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Recent developments on gracefulness of graphs. A survey complemented with chessboard representations

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Abstract
We present a survey of selected new results about graceful labellings of graphs which were published during the last seven years. Among them a proof of famous Ringel-Kotzig Conjecture from the 1960s, which for "large" trees was announced in February 2020, has a prominent role. Many of the new results are complemented by our own representations of the discovered graceful labellings of graphs via their graph chessboards and labelling tables. The aim of creating these representations has been to provide an extra value of visualization, in particular to allow seeing better a pattern of the graceful labelling in graph chessboards or in labelling sequences.

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1 Introduction

The study of graph labellings started in the late 1960s. Since then a lot of methods and techniques on graph labellings have been studied in almost 3000 research papers, surveys and theses. The best source of information on results concerning the graph labellings is the electronic book *Dynamic Survey of Graph Labeling* by Gallian [10]. Our survey is mainly, though not entirely, based on the information provided in this book.

The history of the study of graph labellings began with a problem on decompositions of a complete graph into trees. In 1963 Ringel conjectured at a conference in Smolenice, Slovakia [40] that for any tree of size m the complete graph K_{2m+1} can be decomposed into $2m + 1$ copies of the given tree. Kotzig conjectured (as far as we know at the same conference) that this decomposition can be cyclic. A proof of the Ringel-Kotzig Conjecture has recently been announced in [32] for *large* trees. (By "large" is meant that the size of the tree is comparable with the size of the complete graph.)

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With the aim to give a better insight to the Ringel-Kotzig Conjecture, in 1965 Rosa in his dissertation [42], and two years later in his seminal paper [43], defined four new labellings of graphs: $\alpha, \beta, \sigma, \rho$. Here α is the strongest and ρ is the weakest labelling. A graph with m edges has a β-labelling if its vertices can be assigned different labels from the set $\{0, 1, ..., m\}$ such that the absolute values of the differences in the vertex labels between adjacent vertices form exactly the set $\{1, ..., m\}$. Later on Golomb [12] called β-labellings *graceful labellings* and the graphs possesing graceful labellings are called *graceful graphs*. The famous *Graceful Tree Conjecture* stated by Rosa in [42] and [43], which implies the Ringel-Kotzig Conjecture, says that *every tree is graceful*, that is, every tree can be gracefully labelled. The conjecture, due to its close relationship with the Ringel-Kotzig Conjecture, which we explain later on, is known also as the Ringel-Kotzig-Rosa Conjecture (see also [32, Conjecture 8.1]).

In this survey of recent developments on gracefulness of graphs we mapped selected new results on gracefully labelled graphs over the last seven years. We divided these results into four sections. The first section relates to the mentioned recent proof of the Ringel-Kotzig Conjecture for *large* trees and explains some background related to it. The second section informs about selected new results on gracefulness of certain trees, among them specific trees of diameter six, spider graphs, symmetrical trees and specific caterpillars and lobsters. The third section focuses on recent results on graceful cyclic graphs such as linear cyclic snakes, certain cycle related graphs, unicyclic graphs and *corona* product of an aster flower graph. The last section is about recent results on graceful subdivisions of selected graphs such as complete bipartite graphs and wheels. We finalize our survey with so-called shell and bow graphs.

Most of the presented results are complemented by our own representations of the given graceful labelling of a graph by its simple chessboard, labelling relation and labelling sequence. They have been created in order to provide the extra value of visualization and to allow seeing better a certain pattern in the graceful labelling. These representations have not been done in each case, only when the corresponding simple chessboards to the graceful graphs have *reasonable* sizes enabling their presentations (considering up to 45 vertices). The diagrams of the presented gracefully labelled graphs were taken from the original papers or created, by applying the formulas for the graceful labellings provided in the papers, with the help of a *Graph processor* – a computer program which was developed by and is presented in Haviar and Ivaška [17, Chapter 7].

2 Preliminaries

We note that all basic concepts and facts in this chapter concerning graphs are taken from [17] and [29].

By a graph in this paper we mean what is called a *simple graph*, that is, an undirected finite graph without loops and multiple edges. To denote the vertex set of some known graph G, we use the symbol V_G and to denote the edge set of some known graph G, we use the symbol E_G.

The *order* of a graph G is the number of vertices in G. The *size* of a graph G is the number of edges in G.

Definition 2.1. ([17, Definition 1.2.1]) A **vertex labelling** f of a graph G is a mapping of its vertex set V_G into the set of non-negative integers (which are called **vertex labels**).

Throughout our survey by a *labelling* we mean a *vertex labelling*. If $f(u), f(v)$ are the labels of vertices u, v respectively, then the number $|f(u) - f(v)|$ will be called an *induced label* of the edge uv in the labelling f. Assigning to every edge $uv \in E_G$ the induced label of the edge uv in the labelling f naturally yields the usual understanding of the labelling f as acting also on the set E_G of the edges of G.

The following two labellings play an important role with respect to the Ringel-Kotzig Conjecture and the Graceful Tree Conjecture.

Definition 2.2. ([17, Definition 1.2.5]) Let G be a graph of size $|E_G| = m$ and let f be its one-to-one labelling. Then f is called a **ρ-labelling** if

1. $f(V_G) \subseteq \{0, 1, \dots, 2m\}$, and
2. $f(E_G) = \{x_1, x_2, \dots, x_m\}$, where $x_i = i$ or $x_i = 2m+1-i$, for all $i \in \{0, 1, \dots, m\}$.

Definition 2.3. ([17, Definition 1.2.3]) Let G be a graph of size m and let f be its one-to-one labelling. Then f is called a **graceful labelling** (in the old terminology a β-labelling) if

1. $f(V_G) \subseteq \{0, 1, \dots, m\}$, and
2. $f(E_G) = \{1, 2, \dots, m\}$.

The Ringel-Kotzig Conjecture ([40], [43]) says:

Conjecture 2.4. (**Ringel-Kotzig Conjecture**): For any tree of size m the complete graph K_{2m+1} has a cyclic decomposition into $2m + 1$ copies of the given tree.

It is important to note that Rosa showed ([42], [43]) that the Ringel-Kotzig Conjecture is equivalent to the existence of the ρ-labelling of every tree.

The *Graceful Tree Conjecture*, which is due to Rosa ([42], [43]) says:

Conjecture 2.5. (**Graceful Tree Conjecture**): All trees are graceful.

In Figure 1 we see an example of a graph with its graceful labelling.

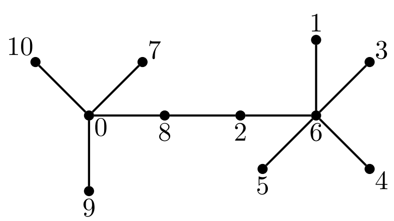


Figure 1. An example of a gracefully labelled graph

Since the ρ-labelling is weaker than the graceful labelling, it follows immediately that the Ringel-Kotzig Conjecture is weaker than the Graceful Tree Conjecture.

In 2016 the authors of [2] proved that the Graceful Tree Conjecture holds asymptotically for trees of maximum degree at most $\frac{n}{\log n}$. Almost all studies on the graph labellings since the 1960s have been devoted to the graceful labellings and to the Graceful Tree Conjecture, and its elder *cousin*, the Ringel-Kotzig Conjecture, and the corresponding ρ-labellings of trees, have received much less attention. However, recent progress has been mainly made on the Ringel-Kotzig Conjecture as we shall see in Section 3.

In [17] the second author of this survey together with his former student Ivaška described the idea that every labelled graph of order n can be visualized by a *simple chessboard* (called also a *graph chessboard* or just a *chessboard*). It is a table with n rows and n columns, in which every edge uv is represented by a pair of dots with coordinates

$[u,v]$ and $[v,u]$. (In Figure 2 we see a graph and its corresponding chessboard.) One can also obtain such a graph chessboard using the adjacency matrix of a graph by placing dots to the cells corresponding to "ones" in the matrix.

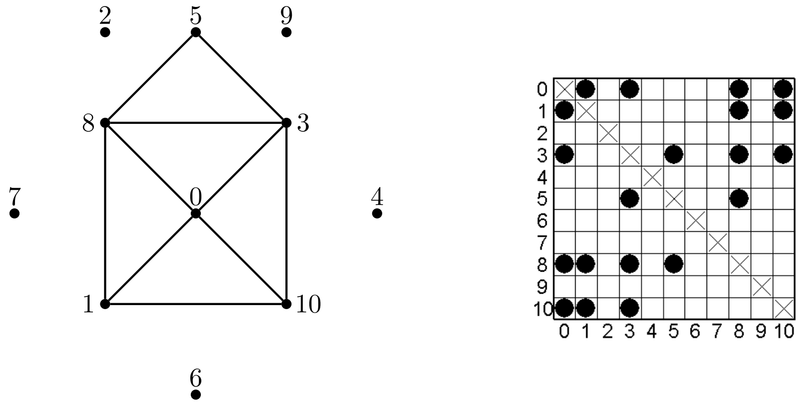


Figure 2. A graph and its corresponding chessboard

Let G be a graph whose vertices are labelled by distinct numbers from the set $\{0,1,2,\ldots,n-1\}$. Consider a chessboard of size n, i.e. table with n rows and n columns. Let the *r-th diagonal* (or the *diagonal with value r*) be the set of all cells with the coordinates $[i,j]$ where $i-j=r$ and $i \geq j$. The 0-th diagonal is called the *main diagonal* of the chessboard and the other diagonals are called *associate diagonals*. We do not need to consider the diagonals "above" the main diagonal, since the chessboard is *symmetric* with respect to the main diagonal.

A simple chessboard will be called *graceful* if there is exactly one dot on each of its associate diagonals.

Example 2.6. In Figure 3 we see a graceful labelling of a graph G and its coresponding chessboard. We can clearly see the gracefulness of the graph because on each of the associate diagonals there is exactly one dot.

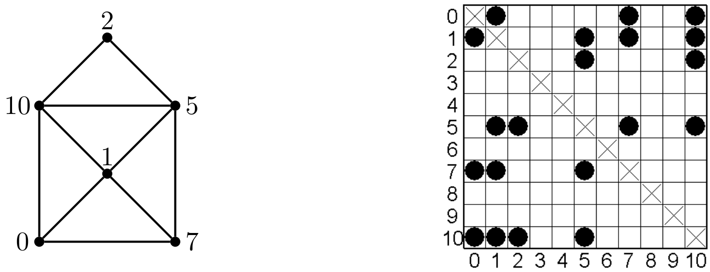


Figure 3. Graceful labelling of graph G and its graceful chessboard

Each gracefully labelled graph can be represented by *labelling sequence*, whose concept was introduced by Sheppard in [55]. He proved there that there is unique correspondence between gracefully labelled graphs and labelling sequences. In [17] Haviar and Ivaška introduced and studied the graph chessboards and *labelling relations*, and showed a one-to-one correspondence between these two concepts and the labelling sequences. Let us now give more details on this.

Definition 2.7. ([55], [17, Definition 3.1.1]) For a positive integer m, the sequence of integers $(j_1, j_2, \dots, j_m)$, denoted (j_i), is a **labelling sequence** if

$$0 \le j_i \le m - i \quad \text{for all } i \in \{1, 2, \dots, m\}. \tag{LS}$$

The labelling sequences can be understood as a tool to encode graceful labellings of graphs. The correspondence between gracefully labelled graphs (without isolated vertices) and the labelling sequences is described in the following theorem.

Theorem 2.8. ([55], [17, Theorem 3.1.2]) There is a one-to-one correspondence between graphs with m edges having a graceful labelling f and between labelling sequences (j_i) of m terms (entries). The correspondence is given by

$$j_i = \min\{f(u), f(v)\}, \quad i \in \{1, 2, \dots, m\},$$

where u, v are the end-vertices of the edge labelled i.

Since the graceful simple chessboards also encode gracefully labelled graphs, it is natural that also the following result holds:

Proposition 2.9. ([17, Proposition 3.1.3]) There is a one-to-one correspondence between all graceful simple chessboards and all labelling sequences.

Now we turn to the concept of a labelling relation which is the third main tool of [17] to encode gracefully labelled graphs.

1	2	3	4	5	6	7	8	9	10
0	5	2	1	5	1	0	2	1	0
1	7	5	5	10	7	7	10	10	10

Figure 4. The labelling table of graph G above

Definition 2.10. ([17, Definition 3.5.1]) Let $L = (j_1, j_2, \dots, j_m)$ be a labelling sequence. Then the relation $A(L) = \{[j_i, j_i + i], i \in \{1, 2, \dots, m\}\}$ is called a **labelling relation** assigned to the labelling sequence L.

From the book [17] we also use the concept of a *labelling table* to visualize a labelling relation (for particular case see Figure 4 above and for a general case see Figure 5 below).

1	2	3	...	m
j_1	j_2	j_3	...	j_m
$j_1 + 1$	$j_2 + 2$	$j_3 + 3$	...	$j_m + m$

Figure 5. Displaying a labelling relation in a table (taken from [17, Figure 3.3])

The table header contains the numbers $1, 2, \dots, m$. The numbers from the labelling sequence are situated in the first row and the sums of numbers from the heading and the first row are in the second row. The pairs from first and second row in each column are then the elements of the labelling relation (and also the edges of the graph).

3 Proof of Ringel's Conjecture for large trees

As we mentioned, the history of the study of graph labellings began with a problem on decompositions of the complete graph into trees. This led to the Ringel's Conjecture that for any tree of size n the complete graph K_{2n+1} can be decomposed into $2n+1$ copies of the given tree [40]. As also mentioned, Kotzig strengthened the conjecture by claiming that this decomposition can be *cyclic*.

In the area of the conjecture only some partial general results have been achieved for almost six decades. As already mentioned, Rosa (cf. [42], [43]) showed that the Ringel-Kotzig Conjecture is equivalent to the existence of the ρ-labelling for any tree. Hence the existence of the stronger graceful labelling for any tree, thus the *Graceful Tree Conjecture*, implies the Ringel-Kotzig Conjecture.

In February 2020, Montgomery, Pokrovskiy and Sudakov published in arXiv a proof of the Ringel-Kotzig Conjecture [32] for *large* trees, where the size of the tree is comparable with the size of the complete graph. In their proof they used a language of *rainbow subgraphs*, which describe the ρ-labellings.

Definition 3.1. ([32, page 2]) A **rainbow copy** of a graph H in an edge-coloured graph G is a subgraph of G isomorphic to H whose edges have different colours.

The Ringel-Kotzig Conjecture is implied by the existence of a rainbow copy of every tree T of size n in a so-called *near distance colouring* of the complete graph K_{2n+1}:

Definition 3.2. ([32, page 2]) Let $\{0, 1, \ldots, 2n\}$ be the vertex set of K_{2n+1}. Colour the edge ij by colour k, where $k \in \{1, \ldots, n\}$, if either $i = j + k$ or $j = i + k$ with addition modulo $2n + 1$. This is called **the near distance (ND) colouring**.

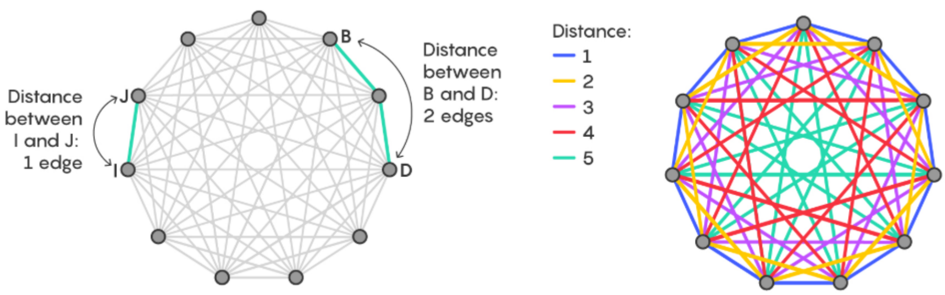


Figure 6. Left is a distance and right the ND-colouring of K_{11} (taken from [16])

Example 3.3. ([16]) Let us consider the complete graph K_{11} of order 11. We color the edges the way that edges of the same distance have the same colour. The distance is defined as the number of edges of circuit we need to move from one vertex to another. No shortcuts through the inside of the circle are allowed (see Figure 6). We always have two options, but we choose the shorter one. Now color the edges of the graph considering distance. All edges connecting vertices of distance 1 paint, say, by blue. All edges connecting vertices of distance 2 paint, say, by yellow. Etc. (See Figure 6.) On the complete graph of order $2n + 1$ we need n different colors to paint the whole graph.

Kotzig realized that this colouring can be helpful to place a given tree over the complete graph. By a placement of a *rainbow copy* of the tree is meant to position the tree so that every edge of the tree has different colour (see Figure 7).

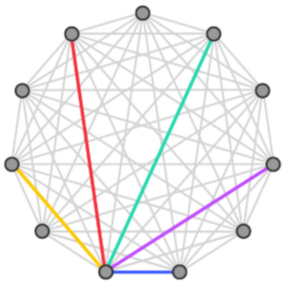


Figure 7. A rainbow copy of a tree (taken from [16])

If the ND-colouring of K_{2n+1} contains a rainbow copy of a tree T, then K_{2n+1} can be decomposed into copies of T by taking $2n+1$ cyclic shifts of the original rainbow copy. This idea and Ringel's Conjecture motivated Kotzig to conjecture that the ND-colouring of K_{2n+1} contains a rainbow copy of every tree of size n. It is important to note that a rainbow copy of a tree T with vertex set $\{0, 1, \ldots, n\}$ in the ND-colouring of K_{2n+1} is equivalent to a graceful labelling of the tree T.

Montgomery, Pokrovskiy, and Sudakov already in 2019 [33] gave a new approach to embedding *large* trees (with no degree restrictions) into edge-colourings of complete graphs, and used this to prove the Ringel's Conjecture asymptotically. In [32] they further developed and refined their approach, combining it with several critical new ideas to prove Ringel's Conjecture for large complete graphs:

Theorem 3.4. ([32, Theorem 1.2]) For every sufficiently large n the complete graph K_{2n+1} can be decomposed into copies of any tree with n edges.

In [32] the authors, instead of working directly with tree decompositions or studying graceful labellings, proved for large n that every ND-coloured complete graph K_{2n+1} contains a rainbow copy of every tree of size n:

Theorem 3.5. ([32, Theorem 2.1]) For sufficiently large n, every ND-coloured K_{2n+1} has a rainbow copy of every n-edge tree.

Then they obtain a decomposition of the complete graph by rotating one copy of a given tree. Hence this gives a proof of the whole Ringel-Kotzig Conjecture for *large* n.

The proof approach of the authors of [32] builds on ideas from the previous research on both graph decompositions and graceful labellings. From the work on graph decompositions, their approach is inspired by *randomized* decompositions and so-called *absorption technique*. The rough idea of the method of "absorption" is as follows (cf. [32]):

(1) Before the embedding of a tree T prepare a *template* which has some useful properties.

(2) Find a partial embedding of the tree T with some vertices removed such that it does not use the edges of the template.

(3) Use the template to embed the remaining vertices extensively since then.

This idea was introduced by Rödl, Rucinski and Szemerédi [41]. Also the proof of Ringel's Conjecture for *bounded degree* trees is based on this method [23].

From the work on graceful labellings, the proof approach of [32], when dealing with trees with very high degree vertices, is based on a completely deterministic approach for finding a rainbow copy of the tree. This approach heavily relies on features of the ND-colouring and produces something very close to a graceful labelling of the tree. Their theorem is the first general result giving a perfect decomposition of a graph into subgraphs with arbitrary degrees. All previous comparable results placed a bound on the maximum degree of the subgraphs into which they decomposed the complete graph. Hence all these techniques encounter some barrier when dealing with trees with arbitrarily large degrees. Having overcome this "bounded degree barrier" for Ringel's Conjecture, the authors of [32] hope that further development of their techniques can help overcome the "bounded degree barrier" also in other problems (cf. [32, page 3]).

The authors of [32] in their concluding remarks return to two other conjectures, the first one is the Graceful Tree Conjecture. They mention that this conjecture was proved for many isolated classes of trees, among them caterpillars, trees with at most 4 leaves, firecrackers, all trees with diameter at most 5, symmetrical trees, trees with at most 35 vertices, and olive trees (see [10]). They also mention that the Graceful Tree Conjecture is known to hold asymptotically for trees of maximum degree at most $\frac{n}{\log n}$ [2]. But as they emphasize, solving the Graceful Tree Conjecture for general trees, even asymptotically, is still wide open.

The second conjecture the authors of [32] mention in their concluding remarks is the *Tree Packing Conjecture* ([13], [32, Conjecture 8.2]):

Conjecture 3.6. (**Tree Packing Conjecture**) Let $T_1, \dots, T_n$ be trees with $|T_i| = i$ for each $i \in \{1, \dots, n\}$. The edges of K_n can be decomposed into n trees which are isomorphic to $T_1, \dots, T_n$ respectively.

In 2018 this conjecture was proved for bounded degree trees by Joos, Kim, Kühn and Osthus [23], but in general it is also wide open. The authors of [32] remark that it would be interesting to see if any of their techniques could be used here to make further progress on the Tree Packing Conjecture.

4 Recent results on graceful trees

4.1 Diameter six trees

In [19] Hrnčiar and Haviar proved that all trees of diameter five are graceful, which is still the best result on gracefulness of all trees with a bounded diameter. Mishra and Panigrahi in [30] and [31] gave a new class of graceful lobsters obtained from diameter four trees. Based on their techniques, in 2015-2017 Mishra and Panda [36] found graceful labellings for some new classes of diameter six trees [34], [35] and [36]. We briefly present the main results of [36].

Definition 4.1. ([36, Definition 1.2]) A **diameter six tree** can be represented as $(a_0; a_1, a_2, \dots, a_m; b_1, b_2, ..., b_n; c_1, c_2, ..., c_r)$, where a_0 is the center of the tree; a_i for $i = 1, 2, \dots, m$; b_j for $j = 1, 2, \dots, n$, and c_k for $k = 1, 2, \dots, r$ are the vertices of the trees adjacent to a_0 such that each a_i is a central vertex of some diameter three tree, each b_j is the central vertex of some star, and each c_k is some pendant vertex.

We note that in the above definition the authors mistakenly wrote in [36] that "each a_i is the center of some diameter four tree" while above we correctly write "each a_i is a central vertex of some diameter three tree" (meaning by a_i that of the two central vertices that is adjacent to a_0). Also the authors mistakenly wrote in [36] the following:

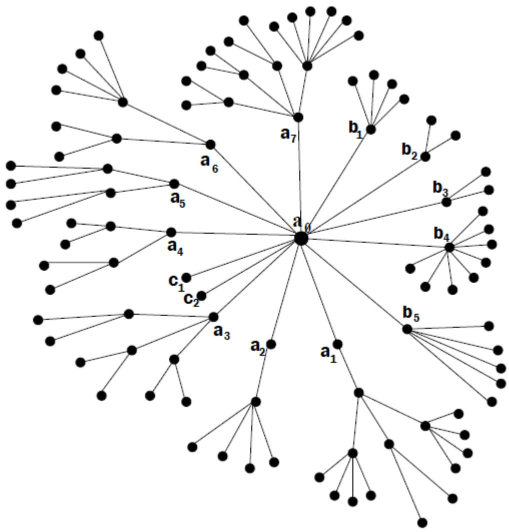


Figure 8. A diameter six tree (a corrected figure from [36])

"It is readily observed that for a diameter six tree with the above representation there are at least two neigbours of a_0 which are the centers of diameter four trees." It should be corrected such that "there are at least two neighbours of a_0 which are central vertices of diameter three trees".

In summary, in [36] graceful labellings were given for new classes of diameter six trees in which the diameter three trees adjacent with the center a_0 consist of six different combinations of odd, even, and pendant branches.

Example 4.2. In Figure 9 we see a diameter six tree D_6 with its graceful labelling found by Mishra and Panda in [34]. The size of the graph is 90, the degree of a_0 is 11.

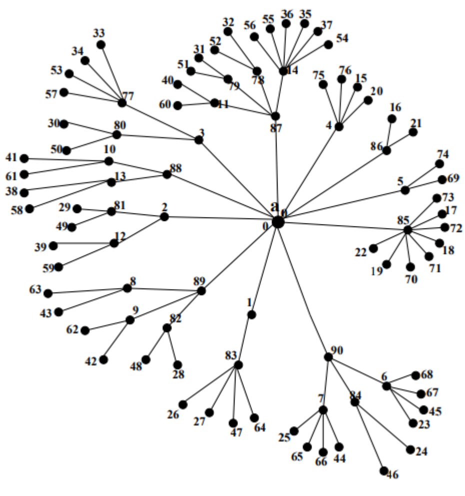


Figure 9. D_6 of order 91 (taken from [34, Figure 2(b)])

4.2 Spider graphs

In the early 1980s graceful labellings were found for all *spider graphs* with three or four legs [20]. Ten years ago it was proved in [5] that a spider graph for which the lengths of all legs (paths from the center to a leaf) differ by at most one is graceful. In 2014 in [21] some other classes of spiders were shown to be graceful, too.

Definition 4.3. A tree with at most one vertex of degree greater than two is called a **spider**, and this vertex is called a **branch vertex**. A path from the branch vertex to a leaf is called a **leg** of the spider.

Let us denote by $S_n(m_1, m_2, \ldots, m_k)$ the spider with n legs such that $n \geq k$ and the legs have lengths one except for k legs of the lengths $m_1, m_2, \ldots, m_k$, where $m_i \geq 2$ for all $i = 1, 2, \ldots, k$.

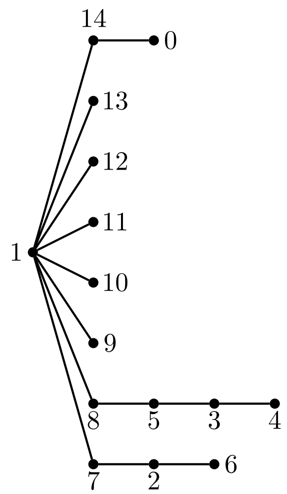
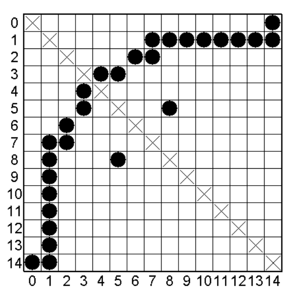


Figure 10. A graceful labelling of a spider $S_8(2, 4, 3)$

In 2016 in [37] graceful labellings were found for all spiders with at most four legs of lengths greater than one.



1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	3	5	2	2	1	1	1	1	1	1	1	1	0
4	5	8	6	7	7	8	9	10	11	12	13	14	14

Figure 11. The representations of the gracefully labelled spider $S_8(2, 4, 3)$

Example 4.4. In Figure 10 we see a gracefully labelled spider graph $S_8(2,4,3)$ of order 15 with 8 legs with lengths 2,1,1,1,1,1,4,3. The branch vertex has label 1. Below the graph we added in Figure 11 also the simple chessboard and the labelling relation of this gracefully labelled graph. We see that the labelling sequence is $(3,3,5,2,2,1,1,1,1,1,1,1,1,0)$.

4.3 Symmetrical trees

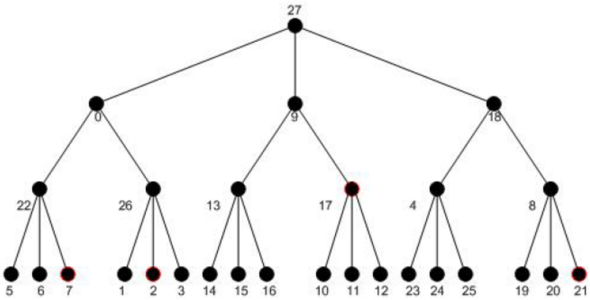
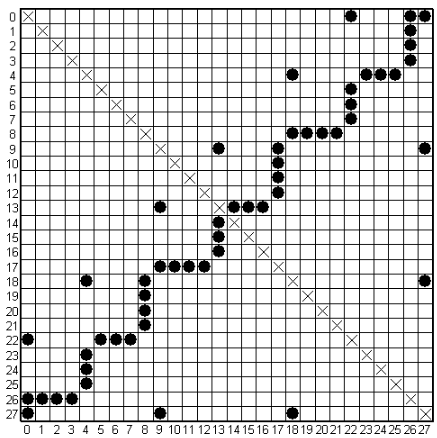


Figure 12. A symmetrical tree (taken from [45, Figure 3])

A **rooted tree** is known as a tree with a countable number of vertices, in which a particular vertex is distinguished from the others and called the **root**.



1	2	3	4	5	6	7	8	9	10	11	12	13	14
13	13	13	9	12	11	10	9	18	8	8	8	8	4
14	15	16	13	17	17	17	17	27	18	19	20	21	18

15	16	17	18	19	20	21	22	23	24	25	26	27
7	6	5	9	4	4	4	0	3	2	1	0	0
22	22	22	27	23	24	25	22	26	26	26	26	27

Figure 13. The representations of the symmetrical tree

For a given vertex, a number of vertices in the path from the root to this vertex is called the **level** of the vertex. A **symmetrical tree** is a rooted tree with k levels, where every level contains vertices of the same degree.

In [38] an algorithm for graceful labelling of symmetrical trees was given. In 2018 Sandy, Rizal, Manurung, and Sugeng [45] gave an alternative construction of graceful symmetrical trees.

Example 4.5. In Figure 12 we see a symmetrical tree with graceful labelling. Again, below the graph we added in Figure 13 the simple chessboard and the labelling relation of this gracefully labelled graph. One can see that the labelling sequence is $(13, 13, 13, 9, 12, 11, 10, 9, 18, 8, 8, 8, 8, 4, 7, 6, 5, 9, 4, 4, 4, 0, 3, 2, 1, 0, 0)$. Here we very well see that creating the graph chessboard provides an extra value of visualization to the graceful labelling, and enables us seeing better a certain pattern of the graceful labelling in the graph chessboard.

4.4 Caterpillars and lobsters

Definition 4.6. ([17, page 72]) A **caterpillar** is a tree with the property that the removal of its vertices of degree one leaves a path.

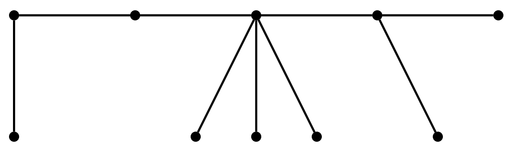


Figure 14. Example of a caterpillar

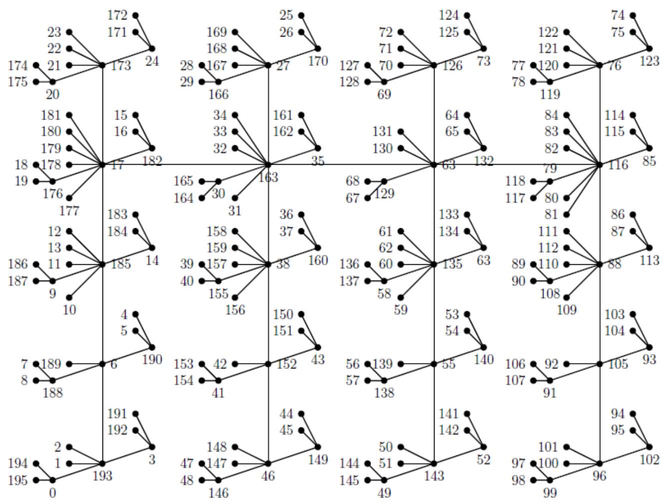


Figure 15. A graceful labelling of a tree (taken from [58, Figure 9])

By combining known graceful trees one can construct larger graceful trees. This idea was used by Sethuraman and Murugan [58] in 2016 and they constructed graceful trees from caterpillars in a specific way. An example of a gracefully labelled tree obtained from caterpillars by their method is seen in Figure 15. Also in this case the representations by the simple chessboard, the labelling relation and the labelling sequence would be possible, but we do not provide them here due to the enormous size of the graph.

Definition 4.7. ([47, Definition 1.2]) For each vertex v of a graph G, take a new vertex v' and join v' to all vertices of G adjacent to v. The graph thus obtained is called the **splitting graph** of G and denoted $S'(G)$.

Sekar in [48] found graceful labellings of $S'(P_n)$ for all n (where P_n is a path) and $S'(C_n)$ for $n \equiv 0, 1 \pmod 4$ (where C_n is a cycle). A gracefulness of the splitting graph of a bistar and a star was proved in [57]. Latest result from 2017 is proved in [47] and it says that the splitting graphs of caterpillars are graceful.

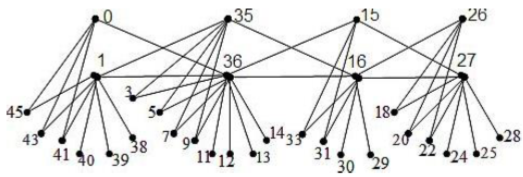


Figure 16. A graceful labelling of a splitting graph (taken from [47, Figure 3])

Example 4.8. In Figure 16 we see an illustration of a splitting graph constructed to a caterpillar by the above definition and its graceful labelling according to [47].

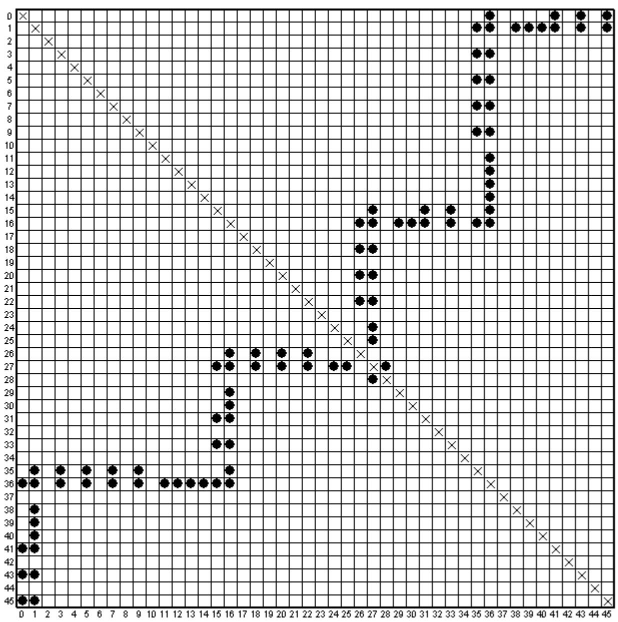


Figure 17. The chessboard of the splitting graph

And in Figures 17 and 18 we added its representations by the simple chessboard and the labelling relation, respectively. The labelling sequence of this gracefully labelled splitting graph ('split' according to the labelling table below) is

$$(27, 25, 24, 22, 22, 20, 20, 18, 18, 16, 16, 15, 16, 16, 16, \\ 15, 16, 15, 16, 16, 15, 14, 13, 12, 11, 9, 9, 7, 7, 5, 5, 3, 3, \\ 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 0).$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
27	25	24	22	22	20	20	18	18	16	16	15	16	16	16
28	27	27	26	27	26	27	26	27	26	27	27	29	30	31

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
15	16	15	16	16	15	14	13	12	11	9	9	7	7	5
31	33	33	35	36	36	36	36	36	36	35	36	35	36	35

31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
5	3	3	1	1	0	1	1	1	1	0	1	0	1	0
36	35	36	35	36	36	38	39	40	41	41	43	43	45	45

Figure 18. The chessboard of the splitting graph

Definition 4.9. ([10]) A **lobster** is a tree with the property that the removal of the vertices of degree 1 leaves a caterpillar.

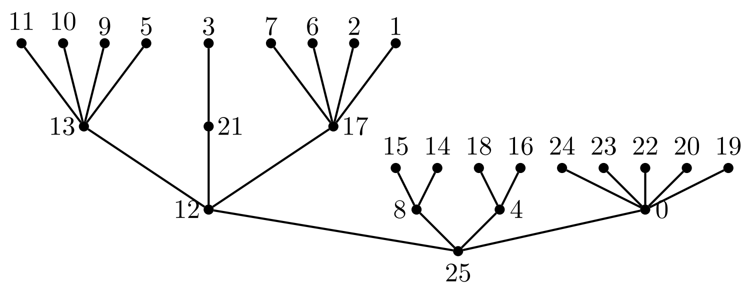
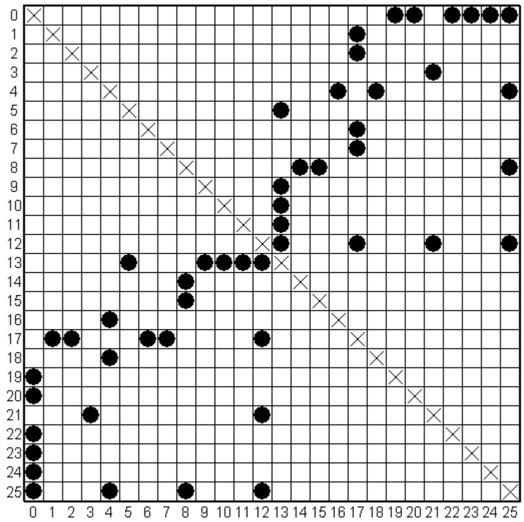


Figure 19. A graceful labelling of a lobster (taken from [11, Figure 18])

Bermond in [7] conjectured that all lobsters are graceful. Then Ghosh in [11] gave some methods how to join graceful graphs and graphs with the α-labeling. He defined three special classes of gracefully labelled lobsters. In 2015 Krop in [27] showed gracefulness of each lobster that has a *perfect matching* that covers all but one vertex. Some new constructions of graceful classes of caterpillars and lobsters were given in 2018 in [56] by Suparta and Ariawan.

Example 4.10. In Figure 19 we see an example of a gracefully labelled lobster. We added its representations by the simple chessboard and the labelling relation that are seen in Figure 20. The labelling sequence representing this gracefully labelled graph is $(12, 11, 10, 9, 12, 8, 8, 5, 12, 7, 6, 4, 12, 4, 2, 1, 8, 3, 0, 0, 4, 0, 0, 0, 0)$.



1	2	3	4	5	6	7	8	9	10	11	12
12	11	10	9	12	8	8	5	12	7	6	4
13	13	13	13	17	14	15	13	21	17	17	16

13	14	15	16	17	18	19	20	21	22	23	24	25
12	4	2	1	8	3	0	0	4	0	0	0	0
25	18	17	17	25	21	19	20	25	22	23	24	25

Figure 20. The representations of the gracefully labelled lobster

5 Recent results on graceful cyclic graphs

5.1 Linear cyclic snakes

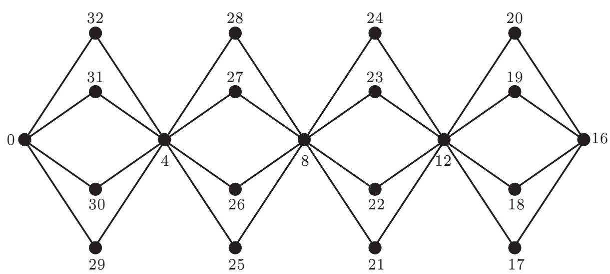
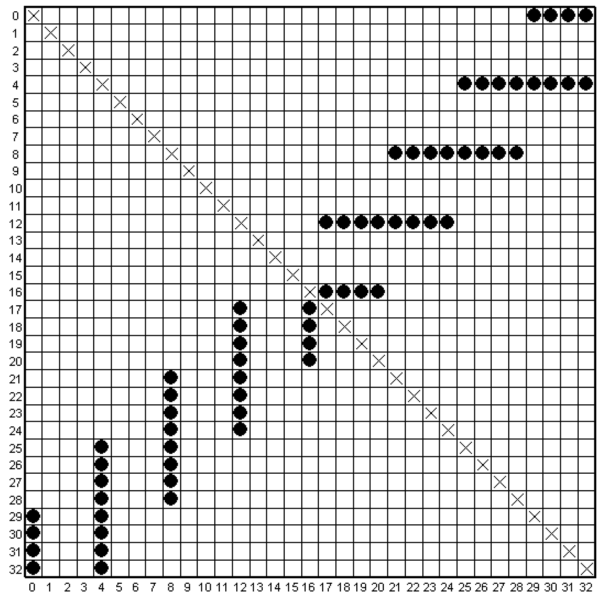


Figure 21. The graceful labelling of $(2,4)C_4$ (taken from [3, Figure 3] and corrected)

Recalling briefly a history of *linear cyclic snakes*, we start with Barrientos who in [6] gave graceful labelings of *cyclic snakes*. Rosa in [44] glued together triangles in a special way and called it a *triangular snake*.

In 2015 Badr proved gracefulness of linear cyclic snakes $(1,k)C_4$, $(2,k)C_4$, $(1,k)C_8$ and $(2,k)C_8$ and showed that every linear cyclic snake of type $(m,k)C_n$ for $m \equiv 0$

(mod 4) and $m \equiv 3 \pmod 4$ is graceful [3].



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
16	16	16	16	12	12	12	12	12	12	12	12	8	8	8	8
17	18	19	20	17	18	19	20	21	22	23	24	21	22	23	24

17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
8	8	8	8	4	4	4	4	4	4	4	4	0	0	0	0
25	26	27	28	25	26	27	28	29	30	31	32	29	30	31	32

Figure 22. The representations of the gracefully labelled linear cyclic snake

We notice that the way Badr in [3] defined his “linear cyclic snakes” is rather badly written and hardly understandable. That is why we do not present his definition and try to explain the notation $(m,k)C_n$ in our own words via the example below.

Example 5.1. In Figure 21 we see a linear cyclic snake $(2,4)C_4$ obtained by joining 4 copies of C_4 graphs in such a way that each of them contains inside an another copy of C_4.

This graph is gracefully labelled. We added its representations by the simple chessboard and the labelling relation which can be seen in Figure 22. The labelling sequence representing this gracefully labelled graph is

$$(16,16,16,16,12,12,12,12,12,12,12,12,8,8,8,8,8,8,8,8,4,4,4,4,4,4,4,4,0,0,0,0).$$

This is an excellent example where one can see that creating the graph chessboard provides the mentioned extra value of visualization to the graceful labelling, and allows to see, very clearly in this case in the graph chessboard, the pattern of the graceful labelling.

5.2 Cycle related graphs

Let C_n be a cycle of length n.

Definition 5.2. ([59, Definition 1.1]) A **chord** of the cycle is an edge connecting two non-neighbouring vertices of the cycle.

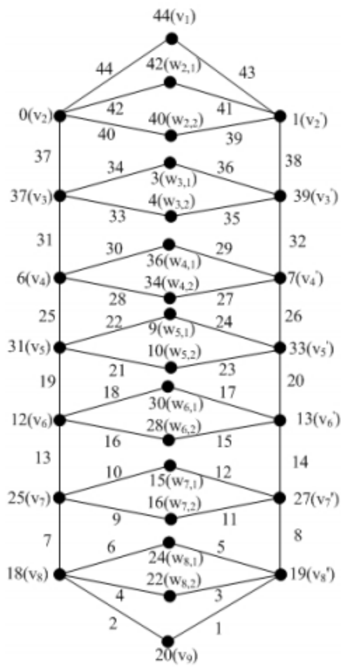


Figure 23. Graceful labelling of $C^{+}_{16,4}$ (taken from [59, Figure 4])

Recalling briefly a history of *cycle related graphs*, we start with Rosa, who in [43] showed that a cycle C_n is graceful if and only if $n \equiv 0$ or $3 \pmod 4$. Later the authors of [8] proved gracefulness of a cycle with a chord. The authors of [25] proved that each cycle with P_3-chord is graceful and conjectured that, more generally, each cycle with P_k-chord is graceful. (We recall that a cycle with a P_k-chord is a cycle with the path P_k joining two nonconsecutive vertices of the cycle.)

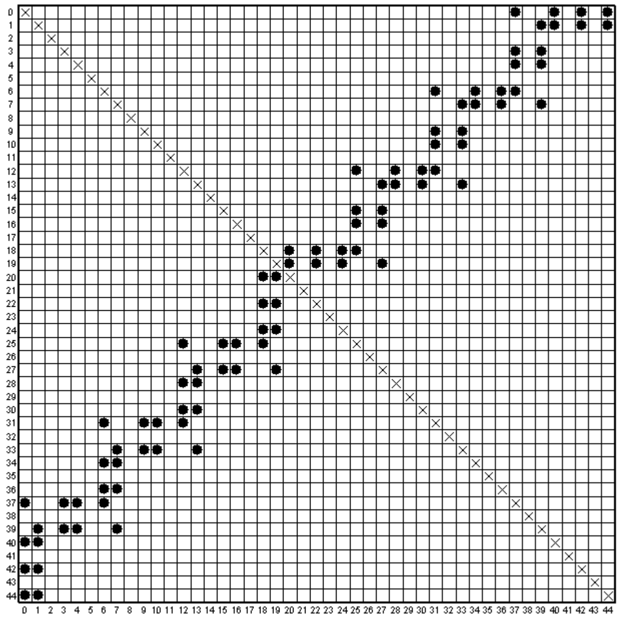
The mentioned conjecture was proved in [39] for all $k \geq 4$. In [49] the authors defined a graph obtained from a cycle C_n ($n \geq 6$) so that disjoint paths P_k (where $k \geq 3$ is fixed) are added between each pair of non-adjacent vertices of C_n and they call it a *cycle with parallel P_k chords*. They verified that each cycle C_n (where $n \geq 6$) with parallel P_k chords is graceful in cases $k = 3, 4, 6, 8$ and 10.

Definition 5.3. ([59, Definition 1.2]) A graph acquired from the cycle C_n by adding the cycle C_k between every non-adjacent vertices is called a **cycle with** $C_k - chord$ and denoted $C_{n,k}$.

Definition 5.4. ([59, Definition 1.3]) A graph acquired from the cycle C_n by adding the cycle C_k between every pair of non-neighbouring vertices $(v_2, v_n), (v_3, v_{n-1}), \ldots, (v_a, v_b)$ where $a = \lfloor \frac{n}{2} \rfloor, b = \lfloor \frac{n}{2} \rfloor + 2$ if n is even, and $a = \lfloor \frac{n}{2} \rfloor, b = \lfloor \frac{n}{2} \rfloor + 3$ if n is odd, is called a **parallel cycle with** $C_k - chord$ and denoted $C^{+}_{n,k}$.

Latest result in this direction was proved in 2017 by Venkatesh and Sivagurunathan in [59]. It says that graphs $C_{n,4}$ and $C^+_{n,4}$ for each $n \equiv 0 \pmod 4$ and $C_{n,6}$ for each odd $n \geq 5$ are graceful.

Example 5.5. In Figure 23 we see a gracefully labelled parallel cycle C_{16} with $C_4-chord$.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
19	18	19	18	19	18	18	19	16	15	16	15	12	13	13
20	20	22	22	24	24	25	27	25	25	27	27	25	27	28

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
12	13	12	12	13	10	9	10	9	6	7	7	6	7	6
28	30	30	31	33	31	31	33	33	31	33	34	34	36	36

31	32	33	34	35	36	37	38	39	40	41	42	43	44
6	7	4	3	4	3	0	1	1	0	1	0	1	0
37	39	37	37	39	39	37	39	40	40	42	42	44	44

Figure 24. The representations of the gracefully labelled of $C^+_{16,4}$

We added its representations by the simple chessboard and the labelling relation that are seen in Figure 24. The labelling sequence representing this gracefully labelled graph is

$$(19, 18, 19, 18, 19, 18, 18, 19, 16, 15, 16, 15, 12, 13, 13, 12, 13, 12, 12,$$
$$13, 10, 9, 10, 9, 6, 7, 7, 6, 7, 6, 6, 7, 4, 3, 4, 3, 0, 1, 1, 0, 1, 0, 1, 0).$$

5.3 Corona product of aster flower graph

Definition 5.6. ([24, Definition 1]) An **aster flower graph** $(A_{(m,n)})$ is a graph which is generated from a cycle graph C_m $(m \geq 3)$ by connecting path graphs P_{n+1} $(n \geq 1)$ at two adjacent vertices. A **corona product** ($A_{(m,n)} \odot \bar{K}_r$) of aster flower graph is a graph which is generated from an aster graph ($A_{(m,n)}$ ($m \geq 3, n \geq 1$) by adding r leaf vertices on each vertex.

In [14] the gracefulness was proved for *corona product* of two graphs. Later in [9] it was proved that any cycle with a leaf connected at each vertex is graceful. In 2018 in [24] Khairunnisa and Sugeng found graceful labelling for each corona product ($A_{(3,1)} \odot \bar{K}_r$) of aster flower graph (for $r \geq 1$).

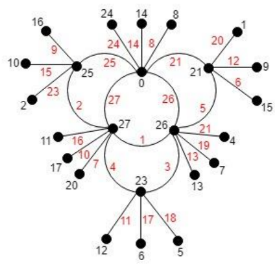
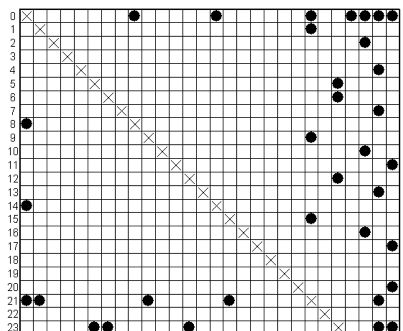


Figure 25. A corona product ($A_{(3,1)} \odot \bar{K}_3$) (taken from [24, Figure 4])

Example 5.7. In Figure 25 we see a gracefuly labelled corona product ($A_{(3,1)} \odot \bar{K}_3$) of aster flower graph. We added its representations by the simple chessboard and the labelling relation which can be seen in Figure 26. The labelling sequence representing this gracefully labelled graph is

$$(26, 25, 23, 23, 21, 15, 20, 0, 16, 17, 12, 9, 13, 0, 10, 11, 6, 5, 7, 1, 0, 4, 2, 0, 0, 0, 0).$$



1	2	3	4	5	6	7	8	9	10	11	12	13	14
26	25	23	23	21	15	20	0	16	17	12	9	13	0
27	27	26	27	26	21	27	8	25	27	23	21	26	14

15	16	17	18	19	20	21	22	23	24	25	26	27
10	11	6	5	7	1	0	4	2	0	0	0	0
25	27	23	23	26	21	21	26	25	24	25	26	27

Figure 26. The representations of the gracefully labelled ($A_{(3,1)} \odot \bar{K}_3$)

5.4 Unicyclic graphs

Definition 5.8. ([15, page 41]) A graph is **unicyclic** if it contains just one cycle and is connected.

In Figure 27 we see an example of unicyclic graph.

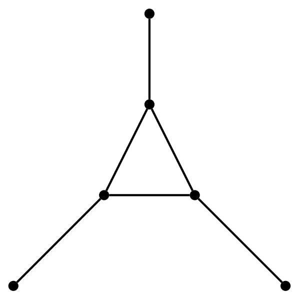


Figure 27. An example of unicyclic graph

Recalling briefly a history of embedding graphs into graceful graphs (see [53, page 11]), we start with Acharya who in [1] proved that each connected graph can be embedded in a graceful graph. Later, the authors of [50] generalized this result and showed that any set of graphs can be "packed" into a graceful graph.

In 2015 Bagga, Fotso, Max, and Arumugam in [4] explored the gracefulness of graphs with only one cycle with some pendant caterpillars at two neighbouring vertices of cycle and pendant edges at some other vertices of the cycle. A cycle with a pendant caterpillar is obtained by identifying a vertex of the cycle with a leaf of caterpillar.

In 2016 Sethuraman in [51] showed that every tree can be embedded in a graceful tree. This inspired Sethuraman and Murugan who proved in 2019 in [53] that any acyclic graph can be embedded in a unicyclic graceful graph. The authors found an algorithm that from any acyclic graph constructs a graceful unicyclic graph.

Also in 2019 Sethuraman and Murugan [52] presented a construction of graceful labeling of a graph G from a graceful tree T in case the number of vertices of G is equal to number of vertices of T. The constructed graph is unicyclic.

6 Recent results on graceful subdivisions of graphs

6.1 Complete bipartite graphs

Definition 6.1. ([46]) If in a graph G an edge uv is replaced by the path $P : uwv$, where w is the new vertex, then the edge uv is called **subdivided**. A **subdivision** of a graph G is the graph obtained by subdividing each edge of the graph G and it is denoted by $S(G)$.

In 2016 Sankar and Sethuraman in [46] proved that each subdivision of the complete bipartite graph $K_{2,n}$ is graceful for every $n \geq 1$.

Example 6.2. In Figure 28 we see the subdivisiom graph $S(K_{2,4})$ and its graceful labeling. Its representations by the simple chessboard and the labelling relation can be seen in Figure 29. The labelling sequence representing this gracefully labelled graph is $(8, 8, 8, 8, 7, 5, 3, 1, 7, 5, 3, 1, 0, 0, 0, 0)$. This is an another excellent example where one can see that creating the graph chessboard allows to see very clearly the pattern of the graceful labelling.

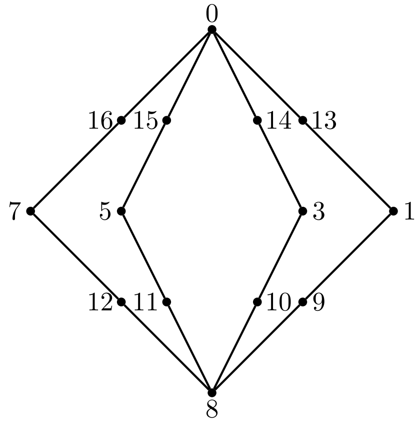
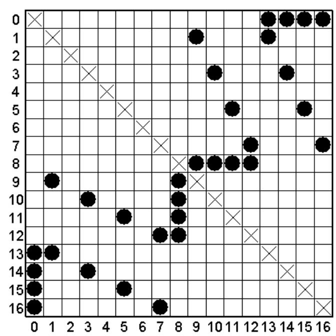


Figure 28. A graceful labeling of $S(K_{2,4})$



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
8	8	8	8	7	5	3	1	7	5	3	1	0	0	0	0
9	10	11	12	12	11	10	9	16	15	14	13	13	14	15	16

Figure 29. The representations of the gracefully labelled $S(K_{2,4})$

6.2 Wheels

Definition 6.3. ([54]) A **wheel** is a graph obtained by connecting a single vertex K_1 to all vertices of a cycle C_n. A wheel W_n is the graph $C_n + K_1$ for $n \geq 3$.

Some authors use the symbol W_n to denote the wheel with n vertices.

In [18] it was proved that all wheels for $n \geq 3$ are graceful. In [28] graceful labellings of directed wheels were presented. In 2015 in [54] Sethuraman and Sankar proved that the subdivision $S(W_n)$ of the wheel W_n is graceful for even numbers $n \geq 4$.

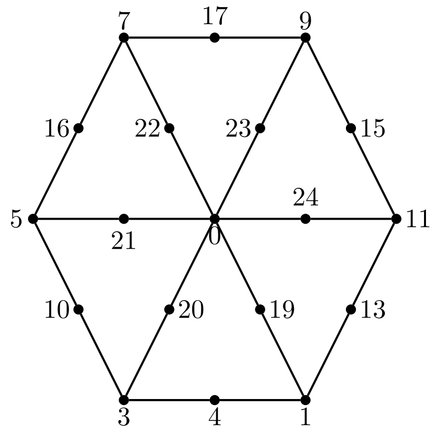


Figure 30. Gracefully labeled wheel $S(W_6)$

Example 6.4. In Figure 30 we see a gracefully labelled subdivison $S(W_6)$ of order 19 and size 24.

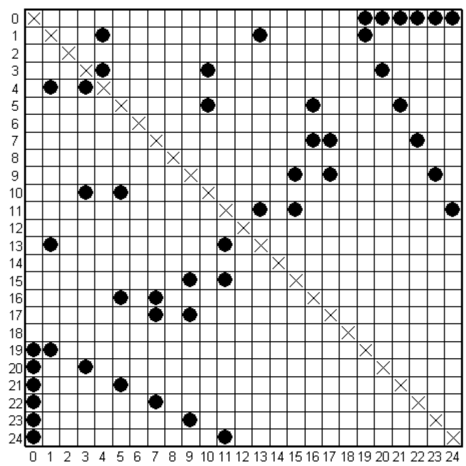


Figure 31. The chessboard of the gracefully labelled $S(W_6)$

We added its representations by the simple chessboard in Figure 31 and the labelling table which can be seen in Figure 32. The labelling sequence representing this gracefully labelled graph is $(3, 11, 1, 11, 5, 9, 3, 9, 7, 7, 5, 1, 11, 9, 7, 5, 3, 1, 0, 0, 0, 0, 0, 0)$.

1	2	3	4	5	6	7	8	9	10	11	12
3	11	1	11	5	9	3	9	7	7	5	1
4	13	4	15	10	15	10	17	16	17	16	13

13	14	15	16	17	18	19	20	21	22	23	24
11	9	7	5	3	1	0	0	0	0	0	0
24	23	22	21	20	19	19	20	21	22	23	24

Figure 32. The labelling table of the gracefully labelled $S(W_6)$

6.3 Shell and bow graphs

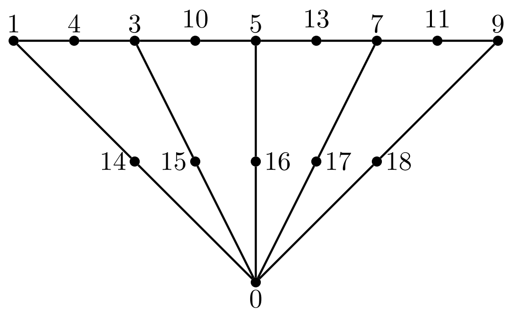


Figure 33. The graph $S(C(6,3))$

Definition 6.5. ([46]) A **shell graph** is a cycle $C_n(v_0, v_1, v_2, \ldots, v_{n_1})$ with $(n-3)$ chords connecting vertex v_0, we denote it $C(n; n-3)$. The vertex v_0 is called *apex* of the shell graph.

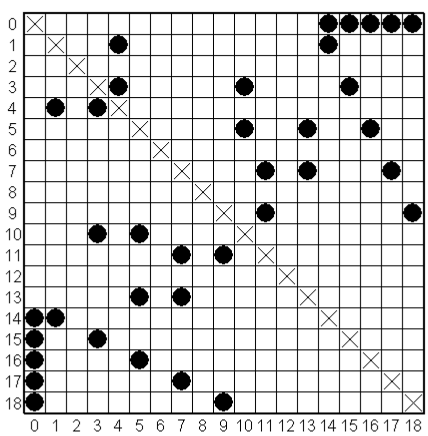


Figure 34. The chessboard of the gracefully labelled $S(C(6,3))$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	9	1	7	5	7	3	5	9	7	5	3	1	0	0	0	0	0
4	11	4	11	10	13	10	13	18	17	16	15	14	14	15	16	17	18

Figure 35. The labelling table of the gracefully labelled $S(C(6,3))$

Example 6.6. In Figure 33 we see a gracefully labelled subdivision of the shell graph $C(6,3)$. The graph $S(C(6,3))$ is of size 18. Its representations by the simple chessboard can be seen in Figure 34 and the labelling table can be seen in Figure 35. The labelling sequence representing this gracefully labelled graph is

$$(3,9,1,7,5,7,3,5,9,7,5,3,1,0,0,0,0,0).$$

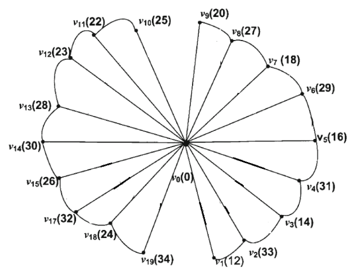


Figure 36. A uniform bow graph (taken from [22, Figure 2])

Definition 6.7. ([22]) A **bow graph** is a graph consisting of two shells of any orders. If each shell has the same order, we call it a **uniform bow graph**. A special case of a bow graph is a **shell butterfly graph**. This is a bow graph with two special edges from the apex.

In 2015 Jesintha and Hilda in [22] proved gracefulness of all uniform bow graphs. In Figure 36 we see a gracefully labelled uniform bow graph of size 34.

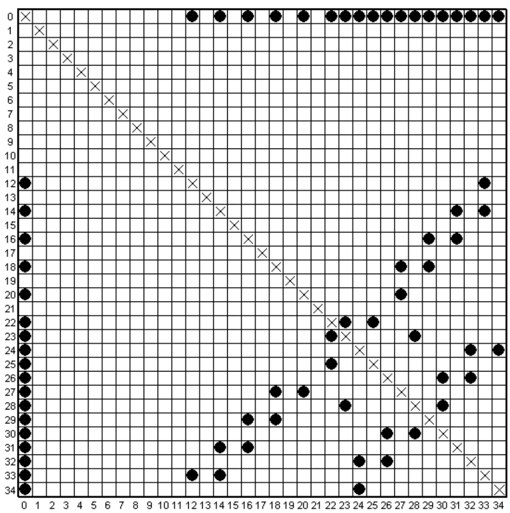


Figure 37. The chessboard of the gracefully labelled uniform bow graph

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
22	28	22	26	23	26	20	24	18	24	18	0	16	0	16	0	14
23	30	25	30	28	32	27	32	27	34	29	12	29	14	31	16	31

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
0	14	0	12	0	0	0	0	0	0	0	0	0	0	0	0	0
18	33	20	33	22	23	24	25	26	27	28	29	30	31	32	33	34

Figure 38. The labelling sequences of the gracefully labelled uniform bow graph

Its representations by the simple chessboard and the labelling relation can be seen in Figures 37 and 38. The labelling sequence representing this gracefully labelled graph is

$$(22, 28, 22, 26, 23, 26, 20, 24, 18, 24, 18, 0, 16, 0, 16,$$
$$0, 14, 0, 14, 0, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

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