to 1}\nu(r)=0$. Then we define the corresponding weight by

$$v(z)=\nu(|z|) \text{ for every } z\in\mathbb{D}.$$

Moreover, we assume that $|\nu(z)|\geq\nu(|z|)$ for every $z\in\mathbb{D}$. The most relevant weights, such as the standard weights, the logarithmic weights and the exponential weights satisfy these conditions.

Such weights may be written as

$$v(z)=\min\{|\nu(\lambda z)|,\ |\lambda|=1\}.$$

For a better understanding we will give the proof. First, we use polar coordinates

$$\min\{|\nu(\lambda z)|,\ |\lambda|=1\}=\min\{|\nu(\lambda re^{i\Theta})|,\ |\lambda|=1\}\leq|\nu(e^{-i\Theta}re^{i\Theta})|=|\nu(r)|=\nu(|z|)=v(z).$$

On the other hand, for every $\lambda\in\partial\mathbb{D}$ we obtain for every $z\in\mathbb{D}$

$$|\nu(\lambda z)|\geq\nu(|\lambda z|)=\nu(|z|)=v(z).$$

We close this section with stating a very useful lemma, which can be easily derived from [5] Proposition 3.11.

Lemma 1. *Let v and w be weights. Then the operator $I_{g,\varphi}:H_v^\infty\to H_w^\infty$ is compact if and only if it is bounded and for every bounded sequence $(f_n)_n$ in H_v^∞ which converges to zero uniformly on the compact subsets of $\mathbb{D}$, $I_{g,\varphi}f_n$ tends to zero in H_w^∞ if $n\to\infty$.*

3 Results

Proposition 2. *The operator $I_{g,\varphi}:H_v^\infty\to H_w^\infty$ is bounded if and only if $\sup_{z\in\mathbb{D}}w(z)\left|\int_0^1\frac{g'(tz)}{v(\varphi(tz))}z\,dt\right|<\infty$.*

Proof. First, we assume that the operator $I_{g,\varphi}:H_v^\infty\to H_w^\infty$ is bounded. As we know, the weight v may be presented as

$$v(z)=\min\{|\nu(\lambda z)|,\ |\lambda|=1\},$$

where ν is a holomorphic function. Now, for fixed $\lambda\in\partial\mathbb{D}$ let

$$h_\lambda(z)=\frac{1}{\nu(\lambda z)}$$

for every $z\in\mathbb{D}$. Then $\|h_\lambda\|_v=\sup_{z\in\mathbb{D}}\frac{v(z)}{|\nu(\lambda z)|}\leq\sup_{z\in\mathbb{D}}\frac{v(z)}{\min_{|\lambda|=1}|\nu(\lambda z)|}=\sup_{z\in\mathbb{D}}\frac{v(z)}{v(z)}=1$ for every $\lambda\in\partial\mathbb{D}$. Now, we arrive at

$$\begin{aligned}\sup_{z\in\mathbb{D}}w(z)\left|\int_0^1\frac{g'(tz)z}{\nu(\lambda\varphi(tz))}\,dt\right|=\sup_{z\in\mathbb{D}}w(z)\left|\int_0^1h_\lambda(\varphi(tz))g'(tz)z\,dt\right|&\leq\|I_{g,\varphi}\|\|h_\lambda\|_v\\&\leq\|I_{g,\varphi}\|<\infty\end{aligned}$$

for every $\lambda\in\partial\mathbb{D}$. Hence, since $\lambda\in\partial\mathbb{D}$ is arbitrary, we obtain

$$\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| < \infty$$

as desired.

Conversely, we assume that $\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| < \infty$. For every $f \in H_v^\infty$ we have

$$\begin{aligned}\|I_{g,\varphi} f\|_w &= \sup_{z\in\mathbb{D}}\left|\int_0^1 f(\varphi(tz))g'(tz)z\, dt\right| = \sup_{z\in\mathbb{D}}\left|\int_0^1 \frac{f(\varphi(tz))}{v(\varphi(tz))} v(\varphi(tz))g'(tz)z\, dt\right| \\ &\le \sup_{z\in\mathbb{D}} w(z) \sup_{t\in[0,1]} v(\varphi(tz))|f(\varphi(tz))| \left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| \\ &\le \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| \|f\|_v.\end{aligned}$$

Hence the operator $I_{g,\varphi}$ must be bounded. □

Remark 3. Let us assume that $I_{g,\varphi} : H_v^\infty \to H_w^\infty$ is bounded. Then

$$\|I_{g,\varphi}\| = \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right|.$$

First, for fixed $\lambda \in \partial\mathbb{D}$ we consider

$$h_\lambda(z) = \frac{1}{\nu(\lambda z)}$$

for every $z \in \mathbb{D}$. We have seen that $\|h_\lambda\|_v \le 1$. Moreover, the proof of Proposition 4 shows that

$$\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)z}{\nu(\lambda\varphi(tz))}\, dt\right| \le \|I_{g,\varphi}\|.$$

Since $\lambda \in \partial\mathbb{D}$ was arbitrary, we also obtain

$$\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)z}{\min_{|\lambda|=1}|\nu(\lambda\varphi(tz))|}\, dt\right| = \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)z}{v(\varphi(tz))}\, dt\right| \le \|I_{g,\varphi}\|.$$

On the other hand the proof of Proposition 4 yields for every $f \in H_v^\infty$

$$\|I_{g,\varphi} f\|_w \le \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| \|f\|_v.$$

but by definition we have $\|I_{g,\varphi}\| = \inf\{M \ge 0,\ \|I_{g,\varphi} f\|_w \le M\|f\|_v \text{ for every } f \in H_v^\infty\}$. Hence $\|I_{g,\varphi}\| \le \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right|$ and the claim follows.

Proposition 4. *The operator $I_{g,\varphi} : H_v^\infty \to H_w^\infty$ is compact if and only if the following conditions are satisfied:*

(a) $\limsup_{|z|\to 1} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| = 0$,

(b) $\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 g'(tz)z\, dt\right| < \infty$.

Proof. First, we assume that the conditions (a) and (b) are fulfilled. Let $(f_n)_n$ be a bounded sequence in H_v^∞ that converges to 0 uniformly on the compact subsets of $\mathbb{D}$ such that $\|f_n\|_v \le M$ for every $n \in \mathbb{N}$. By hypothesis, for every $\varepsilon > 0$, there is $r > 0$ such that if $|z| > r$, then

$$w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| < \varepsilon.$$

Hence

$$w(z)|I_{g,\varphi} f_n(z)| \le w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| \|f_n\|_v < \varepsilon M$$

for every $n \in \mathbb{N}$ and every $z \in \mathbb{D}$ with $|z| > r$.

On the other hand, if $|z| \le r$ there must be $0 < R < 1$ such that $|\varphi(z)| \le R$. Since $f_n \to 0$ uniformly on $\{u;\ |u| \le r\}$, we can find $n_0 \in \mathbb{N}$ such that if $|\varphi(z)| \le R$ and $n \ge n_0$ then $|f_n(\varphi(z))| < \varepsilon$. Hence, we arrive at

$$w(z)|I_{g,\varphi} f_n(z)| = w(z)\left|\int_0^1 g'(tz) f_n(\varphi(tz)) z\, dt\right| < \varepsilon w(z)\left|\int_0^1 g'(tz) z\, dt\right| \le \varepsilon N,$$

where $N = \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 g'(tz)z\, dt\right| < \infty$.

Conversely, we assume to the contrary that the condition (a) does not hold. Then there is a sequence $(z_n)_n \subset \mathbb{D}$ with $|z_n| \to 1$ as $n \to \infty$ such that

$$w(z_n)\left|\int_0^1 \frac{g'(tz_n)}{v(\varphi(tz_n))} z_n\, dt\right| \ge \alpha > 0$$

for every $n \in \mathbb{N}$. Next, we can choose an increasing sequence $(k(n))_n$ of natural numbers with $k(n) \to \infty$ such that $z_n^{k(n)} \ge \frac{1}{2}$ for every $n \in \mathbb{N}$ and for every $n \in \mathbb{N}$ we select $\lambda_n \in \partial\mathbb{D}$ such that $\frac{1}{\nu(\lambda_n \varphi(tz_n))} = \frac{1}{v(\varphi(tz_n))}$. Next, we consider the functions

$$h_{n,\lambda_n}(z) := \frac{z^{k(n)}}{\nu(\lambda_n z)}$$

for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}$. Then obviously, $(h_{n,\lambda_n})_n \subset H_v^\infty$ is bounded, since $\|h_{n,\lambda_n}\|_v = \sup_{z\in\mathbb{D}} v(z)\frac{|z|^{k(n)}}{|\nu(\lambda_n z)|} \le \sup_{z\in\mathbb{D}} |z|^{k(n)} \le 1$. Moreover, $h_{n,\lambda_n} \to 0$ pointwise because of the factor $z^{k(n)}$. Finally,

$$\begin{aligned} \|I_{g,\varphi} h_{n,\lambda_n}\|_v \ge w(z_n)\left|\int_0^1 g'(tz_n) h_{n,\lambda_n}(z_n) z_n\, dt\right| &\ge w(z_n)\left|\int_0^1 g'(tz_n)\frac{z_n^{k(n)}}{\nu(\lambda_n\varphi(tz_n))} z_n\, dt\right| \\ &\ge \frac{1}{2} w(z_n)\left|\int_0^1 \frac{g'(tz_n)}{v(\varphi(tz_n))} z_n\, dt\right| \ge \frac{\alpha}{2} \end{aligned}$$

for every $n \in \mathbb{N}$ which is a contradiction.

It remains to show that condition (b) is satisfied. Since the operator $I_{g,\varphi} : H_v^\infty \to H_w^\infty$ is compact it also must be bounded. Now, take $f(z) = 1$ for every $z \in \mathbb{D}$. Then we have

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| = \sup_{z\in\mathbb{D}} v(z) < \infty$$

since the weight is bounded by hypothesis. Thus, $f \in H_v^\infty$. Finally, we arrive at

$$\|I_{g,\varphi} f\|_w = \sup_{z\in\mathbb{D}} w(z) \left|\int_0^1 f(\varphi(tz))g'(tz)z\,dt\right| = \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 g'(tz)z\,dt\right| \leq \|I_{g,\varphi}\|\|f\|_v.$$

□

Example 5. (a) We select $v(z) = w(z) = 1 - |z|$ as well as $\varphi(z) = g(z) = z$ for every $z \in \mathbb{D}$. Thus, we obtain

$$\begin{aligned}\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\,dt\right| &= \sup_{z\in\mathbb{D}}(1-|z|)\left|\int_0^1 \frac{z}{1-t|z|}\,dt\right| \\ = \sup_{z\in\mathbb{D}}(1-|z|)\left|[\ln(1-t|z|)]_0^1\right| &= \sup_{z\in\mathbb{D}}(1-|z|)\left|\ln(1-|z|)\right| < \infty.\end{aligned}$$

Hence the corresponding operator $I_{g,\varphi}$ must be bounded. Moreover, obviously, $\limsup_{|z|\to1} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\,dt\right| = \limsup_{|z|\to1}(1-|z|)\left|\ln(1-|z|)\right| = 0$ and $\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 g'(tz)z\,dt\right| = \sup_{z\in\mathbb{D}}(1-|z|)\left|\int_0^1 z\,dt\right| = \sup_{z\in\mathbb{D}}(1-|z|)|z| \leq 1$. Thus, the operator must be compact.

(b) We choose $w(z) = 1 - |z|, v(z) = (1 - |z|)^2$ and $\varphi(z) = g(z) = z$ for every $z \in \mathbb{D}$. Then, easy calculations show that

$$\begin{aligned}\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\,dt\right| &= \sup_{z\in\mathbb{D}}(1-|z|)\left|\int_0^1 \frac{z}{(1-t|z|)^2}\,dt\right| \\ &= \sup_{z\in\mathbb{D}}(1-|z|)\left[\frac{1}{1-|z|} - 1\right] \leq 1.\end{aligned}$$

But obviously $\limsup_{|z|\to1} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\,dt\right| = \limsup_{|z|\to1}(1-|z|)\left[\frac{1}{1-|z|} - 1\right] = \limsup_{|z|\to1}\left[1-1+|z|\right] = 1$. Hence the operator ist not compact.

(c) We consider $w(z) = 1 - |z|$, $v(z) = (1 - |z|)^3$ and $\varphi(z) = g(z) = z$ for every $z \in \mathbb{D}$. Then

$$\begin{aligned}\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\,dt\right| &= \sup_{z\in\mathbb{D}}(1-|z|)\left|\int_0^1 \frac{z}{(1-t|z|)^3}\,dt\right| \\ &= \sup_{z\in\mathbb{D}} \frac{1}{2}\left[\frac{1}{1-|z|} - 1 + |z|\right] = \infty.\end{aligned}$$

Hence, the corresponding operator is not bounded.

(d) Select $w(z) = v(z) = 1 - |z|$ as well as $\varphi(z) = z$ and $g(z) = \frac{1}{(1-z)^2}$ for every $z \in \mathbb{D}$. Then obviously $g'(z) = \frac{2}{(1-z)^3}$ for every $z \in \mathbb{D}$. Moroever,

$$\begin{aligned}\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\,dt\right| &= \sup_{z\in\mathbb{D}}(1-|z|)\left|\int_0^1 \frac{2z}{(1-tz)^3(1-t|z|)}\,dt\right| \\ \geq \sup_{z\in\mathbb{D}} 2(1-|z|)\left|\int_0^1 \frac{z}{(1-tz)^3}\,dt\right| &= \sup_{z\in\mathbb{D}}(1-|z|)\left|\frac{1}{(1-z)^2} - 1\right| = \infty.\end{aligned}$$

Hence the operator is not bounded.

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Enumeration of coherent configurations of order at most 15

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Abstract

This text describes the computerized enumeration of all coherent configurations of order up to 15, and provides some viewpoints of the results of this enumeration. The main discovery resulting from this enumeration is the unique non-Schurian coherent configuration of order 14. We also provide classification of the association schemes of order at most 30 up to algebraic isomorphism, using the classification up to combinatorial isomorphism of those schemes by Hanaki and Miyamoto.

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1 Introduction

A coherent algebra of order n is a subalgebra of $M_{n\times n}(\mathbb{C})$ that is closed under transposition and Schur-Hadamard products, and contains I_n and J_n (the all one matrix).

A subalgebra $\mathcal{A}$ of $M_{n\times n}(\mathbb{C})$ is a coherent algebra if and only if it has a basis B of $(0,1)$-matrices such that for any $A \in B$, $A^T \in B$, the sum of all matrices in B is J_n, and I_n is in the algebra (or equivalently, I_n is a sum of some matrices in B). B is called the first standard basis of $\mathcal{A}$.

If $B = \{A_1, \dots, A_r\}$, then the rank of $\mathcal{A}$ is r. $C = \sum iA_i$ is the color matrix of $\mathcal{A}$. More generally, we allow any distinct coefficients.

In relational (or combinatorial) language, $\mathfrak{m} = (\Omega, \mathcal{R})$ is a coherent configuration if $\mathcal{R} = \{R_1, \dots, R_r\}$, where the R_i are relations over Ω, and $B = \{A(R_i)|1 \le i \le r\}$ is a first standard basis of a coherent algebra. The R_i are called basic relations of $\mathfrak{m}$. R_i is the set of arcs of a directed graph Γ_i. The Γ_i are called basic graphs of $\mathfrak{m}$.

For a basic graph Γ_i, one of the following holds:

1) All arcs are loops; or

2) the graph is simple; or

3) the graph has no undirected edges.

In other words, the corresponding relations are reflexive, symmetric or anti-symmetric.

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Order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
CCs	1	2	4	10	15	38	57	143	228	492	769	1845	2806	6167	9841
Schurian	1	2	4	10	15	38	57	143	228	492	769	1845	2806	6166	9839

Table 1. Numbers of coherent configurations of each order

$\Delta = \{(x,x)|x \in \Omega\}$ is a union of some relations of $\mathcal{R}$. This union defines a partition of $\Omega = F_1 \cup \cdots \cup F_k$. F_i is called a fiber of $\mathfrak{m}$. For every relation $R_i \in \mathcal{R}$, there exist two fibers F_a, F_b (not necessarily distinct) such that $R_i \subseteq F_a \times F_b$.

The graphs that are within a single fiber are regular. The graphs between two distinct fibers are biregular.

A coherent configuration with a single fiber is called an association scheme.

A color automorphism of a coherent configuration is a permutation $\sigma \in Sym(\Omega)$, such that there exists a permutation $' \in Sym([1,r])$ for which $R_i^\sigma = R_{i'}$ for all $i \in [1,r]$. The set of all color automorphisms of $\mathfrak{m}$ is denoted by $CAut(\mathfrak{m})$.

A (strong) automorphism is a color automorphism for which the $' = id$. The set of all automorphisms of $\mathfrak{m}$ is denoted by $Aut(\mathfrak{m})$.

For more information and details about coherent configurations and association schemes, see e.g. [1, 2, 7].

2 Results

2.1 Enumeration of coherent configurations

The numbers of coherent configurations and of Schurian coherent configurations of each order up to 15 are listed in Table 1.

An interesting consequence of this enumeration can be seen in column 14. There exists a unique non-Schurian coherent configuration of order 14. The existence of such a coherent configuration was an open question until the results of our enumeration, which were announced in [8]. This non-Schurian coherent configuration has two fibers of sizes 6, 8, rank 11 and its automorphism group has rank 12. For a detailed description of this non-Schurian CC, see [8].

The two non-Schurian coherent configurations of order 15 are the well known doubly regular tournament and the order 14 non-Schurian coherent configuration enlarged by a fiber of size 1.

A file with a list of color graphs of the coherent configurations of order up to 15 is available at `http://my.svgalib.org/math-data/ccs1_15n`. The matrices are in GAP ([4]) format. The list does not include coherent configurations with fibers of size 1.

2.2 Algebraic isomorphisms and automorphisms

Recall that a combinatorial (or strong) isomorphism (or simply an isomorphism) of two coherent configurations is a bijection of the underlying sets that maps relations to relations.

An algebraic isomorphism between two coherent configurations is a bijection of the relations of one configuration to the other that preserves the algebraic structure.

An isomorphism of coherent configurations induces naturally an algebraic isomorphism. Algebraic isomorphism that do not arise from combinatorial ones are thus of

Order	#	Classes
16	15	{5,6}, {14,15,16,17}, {18,19}, {20,21}, {32,33}, {49,50}, {54,55}, {58,59}, {77,78,79}, {83,84,85}, {89,90}, {94,95}, {155,156}, {164,165}, {167,168}
19	1	{2,3}
23	1	{2,...,20}
24	20	{53,54,55}, {56,57,58}, {89,...,93}, {94,95,96}, {99,...,103}, {105,106,107,108}, {113,114}, {130,131,132}, {133,134,135}, {163,164,165,166}, {167,...,171}, {175,176,177,178}, {182,...,188}, {189,190,191}, {195,...,201}, {296,297}, {306,307,308}, {382,...,386}, {395,396,397}, {465,466}
25	5	{4,...,11}, {14,15}, {17,18}, {20,21}, {22,23}
26	1	{3,...,12}
27	8	{5,...,378}, {382,383}, {427,428}, {429,430}, {431, 432}, {472,473}, {474,475}, {476,477}
28	5	{5,6,7,8}, {16,...,71}, {74,75}, {109,110}, {175,176}
29	1	{2,...,22}
30	4	{25,26}, {27,28,29,30}, {106,107}, {122,123}

Table 2. Classes of non-isomorphic, algebraically isomorphic association schemes

some interest. In the case of automorphisms, an algebraic automorphism not arising from a (color) automorphism is called a proper algebraic automorphism.

The smallest case of two algebraically isomorphic coherent configurations that are not isomorphic is the pair of two non-isomorphic strongly regular graphs with parameters $(16, 6, 2, 2)$.

The smallest case of a coherent configuration with a proper algebraic automorphism that maps a basic relation to a non-isomorphic basic relation is the doubly regular tournament on 15 points.

As a result of this project, it is now known that the smallest case of a coherent configuration with a proper algebraic automorphism is of order 14 and rank 12. In this case the proper algebraic automorphism exchanges two isomorphic basic relations. A merging of this coherent configuration is the unique non-Schurian coherent configuration of order 14. See [8] for a detailed discussion of this rank 12 coherent configuration.

For comparison, considering only association schemes, and using the list of [9], we found that up to 30 points there are 445 association schemes with proper algebraic automorphisms, see Table 3. Up to 30 points, there are 61 classes of algebraically isomorphic association schemes which are not combinatorially isomorphic, see Table 2.

2.3 Correctness of the results

Since no formal proof that the programs actually implement the correct algorithm is offered, the correctness of the results is not assured. But comparing the results to similar efforts by our predecessors and colleagues (see [8]) may increase the confidence that the results are correct.

For orders up to 8, a different approach may be used (enumeration of subalgebras). The results are the same as the results presented here.

Order	#	Positions
15	1	5
16	5	16, 78, 84, 160, 172
18	1	60
23	18	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19
24	36	53, 54, 55, 58, 72, 130, 131, 132, 134, 163, 164, 165, 166, 168, 169, 170, 175, 176, 177, 178, 188, 261, 272, 273, 276, 277, 383, 384, 385, 386, 458, 593, 597, 598, 600, 601
25	9	4, 5, 6, 7, 8, 9, 10, 27, 34
27	350	5,...,51, 53,...,66, 68,...,73,75,76,77, 79,...,159, 161,...,188, 190,...,207, 209,...,216, 218,...,227, 229,...,266, 268,...,300, 302,...,329, 331,...,341, 343,...,349, 351,352,353, 356,357,358, 360,361, 363, 365,366,367,368, 371,372, 377, 407, 455
28	2	109, 110
29	20	2,...,21
30	3	13, 25, 74

Table 3. List of association schemes with proper algebraic automorphisms

For orders up to 13, the results agree with unpublished results of a similar project by Sven Reichard.

For orders up to 14, no Schurian coherent configuration was missed. A list of Schurian coherent configurations can be easily calculated by GAP for those orders.

3 Algorithm

3.1 The main algorithm

Inducing a coherent configuration on a subset of fibers results again in a coherent configuration. In particular, inducing on a single fiber results in an association scheme. Thus, the color matrix of a coherent configuration with two fibers is of the form

$$C = \left(\begin{array}{c|c} \mathrm{AS}_1 & CB \\ \hline CB^T & \mathrm{AS}_2 \end{array}\right). \qquad \underbrace{}_{n}\ \underbrace{}_{m}$$

Here CB is a (color matrix of a) color partition of the complete (directed) bipartite graph $K_{n\to m}$ into biregular subgraphs.

The association schemes of orders up to 30 were enumerated by Hanaki and Miyamoto [9], thus to enumerate all coherent configurations of order n with two fibers, we use Algorithm 1.

For the crucial step of this algorithm, finding all CBs, we use the two step "good sets" method, originally used in `COCO` [3].

In the first step we enumerate all biregular subgraphs H of the complete (directed) graph $K_{i\to j}$ that may be colors of such a CB. There are two requirements:

i) when H is multiplied by a color of AS_1 or AS_2, the result does not split H;

ii) when H is multiplied by H^T the result does not split a color of AS_1 or AS_2.

Data: n, list of association schemes
Result: List of coherent configurations of order n with two fibers
for $i \le j$, $i + j = n$ **do**
  for AS_1, AS_2 *association schemes of orders* i, j **do**
    find all CB such that C above is a coherent configuration;
    add C to list if not already in list (up to isomorphism);
  end
end

Algorithm 1: Enumeration of coherent configurations with two fibers

Note that by the biregularity condition, if $\gcd(i, j) = 1$, there is only one good set, the one containing the whole $K_{i \to j}$.

In the second step we construct all partitions of $K_{i \to j}$ from good sets, and filter the partitions for ones that produce a coherent configuration.

3.2 Good sets enumeration

The main function of the C program that enumerates the good sets is listed in Appendix A. This function recursively enumerates all biregular subsets of valencies s_1, s_2 of $K_{i \to j}$ (in the C program, $i =$ `ord[0]`, $j =$ `ord[1]`). In level `level` of the recursion the s_1 out neighbors of vertex `level` are selected and added to the partial graph `st`. On the i-th level, the whole graph is already selected and is then checked for coherency.

The number of candidate graphs that need to be checked for coherency is limited by $\binom{j}{s_1}^i$, but since after each element of $[1, k]$ may only be used s_2 times, the actual number is smaller.

The function counts the number of times each element is used (`ns`), and removes the fully used ones from the set of available out neighbors `a`.

Note that $s_1 \cdot i = s_2 \cdot j$, therefore for a given i and j pair, only a small possible set of values may be used.

3.3 Partitions from good sets

The GAP function listed in Appendix B is the main part of the GAP program that implements the second step. This recursive function enumerates all partitions of `set` such that each cell is in `sets`.

Two limitations on the partitions are:

1. Only one representative of each orbit of the action of `group` on the partitions is required.

2. The function `compare` tells whether two sets are allowed to be in the same partition together (compatible).

The function works recursively by selecting element `e` of `set`, taking one representative `s2` containing `e` of each orbit of `group` on `sets`, and calculating all partitions of the set difference `set` $\setminus$ `s2`, from sets which are disjoint from `s2` and compatible with it. Instead of `group`, we are left with the set stabilizer in `group` of `s2`.

The function `compare` in this case checks that multiplying each set by any color of AS1 or AS2 does not split the other set.

The function `compute` is an optimization that speeds up `compare` by pre-computing some products. The extra parameter, `param` contains AS1 and AS2, in a form that, again, speeds up the calculations of `compare`.

The `group` passed in the initial invocation of this function is the direct product $Aut(\mathrm{AS}_1) \times Aut(\mathrm{AS}_2)$.

3.4 Coherent configurations with more than two fibers

To enumerate all coherent configurations of order n with $k+1$ fibers, $k \geq 2$, we start with a list of all coherent configurations of order less than n and exactly k fibers, and add to each of them a fiber of the needed size to complement the order to n.

$$\left(\begin{array}{c|c|c|c} \mathrm{AS}_1 & * & * & CB_1 \\ \hline * & \ddots & * & \vdots \\ \hline * & * & \mathrm{AS}_k & CB_k \\ \hline & & & \mathrm{AS}_{k+1} \end{array}\right)$$

If we start with a coherent configuration corresponding to the first k by k blocks in the above matrix, and try to extend it by AS_{k+1}, we only need to find the CB_i. It is not necessary to follow the steps described above for CB_i, since we know that

$$C_i = \left(\begin{array}{c|c} \mathrm{AS}_i & CB_i \\ \hline {CB_i}^T & \mathrm{AS}_{k+1} \end{array}\right)$$

is a coherent configuration with two fibers, so it is isomorphic to one in our already calculated list. For such coherent configuration, the group of isomorphisms that fix AS_i and AS_{k+1} is $CAut(\mathrm{AS}_i) \times CAut(\mathrm{AS}_{k+1})$.

4 Further discussion

4.1 Further optimization of the program

Some optimizations in the programs were not discussed above.

When looking for good sets, for every biregular graph, its complement is also tested, so we only need to work with $s1 \leq \frac{j}{2}$. If $s1 = \frac{j}{2}$, then the neighbors of 1 are taken out of $[1, j-1]$, instead of $[1, j]$.

The coherency test is independent for AS_1 and AS_2. In fact we calculate sets that are good for AS_1 and any other association scheme of order j, as well as sets that are good for AS_2 and any other association scheme of order i. Those that are good for AS_1 and AS_2, are exactly the intersection of those two sets of good sets.

In both construction of partitions, and search for coherent configurations with more than two fibers, we define a linear order on the association schemes (order of appearance in Hanaki and Miyamoto list), and make sure that in any coherent configuration, if $i < j$ then $\mathrm{AS}_i \leq \mathrm{AS}_j$, thus reducing repetitions.

The current code can enumerate all coherent configurations of order up to 15. For larger orders, further optimizations are required.

Potential optimizations are:

In the search for good sets, it is possible that some partial graphs are so far from good that they cannot be completed to good sets. Testing for such conditions at upper nodes may reduce the number of leaves considerably. Carrying the information currently in variables `ns` and `a` in the nodes may reduce the amount of calculations per node.

The `checkcoherent` function may probably be further optimized as well. While coherency of a set is not equivalent to coherency of its complement, the calculating both together may be faster than calculating each alone, as is done now.

In the construction of partitions, the function `compare` can be optimized to work faster. It may also disqualify more sets, by comparing not only the two sets in question, but also the already selected sets for the partition.

#	Rank	|Aut|	Aut		orbits	|CAut|	|AAut|
25	4	882	E49 : (C3 x S3)	26	$14^2, 2$	882	2
26	4	441	(C7 : C3) x (C7 : C3)	52	$7^4, 1^2$	882	2
27	4	40320	S8	4	30	40320	1
28	4	1152	((((E16:C3):C2):C3):C2):C2	14	$6, 24$	1152	1
29	4	192	((E8 : E4) : C3) : C2	28	$2, 12, 16$	192	1
30	4	168	E8 : (C7 : C3)	28	$1, 7, 8, 14$	168	1
106	7	360	(C15 : C4) x S3	7	30	720	2
107	7	72	(C6 x S3) : C2	27	$6, 24$	144	2
122	8	120	C2 x A5	12	30	720	6
123	8	120	S5	12	30	720	6

Table 4. Information on some association schemes of order 30

The initial group given to the partition construction function may also be based on $CAut$, instead of Aut.

4.2 Towards explanations of the results

Using the nomenclature described in [7] we aim for an explanation of the results presented in Table 2.

The association schemes of order 16 are mainly related to fusions of WFDF coherent configurations of order 16 (compare with similar objects of order 28 in [5]).

Some association schemes of order 16 (as well as of order 25) are amorphic schemes (see [6]).

The ASs of orders 19 and 23, as well as the rank 3 antisymmetric ASs of rank 27, are generated by doubly regular tournaments. The smallest DRT was already mentioned, as the smallest non-Schurian AS. See also the rapid increase in number of those DRTs as order increases.

The ASs of orders 26 and 29, as well as the rank 3 ASs of order 28, correspond to strongly regular graphs. For order 28 those are the classic Chang graphs.

The challenge of explaining the ASs of order 24, as well as those of rank larger than 3 with order 27 and 28, we leave for the future.

We now look into the ASs of rank 30. See Table 4 for some information about those association schemes. The numbers in the fifth column of the table are the Rank (number of orbitals) of the relevant automorphism group.

- $AS.30.25$ and $AS.30.26$ are wreath products of a scheme of order 2 and the DRT of order 15.
- The ASs numbers 27,28,29,30 are generated by symmetric BIBDs with parameters $v = 15$, $k = 7$, $\lambda = 3$. There are five such designs, three of them are self-dual, and a pair of dual designs.
- For the remaining two pairs of algebraically isomorphic schemes, we provide the calculated information in the table, and leave explanation and interpretation to the future.

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Supplements

A C function enumerating good sets

```
void goodsets_r(FILE *f0, FILE *f1, set *st, int s1, int s2, int
    level) {
    int i, r0, r1;
    set *c1;

    if(level==ord[0]) {
        int j;
        set comp[MAXCOLORS];
        comp[0]=ord[0];
        r0=(f0!=NULL);
        r1=(f1!=NULL);
        checkcoherent(st, &r0, &r1);
        if(r0) {
            fprintf(f0,"[");
            gapsetsf(f0,st);
            fprintf(f0,"],\n");
        }
        if(r1) {
            fprintf(f1,"[");
            gapsetsf(f1,st);
            fprintf(f1,"],\n");
        }
        for(j=1;j<=ord[0];j++)
            comp[j]=DIFFERENCE(NBITS(ord[1]),st[j]);
        r0=(f0!=NULL);
        r1=(f1!=NULL);
        checkcoherent(comp, &r0, &r1);
        if(r0) {
            fprintf(f0,"[");
            gapsetsf(f0,comp);
            fprintf(f0,"],\n");
        }
        if(r1) {
            fprintf(f1,"[");
            gapsetsf(f1,comp);
            fprintf(f1,"],\n");
        }
        return;
    }

    if(level==0) {
        if(s1+s1==ord[1]) {
            c1=Combinations(NBITS(ord[1]-1),s1);
        } else {
            c1=Combinations(NBITS(ord[1]),s1);
        }
```

```
    } else {
        int k,m;
        int ns[BITS];
        set a;
        for(k=0;k<ord[1];k++) ns[k]=0;
        for(k=1;k<=level;k++) for(m=0;m<ord[1];m++)
            if (IS_IN(st[k],m)) ns[m]++;
        a=NBITS(ord[1]);
        for(k=0;k<ord[1];k++)
            if(ns[k]==s2) a=DIFFERENCE(a,BITN(k));
        c1=Combinations(a,s1);
    }

    SSET_SETSIZE(st, level+1);
    for(i=1;i<=SSET_SIZE(c1);i++) {
        st[level+1]=c1[i];
        goodsets_r(f0, f1, st, s1, s2, level+1);
    }
    free(c1);
    return;
}
```

B GAP function enumerating partitions

```
PartitionsFromSetsAC := function(group, action, set, sets,
    compare, compute, param)
    local s1,s2,n1,p1,p2,l, orbs,e,i,nset, nsets;
    if set=[] then return [[]]; fi ;
    if sets=[] then return [] ; fi ;
    if not IsSubset(Union(sets),set) then return []; fi ;
    orbs:=Orbits(group,sets,action);
    e:=set[1];
    s1:=Filtered(orbs,x->e in Union(x));
    p1:=[];;
    for i in s1 do
      s2:=First(i,x->e in x);
      nset:=Difference(set, s2);
      l:=compute(s2,[],param);
      nsets:=Filtered(sets, x -> IsSubset(nset,x) and compare(s2
          ,x,param,l));
      Add(p1,List( PartitionsFromSetsAC(Stabilizer(group,s2,
          action), action,nset,nsets,compare,compute,param), x
          -> Union(x,[s2])));
    od ;

    return Union(p1);
end;
```

C GAP function enumerating coherent configurations with more than 2 fibers

```
CCs_morefibers3_1 := function(cc2, cc2b, cc2d, bcc, n)
#cc2[i] = list of CCs of order i with two fibers
# without fibers of size 1, smaller fiber first,
# for two fibers of the same size, the highest AS index is first.
# cc2b = list of pair of AS indexes of CCs
# cc2d = for each pair in cc2b, all CCs with those ASs (not up to isomorphism).
# bcc[i] = CCs of order i to add a fiber to.
# n = order of requested CCs.
  local ccs,ccg,cg,s,t,ii,c_c,a1,l,r1,r2,ast,nf,p1,nc, a2, k, jj, p3,lm,wm;
  ccs:=[];
  ccg:=[];
  for ii in [2..n-2] do
    r1:=[1..Size(bcc[ii])];
    for c_c in r1 do
      ast:=CC_ASFiberType(bcc[ii][c_c]);
      nf:=Size(ast);
      if n-ii>=ast[nf][1] then
        if n-ii=ast[nf][1] then
          r2:=[1..ast[nf][2]];
        else
          r2:=[1..Size(as[n-ii])];
        fi;
        for a1 in r2 do
          nc:=IdentityMat(n);
          nc{[1..ii]}{[1..ii]}:=bcc[ii][c_c]{[1..ii]}{[1..ii]};
          nc{[ii+1..n]}{[ii+1..n]}:=as[n-ii][a1]+100;
          lm:=[[nc]];
          k:=1;
          for t in [1..nf] do
            Add(lm,[]);
            p1:=Position(cc2b,[ast[t],[n-ii,a1]]);
            if p1=fail then
              break;
            fi;
            for a2 in cc2d[p1] do
              wm:=NullMat(n,n);
              jj:=ast[t][1];
              wm{[k..k+jj-1]}{[ii+1..n]}:=a2{[1..jj]}{[jj+1..Size(a2)]}+200*(k+1);
              wm{[ii+1..n]}{[k..k+jj-1]}:=a2{[jj+1..Size(a2)]}{[1..jj]}+200*(k+2);
              Add(lm[Size(lm)],wm);
            od;
            k:=k+jj;
          od;
```

```
        for nc in List(Cartesian(lm),Sum) do
          if Size(Union(nc))>=Size(Union(nc^2)) and Size(Union
             (nc))>=Size(Union(nc^3)) then
            p3:=FromColorMatrix(nc);
            if IsAS(p3) then
              cg:=CAut_Graph(nc);
              if ForAll(ccg,x->not BlissIsIsomorphicGraph(x,cg
                 )) then
                Add(ccs, NormalizeColorMatrix(nc));
                Add(ccg, cg);
              fi;
            fi;
          fi;
        od;
      od;
    fi;
  od;
od;
return ccs;
end;
```

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