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On selected developments in the theory of natural dualities

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Abstract

This is a survey on selected developments in the theory of natural dualities where the author had the opportunity to make with his foreign colleagues several breakthroughs and move the theory forward. It is aimed as author's reflection on his works on the natural dualities in Oxford and Melbourne over the period of twenty years 1993-2012 (before his attention with the colleagues in universal algebra and lattice theory has been fully focused on the theory of canonical extensions and the theory of bilattices). It is also meant as a remainder that the main problems of the theory of natural dualities, *Dualisability Problem* and *Decidability Problem for Dualisability*, remain still open.

Theory of natural dualities is a general theory for quasi-varieties of algebras that generalizes 'classical' dualities such as *Stone duality* for Boolean algebras, *Pontryagin duality* for abelian groups, *Priestley duality* for distributive lattices, and *Hofmann-Mislove-Stralka duality* for semilattices. We present a brief background of the theory and then illustrate its applications on our study of Entailment Problem, Problem of Endodualisability versus Endoprimality and then a famous Full versus Strong Problem with related developments.

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1 Introduction

In 1936 M.H. Stone published a seminal work on duality theory, exhibiting a dual equivalence between the category of all Boolean algebras and the category of all Boolean spaces [44]. Almost at the same time L. Pontryagin showed that the category of abelian groups is dually equivalent to the category of compact topological abelian groups [37], [38]. The most important step toward the development of general duality theory was Priestley's duality for distributive lattices: the category of all distributive lattices was shown to be dually equivalent to the category of all compact totally-order disconnected ordered topological spaces (since then called Priestley spaces) [41], [42]. Shortly after that, K.H. Hofmann, M. Mislove and A. Stralka developed a duality for semilattices [34]. The general duality theory, called *Natural duality theory*, grew out from these four dualities, in a monumental work by B.A. Davey and H. Werner [26]. Its rapid development over the next two decades is covered in the survey papers by B. A. Davey [4] and by H. A. Priestley [43], and in the monographs by D. M. Clark and B. A. Davey [2] and by J. G. Pitkethly and B. A. Davey [36]. The author's focus here is on selected developments

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in the theory over the period of twenty years 1993-2012 where he had the opportunity and privilege to make, mainly with H. A. Priestley and B. A. Davey in Oxford and Melbourne, certain breakthroughs and move the theory forward.

The theory has proven to be a valuable tool in algebra, algebraic logic, certain parts of computer science, and even in theoretical physics as demonstrated by the author's survey in this journal on free orthomodular lattices [31]. This year's second (and expectedly final) survey is also meant as a remainder that the main problems of the theory, the *Dualisability Problem* and the Decidability Problem for Dualisability, remain still open.

Generally speaking, the theory of natural dualities concerns the topological representation of algebras. The main idea of the theory is that, given a quasi-variety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ of algebras generated by an algebra $\mathbf{M}$, one can often find a topological relational structure $\underset{\sim}{M}$ on the underlying set M of $\mathbf{M}$ such that a dual equivalence exists between $\boldsymbol{\mathcal{A}}$ and a suitable category $\mathfrak{X}$ of topological relational structures of the same type as $\underset{\sim}{M}$. Requiring the relational structure of $\underset{\sim}{M}$ to be *algebraic over* $\mathbf{M}$, all the requisite category theory "runs smoothly" (we refer to [2]). A uniform way of representing each algebra $\mathbf{A}$ in the quasi-variety $\boldsymbol{\mathcal{A}}$ as an algebra of continuous structure-preserving maps from a suitable structure $\mathbf{X} \in \mathfrak{X}$ into $\underset{\sim}{M}$ can be obtained. In particular, the representation is relatively simple and useful for free algebras in $\boldsymbol{\mathcal{A}}$ as was demonstrated also in [31].

The motivation for the natural duality theory goes back to the question "Why in 1614 did the Scottish philosopher and mathematician John Napier, Laird of Merchiston in Scotland, invent the logarithm?" ([6]). To quote from his 1619 book [35]:

"Seeing there is nothing (right well-beloved Students of the Mathematics) that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. ... I found at length some excellent brief rules ... which together with the hard and tedious multiplications, divisions, and extractions of roots, doth also cast away from the work itself even the very numbers themselves that are to be multiplied, divided and resolved into roots, and putteth other numbers in their place which perform as much as they can do, only by addition and subtraction, division by two or division by three."

A *natural duality* is a form of logarithm which is applied to algebraic structures rather than to numbers: it takes difficult problems concerning algebras and converts them into simpler yet equivalent problems concerning completely different mathematical structures just as a logarithm converts a difficult multiplication of positive real numbers into a simpler yet equivalent addition of entirely different (and not necessarily positive) real numbers. Given a finite algebra $\mathbf{A}$, a natural duality based on $\mathbf{A}$ is the exact analogue of a logarithm, $\log_a$, to the base a for some positive real number $a \neq 1$ and $\mathbf{A}$ is said to *admit a natural duality* if a natural duality based on $\mathbf{A}$ exists. Just as $\log_a$ does not exist if a is not positive or $a = 1$, a natural duality based on $\mathbf{A}$ need not exist. ([6])

In Section 2 we present a brief background of the theory of natural dualities with its main two open problems, the *Dualisability Problem* and the Decidability Problem for Dualisability. In Sections 3 and 4 we illustrate the application of the theory on the study of entailment and endodualisability developed by the author in a close collaboration with H.A. Priestley and B.A. Davey. In Section 5 we give an overview of later developments of the theory in the author's collaboration with B. Davey's research group, where our focus is mainly on a famous Full versus Strong Problem.

2 The basic scheme of the theory of natural dualities and its main open problems

We now recall the basic scheme of the theory more precisely. Let $\mathbf{M} = (M; F)$ be a finite algebra. Let $\underset{\sim}{M} = (M; G, H, R, \mathcal{T})$ be a discrete topological structure, i.e. a non-empty set M endowed with (finite) families G, H and R of operations, partial operations and relations, respectively, and with a discrete topology $\mathcal{T}$. We recall that the graph of an n-ary (partial) operation $g\colon M^n \to M$ is the $(n+1)$-ary relation

$$\operatorname{graph}(g) = \{\, (x_1, \ldots, x_n, g(x)) \mid (x_1, \ldots, x_n) \in M^n \,\} \subseteq M^{n+1}.$$

We say that the structure $\underset{\sim}{M}$ is *algebraic over* $\mathbf{M}$ if the relations in R and the graphs of the operations and partial operations in $G \cup H$ are subalgebras of appropriate powers of $\mathbf{M}$. Hence a unary (partial) operation is algebraic over $\mathbf{M}$ if and only if it is a (partial) endomorphism of $\mathbf{M}$.

Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ be the quasi-variety generated by a finite algebra $\mathbf{M}$ and assume that $\underset{\sim}{M} = (M; G, H, R, \mathcal{T})$ is algebraic over $\mathbf{M}$. Let $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{M})$ be the 'topological quasi-variety' generated by $\underset{\sim}{M}$, i.e. the class of all structures which are embeddable as closed substructures into powers of $\underset{\sim}{M}$. For any algebra $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, let $D(\mathbf{A})$ denote the set of all $\boldsymbol{\mathcal{A}}$-homomorphisms $\mathbf{A} \to \mathbf{M}$. Since $\underset{\sim}{M}$ is algebraic over $\mathbf{M}$, $D(\mathbf{A})$ can naturally be understood as a substructure of $\underset{\sim}{M}^A$, and so as a member of $\mathfrak{X}$.

Let $X \subseteq M^I$ for some non-empty set I and let $r \subseteq M^n$ be an n-ary relation on M. We say that a map $\varphi\colon X \to M$ preserves the relation r if $[\varphi(\tilde{\mathrm{x}}_1), \ldots, \varphi(\tilde{\mathrm{x}}_n)] \in r$ for all $\tilde{\mathrm{x}}_1 = (x_{1i})_{i\in I}, \ldots, \tilde{\mathrm{x}}_n = (x_{ni})_{i\in I}$ such that $[x_{1i}, \ldots, x_{ni}] \in r$ for every $i \in I$. We say that φ preserves an n-ary (partial) operation if φ preserves its graph as an $(n+1)$-ary relation.

Let $\mathbf{X}$ be a structure in $\mathfrak{X}$. By an $\mathfrak{X}$-morphism $\varphi\colon \mathbf{X} \to \underset{\sim}{M}$ we mean a continuous structure-preserving map, i.e. a continuous map preserving all (partial) operations in $G \cup H$ and all relations in R. Let $E(\mathbf{X})$ be the set of all $\mathfrak{X}$-morphisms $\mathbf{X} \to \underset{\sim}{M}$. Again, since $\underset{\sim}{M}$ is algebraic over $\mathbf{M}$, $E(\mathbf{X})$ can be understood as a subalgebra of $\mathbf{M}^X$, i.e. a member of $\boldsymbol{\mathcal{A}}$.

The (hom-)functors $D\colon \boldsymbol{\mathcal{A}} \to \mathfrak{X}$ and $E\colon \mathfrak{X} \to \boldsymbol{\mathcal{A}}$ are contravariant and dually adjoint. Moreover, for any $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and for any $\mathbf{X} \in \mathfrak{X}$, we have maps $e_A : \mathbf{A} \to ED(\mathbf{A})$ and $\varepsilon_X : \mathbf{X} \to DE(\mathbf{X})$ given by evaluation, *viz.*

$$\begin{aligned} e_A(a)(h) &= h(a) \quad \text{for every } a \in A \text{ and } h \in D(\mathbf{A}),\\ \varepsilon_X(y)(\varphi) &= \varphi(y) \quad \text{for every } y \in X \text{ and } \varphi \in E(\mathbf{X}), \end{aligned}$$

which are embeddings. We say that $\underset{\sim}{M}$ *yields a pre-duality on* $\boldsymbol{\mathcal{A}}$. In general, such a scheme provides us with a canonical way of constructing, via hom-functors, a dual adjunction between a category of algebras $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$, generated by a finite algebra $\mathbf{M}$, and a category $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{M})$ of structured topological spaces, generated by the alter ego $\underset{\sim}{M}$ of the algebra $\mathbf{M}$.

Let $\underset{\sim}{M} = (M; G, H, R, \mathcal{T})$ be an algebraic structure over $\mathbf{M}$, so that $\underset{\sim}{M}$ yields a pre-duality on $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$. We say that $\underset{\sim}{M}$ *yields a natural duality on* $\boldsymbol{\mathcal{A}}$ if for every $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ the embedding e_A is an isomorphism, i.e. the evaluation maps $e_A(a)$ $(a \in A)$ are the only $\mathfrak{X}$-morphisms from $D(\mathbf{A})$ to $\underset{\sim}{M}$; we notice that they represent then the elements a of $\mathbf{A}$. Sometimes we say that $G \cup H \cup R$ *yields a* (*natural*) *duality on* $\boldsymbol{\mathcal{A}}$ or that $\underset{\sim}{M}$ *is dualisable.* We further say that $\underset{\sim}{M}$ (or $G \cup H \cup R$) yields a *full duality* on $\boldsymbol{\mathcal{A}}$ if $\underset{\sim}{M}$ yields a duality on $\boldsymbol{\mathcal{A}}$ and for every $\mathbf{X} \in \mathfrak{X}$ the embedding ε_X is also an isomorphism. In such a case the categories $\boldsymbol{\mathcal{A}}$ and $\mathfrak{X}$ are dually equivalent via categorical anti-isomorphisms D and E which are inverse to each other. Finally, we say that $\underset{\sim}{M}$ (or $G \cup H \cup R$) yields a *strong* *duality* on $\boldsymbol{\mathcal{A}}$ if it yields a full duality on $\boldsymbol{\mathcal{A}}$ and $\underset{\sim}{M}$ is injective in the category $\boldsymbol{\mathfrak{X}}$ (with respect to embeddings). A famous *Full versus Strong Problem*, which dated back to the beginnings of the theory of natural dualities and was open for about twenty-five years asked:

Problem 2.1. (Full versus Strong Problem) *Is every full duality strong?*

We have not claimed above that it is always possible, for a given algebra $\mathbf{M}$, to choose a structure $\underset{\sim}{M}$ on M yielding a duality on $\mathbb{ISP}(\mathbf{M})$. Indeed, for some finite algebras $\mathbf{M}$ there is no choice of alter ego $\underset{\sim}{M}$ for which the resulting dual adjunction yields a duality between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathfrak{X}}$; for example, the two-element implication algebra $\mathbf{I} = (\{0, 1\}; \rightarrow)$, see [2, Chapter 10]. In fact, the main problem of the theory of natural dualities, the *Dualisability Problem*, remains still open:

Problem 2.2. (Dualisability Problem) *Which finite algebras are dualisable?*

At present, the Dualisability Problem seems to be unsolvable (cf. [36, page viii]). There are algebras $\mathbf{M}$ which fail to be dualizable (we refer to [26] or [4]). However, for a very wide range of algebras dualities do exist. For example, the NU-Duality Theorem ([26], Theorem 1.18 or [4], Theorem 2.8) guarantees that a duality on $\mathbb{ISP}(\mathbf{M})$ is available whenever $\mathbf{M}$ has a lattice reduct. Many further theorems which say how to choose an appropriate structure $\underset{\sim}{M}$ on M to obtain a duality, or a strong (thus full) duality, on $\mathbb{ISP}(\mathbf{M})$ can be found in [2] and in [36]. The Dualisability Problem might be formally undecidable, and in fact, the "holy grail" (cf. [36, page viii]) of some natural-duality theoreticians is the Decidability Problem for Dualisability:

Problem 2.3. (Decidability Problem for Dualisability) *Is there an algorithm for deciding whether or not any given finite algebra is dualisable?*

3 Entailment in natural dualities and our solution of the Entailment problem

Again assume a structure $\underset{\sim}{M} = (M; G, H, R, \mathcal{T})$ is algebraic over a finite algebra $\mathbf{M}$ and let r be an n-ary algebraic relation on M (i.e. a subalgebra of $\mathbf{M}^n$). We say that the structure $\underset{\sim}{M}$, or more often just $G \cup H \cup R$, *entails* r if for every $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$, each $\boldsymbol{\mathfrak{X}}$-morphism $\varphi\colon \mathbf{X} \to \underset{\sim}{M}$ preserves r; we write $G \cup H \cup R \vdash r$. For relations r and s we write $r \vdash s$ in place of $\{r\} \vdash s$. We say that $G \cup H \cup R$ entails an n-ary (partial) operation h if it entails its graph as an $(n + 1)$-ary relation, and that it entails a set R' of relations and (partial) operations if it entails each $r \in R'$.

3.1 Test Algebra Lemma and the Entailment problem

Central to the identification of the relations entailed from certain set $G \cup H \cup R$ is so-called Test Algebra Lemma. (It is formulated in entailment terms in [25], Lemma 2.3 and in [2], Lemma 8.1.3.) We present this statement and we notice that $\mathbf{s}$ always denotes the algebraic relation s considered as an algebra in $\boldsymbol{\mathcal{A}}$.

Theorem 3.1. (Test Algebra Lemma) Let $\mathbf{M}$ be a finite algebra, let G, H, R be, respectively, sets (possibly empty) of operations, partial operations and relations which are algebraic over $\mathbf{M}$, and let s be an algebraic relation. Then the following are equivalent:

(1) $G \cup H \cup R$ entails s;

(2) $G \cup H \cup R$ entails s on $D(\mathbf{s})$.

Moreover, $G \cup H \cup R$ entails s whenever $G \cup H \cup R$ yields a duality on $\mathbf{s}$.

We often use the term *test algebra* for an algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ witnessing the failure of the structure $\underset{\sim}{M}$ to yield a duality on $\mathbb{ISP}(\mathbf{M})$.

It is important that provided a set $G \cup H \cup R$ yields a duality on $\boldsymbol{\mathcal{A}}$ then the duality is not destroyed by deleting from $G \cup H \cup R$ any element which is entailed by the remaining members. This is the key to obtaining so-called *economical dualities* which are easy to work with. A full discussion of the central role played by entailment in duality theory is presented in the paper [17]. In this paper we solved the Entailment Problem of duality theory that was formulated as follows:

Problem 3.2. (Entailment Problem) *Find an intrinsic description of the relations entailed by $G \cup H \cup R$.*

This problem was formulated as the first open problem of the natural dualities in the famous survey paper [4]. When this problem was firstly introduced, it was expected that the solution would be a semantic one in terms of a preservation theorem providing a list of finitary constructs which preserve entailment. By this is meant that if $(G \cup H \cup R) \vdash s$ then s would be obtainable from the set $G \cup H \cup R$ via a finite sequence of finitary constructs. In our solution to the problem in [17] we indeed firstly discovered a semantic solution, which was similar to the characterisation of the well-known *clone closure* Inv(Pol(R)) of a set of relations R (all 'invariants' of 'polymorphisms' preserving R) originally obtained in the famous pair of papers [1] by V. Bodnarčuk, L.A. Kalužnin, V.N. Kotov and B.A. Romov. Later on, we noticed that our semantic solution also arises as a direct application of a syntactic solution: a description of relations entailed by $G \cup H \cup R$ in terms of the first-order formulæ of the language with equality, $\boldsymbol{\mathcal{L}}_{\underset{\sim}{M}}$, associated with $\underset{\sim}{M}$. An important step towards the solution was the recognition that on a given set Ω of finitary algebraic relations on $\mathbf{M}$ the map $R \longmapsto \overline{R} := \{\, s \in \Omega \mid R \vdash s \,\}$ is a closure operator (*entailment closure*). And also the recognition that this closure operator is algebraic, in the sense that the closure of any set R is the union of the closures of its finite subsets (so that the lattice of closed sets is algebraic). This provided indirect evidence for a positive solution to the Entailment Problem.

3.2 Our syntactic solution of the Entailment problem

In [25] the important fact that entailment closure is algebraic was deduced as a corollary of the Test Algebra Lemma. In the paper [17] we extended the Test Algebra Lemma, upgrading it to the Test Algebra Theorem. This theorem provides our syntactic solution to the Entailment Problem:

Theorem 3.3. (The Test Algebra Theorem or Entailment in the duality sense) Let $\mathbf{M}$ be a finite algebra and let a structure $\underset{\sim}{M} = (M; G, H, R, \mathfrak{T})$ be algebraic over $\mathbf{M}$. Then the following are equivalent:

(1) $G \cup H \cup R$ entails s;

(2) $G \cup H \cup R$ entails s on $D(\mathbf{s})$;

(3) some finite subset of $G \cup H \cup R$ entails s on $D(\mathbf{s})$;

(4) $s = \{\, (u(\rho_1), \dots, u(\rho_n)) \mid u : D(\mathbf{s}) \to M$ preserves $G \cup H \cup R \,\}$;

(5) there exists a primitive positive formula $\Phi(x_1, \dots, x_n)$ in the language $\boldsymbol{\mathcal{L}}_{\underset{\sim}{M}}$ such that

  (i) $D(\mathbf{s}) \vdash \Phi(\rho_1, \dots, \rho_n)$ and
  (ii) $s = \{\, (c_1, \dots, c_n) \in M^n \mid M \vdash \Phi(c_1, \dots, c_n) \,\}$.

The most important part of our syntactic solution is that $(G \cup H \cup R) \vdash s$ if and only if there is a primitive positive formula Φ in the language $\mathcal{L}_{\underset{\sim}{M}}$ such that s may be obtained from $G \cup H \cup R$ via a *primitive positive construct.* We may take Φ to be the primitive positive type of $\rho_1, \ldots, \rho_n$ in $D(\mathbf{s})$.

In duality theory, a set R of finitary algebraic relations on a finite algebra $\mathbf{M}$ entails a finitary algebraic relation s on the powers of $\underset{\sim}{M}$ (which are the duals of free algebras in the associated quasivariety $\boldsymbol{\mathcal{A}}$; see, for example, [26]) if and only if s can be obtained from R in the clone-theoretic case.

Therefore applying our results in the clone setting we derive a famous consequence due to V. Bodnarčuk, L.A. Kalužnin, V.N. Kotov and B.A. Romov [1]:

Theorem 3.4. (Entailment in the clone sense) Let R be a family of finitary relations on a finite set M and let $s \subseteq M^n$. Then the following are equivalent:

(1) $s \in \mathrm{Inv}(\mathrm{Pol}(R))$;

(2) R entails s on M^s;

(3) $s = \{(u(\rho_1), \ldots, u(\rho_n)) \mid u : M^s \to M \text{ preserves } R\}$;

(4) there is some finite structure Z of type $(M; R)$ and elements $z_1, \ldots, z_n \in Z$ such that $s = \{\, (u(z_1), \ldots, u(z_n)) \mid u : Z \to M \text{ preserves } R\,\}$;

(5) $s = \{\, (c_1, \ldots, c_n) \in M^n \mid M \vdash \Phi(c_1, \ldots, c_n)\,\}$ for some primitive positive formula $\Phi(x_1, \ldots, x_n)$ (in the language of the relational structure $(M; R)$).

3.3 Our semantic solution of the Entailment problem

Through the Test Algebra Theorem we are able to convert our syntactic solution to the Entailment Problem to a semantic solution, so obtaining a set of constructs sufficient to describe entailment. We only summarise the results below and sketch the main steps of our semantic solution while for all details of it and definitions of the constructs we refer to our paper [17] or to [2, 2.4.5 and 9.2.1].

In case $G \cup H = \emptyset$, the list of entailment constructs may be taken to be: *trivial relations*, *repetition removal*, *intersection*, *product*, and *retractive projection* (in which the natural projection map is required to be a retraction). As a consequence in the clone setting we have the result of [1] that $\mathrm{Inv}(\mathrm{Pol}(R))$ can be obtained from R by a finite number of applications of trivial relations, intersection, repetition removal, product and projection.

As is well known, arbitrary projection is not necessarily an allowable construct on structures of the form $D(\mathbf{A}) = \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$. If it were, we could form the relational product of two relations, which is not guaranteed to lift to structures $D(\mathbf{A})$ which are not full powers. This explains why a set R of algebraic relations on $\mathbf{M}$ which determines the clone of term functions on $\mathbf{M}$ will not necessarily yield a duality on $\boldsymbol{\mathcal{A}}$. This is illustrated in [4, p.102] in case $\boldsymbol{\mathcal{A}}$ is the variety $\boldsymbol{\mathcal{K}}$ of Kleene algebras; for a more extended discussion we refer to [25, Section 5] or [19].

Our semantic solution to the Entailment Problem in [17] was carried out in two stages. Firstly, we showed that the second dual $ED(\mathbf{s})$ of an algebraic relation s can be *concretely* constructed from $G \cup H \cup R$, whether or not $G \cup H \cup R$ entails s (for details again see [17] or [2, 2.4.5 and 9.2.1]). Secondly, we showed that if $G \cup H \cup R$ entails s then s can be obtained from this second dual $ED(\mathbf{s})$ by a retractive projection, which is a bijective projection in case $G \cup H \cup R$ yields a duality on $\mathbf{s}$.

To explain the latter concepts, given an m-ary algebraic relation r on M and an injective mapping $\eta : \{1, \ldots, n\} \to \{1, \ldots, m\}$ $(n \leq m)$ we define the relation

$$r_\eta = \{\, (c_1, \ldots, c_n) \in M^n \mid (\exists d_1 \ldots d_m \in M)\ (d_1, \ldots, d_m) \in r \text{ and } c_i = d_{\eta(i)}\ (1 \leq i \leq n)\,\}$$

(it can be alternatively denoted as the projection $P_{\eta(1),\ldots,\eta(n)}(r)$ of r into its coordinates $\eta(1),\ldots,\eta(n)$). Then we say that the relation $s := r_\eta$ is a *retractive projection of* r if the natural projection map $p : \mathbf{r} \to \mathbf{s}$ is a retraction, that is, there is a homomorphism $q : \mathbf{s} \to \mathbf{r}$ such that $p \circ q = \mathrm{id}_s$. It is called a *bijective projection* (as introduced by L. Zadori [45]) if moreover $q \circ p = \mathrm{id}_r$.

Consider G, H and R as before and let now $Z = \{z_1,\ldots,z_k\}$ be a finite substructure of M^T, for some non-empty set T. By the *graph of* $E(Z)$ (with respect to $G \cup H \cup R$) we mean the relation

$$G[E(Z)] := \{\, (u(z_1),\ldots,u(z_k)) \in M^k \mid u : Z \to M \text{ preserves } G \cup H \cup R \,\}.$$

Thus the graph of $E(Z)$ is simply $E(Z)$, given a fixed labelling of Z. We showed that if Z is a finite subset of M^T for some non-empty set T which is *hom-closed* (for details see [2, p. 66]), then the relation $G[E(Z)]$ can be concretely constructed from $G \cup H \cup R$.

For an n-ary algebraic relation s we take $Z := D(\mathbf{s})$ to be the dual of the algebra $\mathbf{s}$ and enumerate its elements as $\{\rho_1,\ldots,\rho_n,\mathcal{T}_1,\ldots,\mathcal{T}_m\}$. We then assume that

$$G[\mathbf{s}] := \{\, (\rho_1(a),\ldots,\rho_n(a),\mathcal{T}_1(a),\ldots,\mathcal{T}_m(a)) \in M^{n+m} \mid a \in s \,\}$$

encode the evaluation maps from $D(\mathbf{s})$ to M. It is evident that $G[\mathbf{s}]$ is in bijective correspondence with s itself. Now we have that if $G \cup H \cup R$ yields a duality on $\mathbf{s}$ then $G[ED(\mathbf{s})]$ necessarily coincides with $G[\mathbf{s}]$. It is helpful to employ the intuition that the relation $G[ED(\mathbf{s})] \setminus G[\mathbf{s}]$ can be thought of as a measure of how far $G \cup H \cup R$ is from yielding a duality on $D(\mathbf{s})$.

Since by the Test Algebra Theorem we have that an algebraic relation s is the retractive projection of $G[ED(\mathbf{s})]$ onto its first n coordinates, where the dual $D(\mathbf{s})$ of s is labelled as above, we immediately have:

Lemma 3.5. Let $\mathbf{s} \le \mathbf{M}^n$ and $G \cup H \cup R$ entail s. Then s is a retractive projection of the graph $G[ED(\mathbf{s})]$ of $ED(\mathbf{s})$.

A number of consequences can be deduced. The first is the desired Semantic Entailment Theorem of [17]:

Theorem 3.6. (Semantic Entailment Theorem) Let R be a set of algebraic relations on a finite set M, let s be an algebraic relation on M and let $R \vdash s$. Then s can be obtained from R by a finite number of applications of product, intersection, trivial relations and repetition removal, followed by one application of retractive projection.

If a set R of algebraic relations on a finite set M is such that $R \vdash s$ for every algebraic relation s on M, then we say that R is *entailment-dense*. The following result, that can be derived from our semantic solution, was (independently to our investigations) discovered by L. Zádori [45]:

Theorem 3.7. (Special Semantic Entailment Theorem) Let R be a set of algebraic relations on a finite set M and let s be an algebraic relation on M.

(a) If R yields a duality on $\mathbf{s}$, then s can be constructed from R by a finite number of applications of product, intersection, trivial relations, repetition removal and bijective projection.

(b) The following are equivalent:

  (i) R yields a duality on every finite algebra in $\boldsymbol{\mathcal{A}}$;

(ii) R is entailment-dense;

(iii) every algebraic relation s on M can be constructed from R by a finite number of applications of product, intersection, trivial relations, repetition removal and bijective projection.

4 Endoprimality and endodualisability in theory and practice

The relationship between duality entailment and clone-entailment is rather complex. It is known that it is possible for $G \cup H \cup R$ to clone-entail every finite algebraic relation on $\mathbf{M}$ but to fail to dualise $\mathbf{M}$, but the circumstances under which this phenomenon occurs, and what it signifies, are still obscure. In particular, we may ask what it means for $\mathbf{M}$ to be *endoprimal* but not *endodualisable* (we refer to definitions of these concepts below). More explicitly, we may ask what it means for some finitary algebraic relation r on $\mathbf{M}$ to be clone-entailed but not entailed by (the graphs of) the endomorphisms of $\mathbf{M}$. From a semantic viewpoint, a clear difference can be seen: clone-entailment allows all relational products, whereas duality entailment allows only *homomorphic relational products* (for details see [17] or [2, 9.2.1]). Thus one may expect relational products appearing in the construction of r from the endomorphisms of $\mathbf{M}$ to be non-homomorphic relational products. Exactly how this behaviour happens in general is not clear.

4.1 Endoprimality versus endodualisability

In [19] we showed that the relationship between the two entailment concepts also lies at the heart of the relationship between endoprimality and endodualisability. This was nicely demonstrated by the Kleene algebra examples. We note that Kleene algebras were already known to illustrate the distinction between entailment in the clone sense and in the duality sense - we refer to [4, p. 87], [25, Section 5] and [2, pp. 272–273]. In [19] we gave a complete description of endodualisable and endoprimal finite Kleene algebras from the quasi-variety $\mathbb{ISP}(\mathbf{4})$ and showed that there was a plentiful supply of finite Kleene algebras which were endoprimal but not endodualisable.

Let $\mathbf{M} = (M; F)$ be any algebra. The algebra $\mathbf{M}$ is called *k-endoprimal* ($k \geq 1$) if every k-ary End($\mathbf{M}$)-preserving function on $\mathbf{M}$ is a term function of $\mathbf{M}$. Algebras which are k-endoprimal for every $k \geq 1$ are called *endoprimal.* A finite algebra $\mathbf{M}$ is *endodualisable* if End($\mathbf{M}$) yields a duality on the quasivariety $\mathbb{ISP}(\mathbf{M})$.

The relationship between endodualisability on one hand, and endoprimality and k-endoprimality on the other hand, was explored, successively, in [18], [5], [23], [32] and [19]. It was shown that in many quasivarieties a finite algebra is endoprimal if and only if it is endodualisable (we refer to [23], [33] and the papers cited therein).

In [18] we started an intensive study of a general relationship between endodualisability and endoprimality by the following result:

Theorem 4.1. (Endoprimality versus endodualisability for distributive lattices) Let $\mathbf{L} = (L; \vee, \wedge)$ be a finite non-trivial distributive lattice. The following are equivalent:

(1) $\mathbf{L}$ is 3-endoprimal;

(2) $\mathbf{L}$ is endoprimal;

(3) $\mathbf{L}$ is endodualisable;

(4) the retractions of $\mathbf{L}$ onto $\{0, 1\}$ together with the constants $0, 1$ yield a duality on $\mathbb{ISP}(\mathbf{L})$;

(5) $\mathbf{L}$ is not a Boolean lattice.

In case of bounded distributive lattices we obtained a similar result, the only difference is in Condition (1):

Theorem 4.2. (Endoprimality vs endodualisability for bounded distributive lattices) Let $\mathbf{L} = (L; \vee, \wedge, 0, 1)$ be a finite non-trivial bounded distributive lattice. The following are equivalent:

(1) $\mathbf{L}$ is 1-endoprimal;

(2) $\mathbf{L}$ is endoprimal;

(3) $\mathbf{L}$ is endodualisable;

(4) the retractions of $\mathbf{L}$ onto $\{0, 1\}$ together with the constants $0, 1$ yield a duality on $\mathbb{ISP}(\mathbf{L})$;

(5) $\mathbf{L}$ is not a Boolean lattice.

The first examples of finite algebras which are endoprimal but not endodualisable were found by B.A. Davey and J.G. Pitkethly in their paper [23], among algebras with a semilattice reduct. Many other such examples have been found among Kleene algebras in our paper [19].

4.2 A criterion for a finite endoprimal algebra to be endodualisable

In the paper [32] the strategy for finding endoprimal algebras due to B.A. Davey and J.G. Pitkethly [23] is further explored in the finite case. A new theoretical tool, called the *Retraction Test Algebra Lemma*, is used to show that, in many quasivarieties, endoprimality is equivalent to endodualisability for finite algebras which are suitably related to finitely generated free algebras. The main result of [32] is the following theorem.

Theorem 4.3. (Retraction Test Algebra Lemma) Let a finite algebra $\mathbf{D}$ be dualisable via the structure

$$\underset{\sim}{D} = (D; \mathrm{End}(\mathbf{D}), s_1, \dots, s_m, \mathcal{T})$$

where $m \geq 1$ and $s_1, \dots, s_m$ are finitary algebraic relations on $\mathbf{D}$. Let the algebras $\mathbf{s}_1, \dots, \mathbf{s}_m$ be retracts of the k-generated free algebra $\mathbf{F}_{\mathfrak{D}}(k) \in \mathfrak{D}$ where $\mathfrak{D} = \mathbb{ISP}(\mathbf{D})$.

Then for any finite algebra $\mathbf{M} \in \mathfrak{D}$ which has $\mathbf{D}$ as a retract the following are equivalent:

(1) $\mathbf{M}$ is endoprimal;

(2) $\mathbf{M}$ is k-endoprimal;

(3) $\mathbf{M}$ is endodualisable.

The result can be applied to the (quasi-)varieties of distributive lattices (with $k = 3$), bounded distributive lattices ($k = 1$), finite vector spaces of dimension greater than one ($k = 2$), Stone algebras ($k = 2$), abelian groups ($k = 2$), sets ($k = 3$), semilattices ($k = 3$), lower-bounded semilattices ($k = 2$) and median algebras ($k = 3$), which have not been considered before as regards endoprimality.

We explain the applications of our theorem above in several selected cases:

Distributive lattices

The class $\mathfrak{D}$ of distributive lattices is the quasi-variety $\mathbb{ISP}(\mathbf{2})$ generated by the 2-element lattice $\mathbf{2} = (\{0, 1\}; \vee, \wedge)$. It is well-known (by *Priestley duality* presented in [41], [42]) that $\mathbf{2}$ is dualisable via the structure $\underset{\sim}{2} = (\{0, 1\}, 0, 1, \leq, \mathcal{T})$ where $\leq$ is the usual order on $\{0, 1\}$ and the constants 0 and 1 replace the usual unary constant endomorphisms onto 0 and 1, respectively. It is said that $\mathbf{2}$ is *almost endodualisable* with $\leq$ as the extra relation to the endomorphisms in the dualising structure. We notice that $\leq$ is, as a distributive lattice, isomorphic to the 3-element chain $\mathbf{3}$.

It is easy to check that the free algebras $\mathbf{F}_{\mathcal{D}}(1) \cong \mathbf{1}$ and $\mathbf{F}_{\mathcal{D}}(2) \cong \mathbf{2}^2$ do not have $\mathbf{3}$ as a retract while the free algebra $\mathbf{F}_{\mathcal{D}}(3)$ does have $\mathbf{3}$ as a retract. All non-trivial distributive lattices $\mathbf{L} \in \mathcal{D}$ have evidently $\mathbf{2}$ as their retracts. From our theorem above it therefore follows that a finite non-trivial distributive lattice $\mathbf{L}$ is endoprimal iff $\mathbf{L}$ is 3-endoprimal iff L is endodualisable.

Stone algebras

The class of Stone algebras is the quasi-variety $\mathbb{ISP}(\mathbf{3})$ generated by the 3-element Stone algebra $\mathbf{3} = (\{0, a, 1\}; \vee, \wedge, ^\star, 0, 1)$ where $\{0, a, 1\}$ is the 3-element chain, $0^\star = 1$, and $a^\star = 1^\star = 0$. It is well known that the structure $\underset{\sim}{3} = (\{0, a, 1\}, d, \preccurlyeq, \mathcal{T})$ yields a duality on the variety of Stone algebras (cf. e.g. [2, p. 105]) where $\preccurlyeq$ is the order $\{(0, 0), (a, a), (1, 1), (1, a)\}$ and $\mathrm{graph}(d) = \{(0, 0), (1, 1), (a, 1)\}$. It means that $\mathbf{3}$ is almost endodualisable with the extra relation $\preccurlyeq$ which is isomorphic to the 4-element chain algebra $\mathbf{4}$ in $\mathcal{S}$. Now the smallest k-generated free algebra in $\mathcal{S}$ having $\mathbf{4}$ as a retract is known to be $\mathbf{F}_{\mathcal{S}}(2)$ (we refer to [29]). Our theorem can be applied to Stone algebras having $\mathbf{3}$ as a retract. The only Stone algebras which do not have $\mathbf{3}$ as a retract are the Boolean algebras (and these are endodualisable). It follows that a finite non-Boolean Stone algebra $\mathbf{L}$ is endoprimal iff $\mathbf{L}$ is 2-endoprimal iff $\mathbf{L}$ is endodualisable.

Median algebras

The class of median algebras is the quasi-variety $\mathcal{M} = \mathbb{ISP}(\mathbf{M})$ generated by the 2-element median algebra $\mathbf{M} = (\{0, 1\}; m)$ in which the ternary (median) operation m satisfies the equations

$$m(x, y, z) = m(y, x, z) = m(y, z, x),\ m(x, x, y) = x$$

and

$$m(m(x, y, z), u, v) = m(x, m(y, u, v), m(z, u, v)).$$

The duality for $\mathcal{M}$ is given by the structure $\underset{\sim}{M} = (\{0, 1\}; ^*, 0, 1, \leq, \mathcal{T})$, where * is the automorphism reversing 0 and 1 and $\leq$ is the usual order on $\{0, 1\}$ (we refer, for example, to [2, p. 103]). It follows that $\mathbf{M}$ is almost endodualisable with the extra relation $\leq$ which can be considered as a median algebra, say $\mathbf{s}$. In our paper [32] we present a verification in terms of natural duals of the fact that the smallest k-generated free algebra in $\mathcal{M}$ which has the algebra $\mathbf{s}$ as a retract is $\mathbf{F}_{\mathcal{M}}(3)$. Because any non-trivial median algebra $\mathbf{L} \in \mathcal{M}$ has $\mathbf{M}$ as a retract it immediately follows from our theorem that a finite non-trivial median algebra $\mathbf{L} \in \mathcal{M}$ is endoprimal iff $\mathbf{L}$ is 3-endoprimal iff $\mathbf{L}$ is endodualisable.

Abelian groups

Our method allows us to identify also the finite endoprimal abelian groups. Starting from a finite abelian group $\mathbf{A}$, one can choose $\mathcal{D}$ and the generator $\mathbf{D}$ of $\mathcal{D}$ in such a way that $\mathbf{A} \in \mathcal{D}$ and $\mathbf{D}$ is a retract of $\mathbf{A}$. This enables us to apply our theorem.

It is well-known that for any finite abelian group $\mathbf{A}$ there is a cyclic group $\mathbf{Z}_m$ such that $\mathbf{A} \in \mathcal{A}_m$ where $\mathcal{A}_m = \mathbb{ISP}(\mathbf{Z}_m)$ and $\mathbf{Z}_m$ is a direct factor, and hence a retract, of $\mathbf{A}$. It was shown in [26] (we also refer to [2, p. 114]) that the structure $\underset{\sim}{Z}_m = (Z_m; +, ^-, 0, \mathcal{T})$ yields a duality on the quasi-variety $\mathcal{A}_m$. This means that $\mathbf{Z}_m$ is almost endodualisable with $\mathrm{graph}(+)$ as the extra relation, which is, as an algebra, isomorphic to $\mathbf{Z}_m^2$. We have $\mathbf{F}_{\mathcal{A}_m}(2) \cong \mathbf{Z}_m^2$. Hence for the finite abelian group $\mathbf{A}$ and the associated quasivariety $\mathcal{A}_m = \mathbb{ISP}(\mathbf{Z}_m)$ we could apply our theorem with $k = 2$. It follows that a finite abelian group $\mathbf{A}$ is endoprimal iff it is 2-endoprimal iff it is endodualisable.

4.3 Endodualisable and endoprimal finite double Stone algebras

In the paper [33] we give a complete characterisation of the endoprimal finite double Stone algebras. In particular, we have shown that all of these algebras are endodualisable, and found in every case the minimum value of k for which k-endoprimality forces endoprimality. Much more work was involved in completing this analysis than that for the other examples considered in the paper [32], and further duality techniques were required.

Let us present a brief outline of the results. An algebra $\mathbf{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ is called a *double Stone algebra* if $(L; \vee, \wedge, ^\star, 0, 1)$ and $(L; \wedge, \vee, ^+, 1, 0)$ are Stone algebras. The double Stone algebras form a variety $\mathfrak{DS} = \mathbb{ISP}(\mathbf{4})$ which is generated by the 4-element chain algebra $\mathbf{4} = (\{0, a, b, 1\}; \vee, \wedge, ^\star, ^+, 0, 1)$ where $0 < a < b < 1$ and

$$1^\star = b^\star = a^\star = 0,\ 0^\star = 1,\ 0^+ = a^+ = b^+ = 1,\ 1^+ = 0.$$

The proper non-trivial subvarieties of $\mathfrak{DS}$ are generated by the subdirectly irreducible subalgebras $\mathbf{2} = \{0, 1\}$ and $\mathbf{3} = \{0, a, 1\}$. The variety $\mathbb{ISP}(\mathbf{2})$ is just the class of Boolean algebras, while $\mathbb{ISP}(\mathbf{3})$ is the variety of regular double Stone algebras, *alias* three-valued Lukasiewicz algebras. An algebra is *proper* precisely when it has $\mathbf{4}$ as a retract. We have to consider separately the algebras in $\mathbb{ISP}(\mathbf{4}) \setminus \mathbb{ISP}(\mathbf{3})$, which we call proper double Stone algebras, and algebras in $\mathbb{ISP}(\mathbf{3})$. Also, a further splitting into cases is necessary, into algebras with non-empty core and algebras with empty core. The *core* of an algebra $\mathbf{L}$ in $\mathfrak{DS}$ is defined to be $K(\mathbf{L}) = \{\, x \in L \mid x^\star = 0,\ x^+ = 1 \,\}$. A finite algebra $\mathbf{L}$ has empty core if and only if $\mathbf{L}$ has $\mathbf{2}$ as a direct factor. It is easily shown that this occurs if and only if $\mathbf{L} \in \mathbb{ISP}(\mathbf{4} \times \mathbf{2})$. Every k-generated free algebra $\mathbf{F}_{\mathfrak{DS}}(k)$ lies in the subquasivariety $\mathbb{ISP}(\mathbf{4} \times \mathbf{2})$.

The finite non-Boolean algebras in the variety $\mathbb{ISP}(\mathbf{3})$ are exactly those of the form $\mathbf{3}^m \times \mathbf{2}^\ell$ $(m \geq 1,\ \ell \geq 0)$. We could set up a duality for $\mathbb{ISP}(\mathbf{3} \times \mathbf{2})$ in which the only non-endomorphism was isomorphic to $\mathbf{3} \times \mathbf{2}^2$.

We proved the following result:

Theorem 4.4. (Endodualisable finite double Stone algebras) Let $\mathbf{L}$ be a finite non-trivial double Stone algebra and express $\mathbf{L}$ as $\mathbf{J} \times \mathbf{2}^\ell$ where $\mathbf{J}$ does not have $\mathbf{2}$ as a factor and $\ell \geq 0$.

Then $\mathbf{L}$ is endodualisable when $\mathbf{L}$ takes one of the forms described below.

(1) $\mathbf{L}$ has non-empty core and $\mathbf{L}$ satisfies the following equivalent conditions:

  (i) $\mathbf{L}$ has $\mathbf{5}$ as a retract;
  (ii) $K(\mathbf{L})$ is a non-Boolean lattice.

(2) $\mathbf{L}$ is proper, $\mathbf{J}$ has $\mathbf{5}$ as a retract and $\ell \geq 2$.

(3) $\mathbf{L}$ is not proper and takes the form $\mathbf{3}^m \times \mathbf{2}^\ell$ where $m \geq 1$ and $\ell \geq 2$.

(4) $\mathbf{L}$ is Boolean.

Let $\mathbf{L}$ be a finite non-trivial and non-Boolean double Stone algebra which is not shown by above theorem to be endodualisable and assume that $\mathbf{L}$ is expressed as $\mathbf{J} \times \mathbf{2}^\ell$ where $\mathbf{J}$ does not have $\mathbf{2}$ as a factor. The following cases arise:

(A) $\mathbf{L}$ is a Post algebra of order 3 (that is, $\mathbf{L}$ is not proper and $\ell = 0$);

(B) $\mathbf{L}$ has a single factor $\mathbf{2}$ (that is, $\ell = 1$);

(C) $\mathbf{L}$ is proper, $K(\mathbf{L}) \neq \emptyset$ (that is, $\ell = 0$), and $\mathbf{J}$ does not have $\mathbf{5}$ as a retract;

(D) $\mathbf{L}$ is proper, $\mathbf{J}$ does not have $\mathbf{5}$ as a retract and $\ell \geq 2$.

We showed that $\mathbf{L}$ is not endodualisable in each of cases (A)–(D), treating these in turn.

Proposition 4.5. (Non-endodualisable finite double Stone algebras, Case A) Let $\mathbf{L}$ be a finite Post algebra of order 3. Then

(1) $\mathbf{L}$ is not endodualisable, with $\mathbf{2}$ serving as a test algebra;

(2) $\mathbf{L}$ is not 1-endoprimal.

Proposition 4.6. (Non-endodualisable finite double Stone algebras, Case B) Let $\mathbf{L} = \mathbf{J} \times \mathbf{2}$ be a finite non-Boolean double Stone algebra with exactly one factor $\mathbf{2}$. Then

(1) $\mathbf{L}$ is not endodualisable, with $\mathbf{2}^2$ serving as a test algebra;

(2) $\mathbf{L}$ is not 1-endoprimal.

For case (C) we showed that the algebra $\mathbf{L}$ is the retract of a power of a finite indecomposable algebra which is not 3-endoprimal.

Proposition 4.7. (Non-endodualisable finite double Stone algebras, Case C) Let $\mathbf{L}$ be a finite proper double Stone algebra with a non-empty core $K(\mathbf{L}) = [a, b]$ $(a < b)$ which is a Boolean lattice. Then $\mathbf{L}$ is not 3-endoprimal (and hence not endodualisable).

Finally we need to consider algebras which have $\mathbf{2}^\ell$ as a factor, where $\ell \geq 2$ (case (D)).

Proposition 4.8. (Non-endodualisable finite double Stone algebras, Case D) Let $\mathbf{L} = \mathbf{J} \times \mathbf{2}^\ell$, where $\mathbf{J} \in \mathbb{ISP}(\mathbf{4}) \setminus \mathbb{ISP}(\mathbf{3})$ is a finite double Stone algebra with a non-trivial Boolean core and $\ell \geq 2$. Then $\mathbf{L}$ is not 3-endoprimal (and so not endodualisable).

To summarise, we identified firstly various endodualisable finite double Stone algebras and then we showed, considering in turn four cases (A)–(D), that there are no other endodualisable finite double Stone algebras. Here we bring our results together.

Theorem 4.9. (Endodualisability for finite double Stone algebras, Summary) Assume that $\mathbf{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ is a finite proper double Stone algebra with a non-empty core $K(\mathbf{L}) = [a, b]$ $(a < b)$. Then the following are equivalent:

(1) $\mathbf{L}$ is endodualisable;

(2) $\mathbf{L}$ is endoprimal;

(3) $\mathbf{L}$ is 3-endoprimal;

(4) $\mathbf{5}$ is a retract of $\mathbf{L}$;

(5) the core $K(\mathbf{L})$ is a non-Boolean lattice.

For proper double Stone algebras with empty core we have the following theorem.

Theorem 4.10. Let $\mathbf{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ be a finite proper double Stone algebra with empty core. Then the following are equivalent:

(1) $\mathbf{L}$ is endodualisable;

(2) $\mathbf{L}$ is endoprimal;

(3) $\mathbf{L}$ is 3-endoprimal;

(4) $\mathbf{5} \times \mathbf{2}^2$ is a retract of $\mathbf{L}$.

For algebras in $\mathbb{ISP}(\mathbf{3})$ we have, likewise, the following result.

Theorem 4.11. Let $\mathbf{L}$ belong to the variety $\mathcal{R} = \mathbb{ISP}(\mathbf{3})$ of regular double Stone algebras and assume that $\mathbf{L}$ is not Boolean. Then the following are equivalent:

(1) $\mathbf{L}$ is endodualisable;

(2) $\mathbf{L}$ is endoprimal;

(3) $\mathbf{L}$ is 1-endoprimal;

(4) $\mathbf{3} \times \mathbf{2}^2$ is a retract of $\mathbf{L}$.

We record explicitly the following theorem, which is a corollary of our preceding results.

Corollary 4.12. A finite double Stone algebra is endoprimal if and only if it is endodualisable.

5 Full versus Strong Problem in the theory of natural dualities

Every quasi-variety of the form $\mathcal{A} = \mathbb{ISP}(M)$, where $\mathbf{M}$ is a finite lattice-based algebra, has a natural duality. In the case that M is distributive-lattice based, it is possible to use the *restricted Priestley duality* and the natural duality for $\mathcal{A}$ simultaneously. In tandem, these dualities can provide an extremely powerful tool for the study of $\mathcal{A}$: see Clark and Davey [2, Chapter 7]. As well as being a natural area of application of natural duality theory, distributive-lattice-based algebras in general, and distributive lattices in particular, have provided deep insights into the general theory. Important examples have been Heyting algebras, particularly the finite Heyting chains, and Kleene algebras; but here we firstly concentrate on the three-element bounded distributive lattice

$$\mathbf{3} = (\{0, d, 1\}; \vee, \wedge, 0, 1),$$

which was seminal in developments that led to the solution of the *Full versus Strong Problem*, one of the most tantalizing problems in the theory of natural dualities.

5.1 The seminal example of the three-element chain

For a natural-duality viewpoint, Priestley duality for the class $\mathcal{D}$ of bounded distributive lattices is obtained via homsets based on the two-element chain $\mathbf{2}$ and uses the fact that $\mathcal{D} = \mathbb{ISP}(\mathbf{2})$. By using the fact that $\mathcal{D} = \mathbb{ISP}(\mathbf{3})$, in [18] we introduced the following modified Priestley duality for $\mathcal{D}$ as a natural duality based on $\mathbf{3}$. Let f, g be the non-identity endomorphisms of 3 (see Figure 1) and let

$$\underset{\sim}{3} = (\{0, d, 1\}; f, g, \mathcal{T}),$$

where $\mathcal{T}$ is the discrete topology $\mathcal{T}$.

Let $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{3})$ be the class of all isomorphic copies of closed substructures of non-zero powers of $\underset{\sim}{3}$.

In [18] we showed that such a modified Priestley duality for $\mathcal{D}$, in which the order is replaced by endomorphisms, can be based on any finite non-boolean distributive lattice $\mathbf{M}$. We also showed that, while the order relation cannot be removed in the boolean case, it can at least be replaced by any finitary relation on $\mathbf{M}$, which itself, like the order on $\mathbf{2}$, forms a non-boolean lattice.

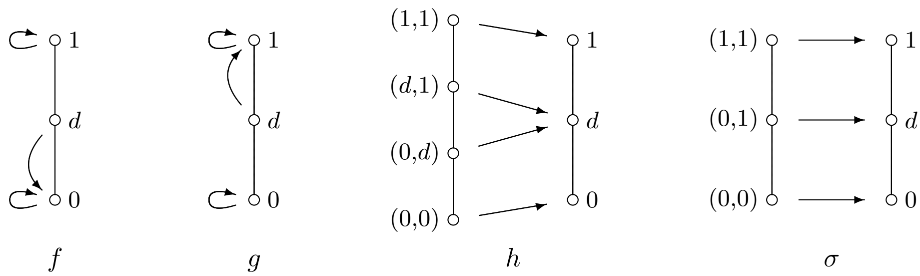

Figure 1. The (partial) operations f, g, h and σ on 3

In [9] we studied the enrichment of $\underset{\sim}{3}$ given by

$$\underset{\sim}{3}_\sigma := (\{0, d, 1\}; f, g, \sigma, \mathcal{T}),$$

and in [21] we explored deeply the enrichments $\underset{\sim}{3}_\sigma$ and

$$\underset{\sim}{3}_h := (\{0, d, 1\}; f, g, h, \mathcal{T}).$$

(The binary partial operations h and σ are also given in Figure 1.) If in the above scheme for the modified Priestley duality for $\mathfrak{D}$ based on $\mathbf{3}$ the alter ego $\underset{\sim}{3}$ of $\mathbf{3}$ is replaced with the alter ego $\underset{\sim}{3}_\sigma$, then not only the map $e_A : \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathfrak{D}$, establishing a duality between $\mathfrak{D} = \mathbb{ISP}(\mathbf{3})$ and $\mathfrak{X}_\sigma = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{3}_\sigma)$, but moreover the map $\varepsilon_X : \mathbf{X} \to DE(\mathbf{X})$ is an isomorphism, for all $\mathbf{X} \in \mathfrak{X}_\sigma$, establishing a full duality between $\mathfrak{D}$ and $\mathfrak{X}_\sigma$. If the hom-functors D, E are restricted to the categories $\mathfrak{A}_{\text{fin}}$ and $\mathfrak{X}_{\text{fin}}$ of finite members of $\mathfrak{A}$ and $\mathfrak{X}$ only, then the concepts of a *finite-level* duality, full duality or strong duality are obtained.

The properties of the modified Priestley dualites for $\mathfrak{D}$ based on 3 given by the alter egos $\underset{\sim}{3}$, $\underset{\sim}{3}_h$ and $\underset{\sim}{3}_\sigma$ are summarized in the following theorem.

Theorem 5.1. Let $\underset{\sim}{3}$, $\underset{\sim}{3}_h$ and $\underset{\sim}{3}_\sigma$ be the alter egos of $\mathbf{3}$ defined above.

(i) $\underset{\sim}{3}$ yields a duality on $\mathfrak{D}$. (Davey, Haviar, Priestley [18])

(ii) $\underset{\sim}{3}_h$ yields a full duality, which is not strong, on the category $\mathfrak{D}_{\text{fin}}$ and yields a duality, which is not full, on the category $\mathfrak{D}$. (Davey, Haviar, Willard [21])

(iii) $\underset{\sim}{3}_\sigma$ yields a strong duality for $\mathfrak{D}$. (Davey, Haviar [9])

(iv) Every full duality on $\mathfrak{D}$ based on $\mathbf{3}$ is strong. (Davey, Haviar, Willard [21])

5.2 Full versus Strong Problem: its local versions and when full implies strong

Since the Full versus Strong Problem in its global version had remained open for twenty-five years, we introduced in [13] local versions of this problem that could prove more tractable and fruitful.

Problem 5.2. For an arbitrary finite algebra $\mathbf{M}$ in your favourite class $\mathfrak{C}$ of algebras, is every full duality based on $\mathbf{M}$ necessarily strong?

We also posed the finite-level version of Problem 5.2.

Problem 5.3. For an arbitrary finite algebra $\mathbf{M}$ in your favourite class $\mathfrak{C}$ of algebras, is every duality based on $\mathbf{M}$ that is full at the finite level necessarily strong at the finite level?

The first solutions to these local versions of the Full versus Strong Problem were given for full dualities based on the three-element chain in the variety of bounded distributive lattices in our paper [21] (as shown in the previous subsection). The answer was shown to be affirmative to Problem 5.2 and negative to Problem 5.3. In [13] we provided affirmative answers to Problems 5.2 and 5.3 for full dualities based on an arbitrary finite algebra in three varieties of algebras: abelian groups, semilattices (with or without bounds) and relative Stone Heyting algebras. We also developed some general conditions under which 'full implies strong' that had the potential to add to the list of solutions. Finally, we answered Problem 5.2 in the affirmative for full dualities based on an arbitrary finite lattice in the variety of bounded distributive lattices.

There is a further, weaker version of Problem 5.2, which deserves to be recorded here.

Problem 5.4. In your favourite class $\mathcal{C}$ of algebras, is every fully dualisable finite algebra necessarily strongly dualisable?

It should be noted that the finite-level variant of this question makes no sense since every finite algebra $\mathbf{M}$ is strongly dualised at the finite level by the alter ego $\underset{\sim}{M} = \langle M; H, \mathcal{T}\rangle$, where H consists of all finitary algebraic partial operation on $\mathbf{M}$. We found in [13] several sufficient conditions for full to imply strong:

Theorem 5.5. Let $\mathbf{D}$ be a finite algebra, let $\mathbf{M}$ be a finite algebra in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{D})$ such that $\mathbf{D}$ is a subalgebra of $\mathbf{M}$. Assume that $\underset{\sim}{D} = \langle D; G^D, H^D, R^D, \mathcal{T}\rangle$ strongly dualises $\mathbf{D}$ [at the finite level] and that D, each relation $r \in R^D$, and $\mathrm{dom}(h)$, for all $h \in H^D$, is an intersection of equalizers of pairs of algebraic total operations on $\mathbf{M}$. Then any alter ego $\underset{\sim}{M}$ that fully dualises $\mathbf{M}$ [at the finite level] strongly dualises $\mathbf{M}$ [at the finite level].

When $R^D = \varnothing$ there is a particularly satisfying simplification of this result that involves assumptions on $\mathbf{D}$ only. We say that $\mathbf{D}$ is a *subretract* of $\mathbf{M}$ if $\mathbf{D}$ is a subalgebra of $\mathbf{M}$ and there is a *retraction* of $\mathbf{M}$ onto $\mathbf{D}$, that is, a homomorphism $\omega : \mathbf{M} \to \mathbf{D}$ with $\omega \restriction D = \mathrm{id}_D$.

Theorem 5.6. Let $\mathbf{D}$ be a finite algebra and let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{D})$. Assume that $\underset{\sim}{D} = \langle D; G^D, H^D, \mathcal{T}\rangle$ strongly dualises $\mathbf{D}$ [at the finite level] and that, for all $h \in H^D$, the set $\mathrm{dom}(h)$ is an intersection of equalizers of pairs of algebraic total operations on $\mathbf{D}$. Let $\mathbf{M}$ be a finite algebra in $\boldsymbol{\mathcal{A}}$ such that $\mathbf{D}$ is a subretract of $\mathbf{M}$. Then any alter ego $\underset{\sim}{M}$ that fully dualises $\mathbf{M}$ [at the finite level] strongly dualises $\mathbf{M}$ [at the finite level].

The version of Theorem 5.6 that applies when $\underset{\sim}{D}$ is a total algebra turned out to be so striking that we stated it as a separate result:

Theorem 5.7. Let $\mathbf{D}$ be a finite algebra, let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{D})$ and let $\mathbf{M}$ be a finite algebra in $\boldsymbol{\mathcal{A}}$ that has $\mathbf{D}$ as a subalgebra. Assume that $\underset{\sim}{D} = \langle D; G^D, \mathcal{T}\rangle$ is a total algebra that strongly dualises $\mathbf{D}$ [at the finite level]. If $\underset{\sim}{M}$ is an alter ego of $\mathbf{M}$ that fully dualises $\mathbf{M}$ [at the finite level], then $\underset{\sim}{M}$ strongly dualises $\mathbf{M}$ [at the finite level].

Also we presented the following special case of Theorem 5.5:

Theorem 5.8. Let $\mathbf{D}$ be a finite algebra. Assume that $\underset{\sim}{D} = \langle D; G^D, H^D, R^D, \mathcal{T}\rangle$ strongly dualises $\mathbf{D}$ [at the finite level] and that each relation $r \in R^D$, and $\mathrm{dom}(h)$, for all $h \in H^D$, is an intersection of equalizers of pairs of algebraic total operations on $\mathbf{D}$. Then any alter ego that fully dualises $\mathbf{D}$ [at the finite level], strongly dualises $\mathbf{D}$ [at the finite level].

We then applied Theorem 5.7 to show that Questions 5.2 and 5.3 have affirmative answers for arbitrary finite algebras in the varieties of abelian groups and semilattices.

Abelian groups Let $\mathbf{M} = \langle M; +, ^-, 0\rangle$ be a finite non-trivial abelian group. Then there is a cyclic subgroup $\mathbf{D}$ of $\mathbf{M}$ such that $\mathbf{D}$ is a direct factor of $\mathbf{M}$ and such that $\mathbf{D}$ and $\mathbf{M}$ generate the same quasi-variety $\boldsymbol{\mathcal{A}}$. Since the total algebra $\underset{\sim}{D} = \langle D; +, ^-, 0, \mathfrak{T}\rangle$ yields a strong duality on $\boldsymbol{\mathcal{A}}$ based on $\mathbf{D}$ (see [2, 4.4.2]), we may apply Theorem 5.7 to obtain that every alter ego $\underset{\sim}{M}$ that fully dualises the finite abelian group $\mathbf{M}$ [at the finite level] also strongly dualises $\mathbf{M}$ [at the finite level]. Hence the answers to Questions 5.2 and 5.3 in the variety of abelian groups are always in the affirmative.

Semilattices Let $\mathbf{D}_K = \langle\{0,1\}; \vee, K\rangle$ be the two-element semilattice with possible bounds $K \subseteq \{0,1\}$, let $\boldsymbol{\mathcal{S}}_K := \mathbb{ISP}(\mathbf{D}_K)$ and let $\mathbf{S}$ be a finite non-trivial semilattice in $\boldsymbol{\mathcal{S}}_K$. We have the following strong dualities on $\boldsymbol{\mathcal{S}}_K := \mathbb{ISP}(\mathbf{D}_K)$ based on $\mathbf{D}_K$ given by total algebras.

(i) $\underset{\sim}{D} := \langle\{0,1\}; \vee, 0, 1, \mathfrak{T}\rangle$ yields a strong duality on $\boldsymbol{\mathcal{S}}$ based on the (unbounded) semilattice $\mathbf{D} = \langle\{0,1\}; \vee\rangle$.

(ii) $\underset{\sim}{D}_0 = \langle\{0,1\}; \vee, 0, \mathfrak{T}\rangle$ yields a strong duality on $\boldsymbol{\mathcal{S}}_0$ based on the semilattice with zero $\mathbf{D}_0 = \langle\{0,1\}; \vee, 0\rangle$.

(iii) $\underset{\sim}{D}_1 = \langle\{0,1\}; \vee, 1, \mathfrak{T}\rangle$ yields a strong duality on $\boldsymbol{\mathcal{S}}_1$ based on the semilattice with one $\mathbf{D}_1 = \langle\{0,1\}; \vee, 1\rangle$.

(iv) $\underset{\sim}{D}_{01} = \langle\{0,1\}; \vee, \mathfrak{T})$ yields a strong duality on $\boldsymbol{\mathcal{S}}_{01}$ based on the bounded semilattice $\mathbf{D}_{01} = \langle\{0,1\}; \vee, 0, 1\rangle$.

According to Theorem 5.7, if $\underset{\sim}{M}$ is an alter ego of $\mathbf{S}$ that fully dualises the finite semilattice $\mathbf{S}$ [at the finite level], then $\underset{\sim}{M}$ also strongly dualises $\mathbf{M}$ [at the finite level]. So Questions 5.2 and 5.3 have affirmative answers for arbitrary finite algebras in these varieties of semilattices (with bounds).

Bounded distributive lattices

Let $\boldsymbol{\mathfrak{D}}$ be the variety of bounded distributive lattices. We proved in [13] the following theorem, thereby showing that Question 5.2 has an affirmative answer for an arbitrary finite algebra in the variety of bounded distributive lattices.

Theorem 5.9. Let $\mathbf{M}$ be a finite non-trivial bounded distributive lattice. If $\underset{\sim}{M}$ is an alter ego of $\mathbf{M}$ that yields a full duality on $\boldsymbol{\mathfrak{D}}$ (based on $\mathbf{M}$), then $\underset{\sim}{M}$ yields a strong duality on $\boldsymbol{\mathfrak{D}}$.

5.3 Full versus Strong Problem: related developments and the solution

The realm of natural dualities that were known to be full but not strong at the finite level was for some time a very small one, consisting of a single example. This example, based on the three-element bounded distributive lattice, was presented in our paper [21]. In our other developments, we extended this realm to the class of all natural dualities based on an arbitrary finite non-boolean bounded distributive lattice [14].

The results in [21] raised new questions and opened up new research paths within the field of natural dualities. More precisely, we were led to ask the following questions (cf. [14]):

(a) Could it be that, for a finite algebra that is strongly dualisable, every full duality on the quasi-variety it generates is strong?

(b) What is it about a finite algebra that allows its full dualities at the finite level to behave so differently from its full dualities at the infinite level?

(c) Which finite algebras generate a quasi-variety for which every duality that is full [at the finite level] is necessarily strong?

(d) Which finite algebras have an alter ego that yields a full but not strong duality at the finite level?

As already mentioned, in [13] we proved that, for each finite abelian group, semilattice and relative-Stone Heyting algebra, every duality that is full [at the finite level] is strong [at the finite level], and, for each finite bounded distributive lattice, every full duality is strong. This provided a partial answer to Question (c) and thereby provided examples with which to study Question (b). While Question (a) could be regarded as wild speculation, it was supported by the limited evidence available to us. In order to make headway on questions such as these, we felt we needed a range of examples of finite algebras that possess a full but not strong duality at the finite level.

In the paper [14] we addressed Question (d). More precisely, we proved the following result:

Theorem 5.10. Let $\mathbf{M}$ be a finite non-boolean bounded distributive lattice. Then there is an alter ego $\underset{\sim}{M}$ of $\mathbf{M}$ such that

(a) $\underset{\sim}{M}$ yields a duality that is not full on the class $\mathfrak{D}$ of all bounded distributive lattices, yet

(b) $\underset{\sim}{M}$ yields a duality that is full but not strong on the class of finite bounded distributive lattices.

Hence our Problem 5.3 was shown to have a negative answer in the variety of bounded distributive lattices by producing full but not strong dualities at the finite level based on an arbitrary finite non-boolean lattice.

The authors had hoped to find a conceptual proof of this last theorem that would indicate possible generalizations beyond distributive lattices. A natural approach would be to proceed as follows: let $\mathbf{M}$ be a finite non-boolean bounded distributive lattice; then $\mathbf{M}$ has the three-element chain $\mathbf{3}$ as a retract; in [21] an alter ego $\underset{\sim}{3}$ for $\mathbf{3}$ was given that yields a full but not strong duality at the finite level; use the retraction from $\mathbf{M}$ onto $\mathbf{3}$ to lift the alter ego $\underset{\sim}{3}$ up to an appropriate alter ego $\underset{\sim}{M}$ for $\mathbf{M}$. Unfortunately, this turned out to be too simple minded. We pursued this and many other approaches but to no avail. The hoped-for conceptual proof eluded us and we were left with the direct computational proof presented in [14]. Nevertheless, our result provided an infinite number of desired examples where previously there was only one.

Now, at last, we briefly present the much-seeked solution to the Full versus Strong Problem that was presented by D. M. Clark, B. A. Davey and R. Willard [3].

Let $\mathbf{R} := (\{0, a, b, 1\}; t, \vee, \wedge, 0, 1)$ be the four-element chain with $0 < a < b < 1$ enriched with the ternary discriminator function t. Let u be the partial endomorphism of $\mathbf{R}$ with domain $\{0, a, 1\}$ given by $u(a) = b$. In [3] the authors showed (via three slightly different approaches, found gradually by each of them) that the algebra $\mathbf{R}$ provides a negative solution to the Full versus Strong Problem of the theory of natural dualities:

Theorem 5.11. The alter ego $\underset{\sim}{R}_{\perp} = (\{0, a, b, 1\}; \mathrm{graph}(u), \mathcal{T})$ yields a full but not strong duality on $\mathbb{ISP}(\mathbf{R})$. (Clark, Davey, Willard [3])

In general, a finite algebra $\mathbf{M}$ admits essentially only one finite-level strong duality, but can admit many different finite-level full dualities. The alter egos $\underset{\sim}{M}$ yielding the finite-level full dualities for $\mathbb{ISP}_{\mathrm{fin}}(\mathbf{M})$ form a doubly algebraic lattice $\mathfrak{F}(\widetilde{M})$ introduced and studied in B. A. Davey, J. G. Pitkethly and R. Willard [24]. The following theorem summarises results in this direction.

Theorem 5.12.

(i) $|\mathfrak{F}(\mathbf{M})| = 1$ for any finite semilattice, abelian group or relative Stone Heyting algebra $\mathbf{M}$. (Davey, Haviar, Niven [13])

(ii) $\mathfrak{F}(\mathbf{M})$ is finite for any finite quasi-primal algebra $\mathbf{M}$; in particular, for the algebra $\mathbf{R}$ defined above, $|\mathfrak{F}(\mathbf{R})| = 17$. (Davey, Pitkethly, Willard [24])

(iii) The lattice $\mathfrak{F}(\mathbf{3})$ is non-modular and has size $2^{\aleph_0}$. (Davey, Haviar and Pitkethly [16]).

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