Acta Universitatis Matthiae Belii, series Mathematics
Volume 29 (2021), 1–38, ISSN 1338-7111
Available at https://actamath.savbb.sk

A survey on chessboard representations of classes of graceful graphs

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Abstract

This is a survey on chessboard representations of classes of graceful graphs. We introduced and studied the concept of chessboard representations of graceful graphs in our book *Vertex labellings of simple graphs* in 2015, however, the book and hence its concepts and methods are not widely accessible. Our survey from the book is complemented by new applications of chessboard representations to the classes of k-enriched fan graphs presented by the first author and Kurtulík in 2021.

The aim of the chessboard representations of classes of graceful graphs has been to provide an important value of visualization. Firstly, it leads to seeing better the pattern of the graceful labellings in already known graceful representations of simple graphs. Secondly, it can lead to creating new classes of graceful graphs as is demonstrated on the very recent example of the classes of k-enriched fan graphs.

Received: December 27, 2021
Accepted: January 24, 2022
Available online: January 24, 2022
Communicated with Alexander Rosa.

Keywords simple graph, graceful labelling, graph chessboard, labelling sequence, labelling relation.
MSC(2010) Primary 05C78; Secondary 05C05.

1 Introduction

Our presentation in the first four sections of this survey paper is following the book *Vertex labellings of simple graphs* [15], while the last Section 5 is based on a recent paper [17] by the first author and Kurtulík.

Said roughly, by a *graph labelling* is meant an assignment of integers to the vertices or edges, or both, of a graph. The theory of graph labellings has its foundations in the late 1960s. Over a half of a century, a huge number of methods and techniques on graph labellings have been presented. In the bibliography of the famous 'Dynamic Survey of Graph Labeling' by Gallian [11] – its twenty-fourth edition was published online on December 9, 2021 – over 3000 papers, theses or books on graph labellings are

*The first author acknowledges a support by Slovak VEGA grant 2/0078/20 and a Visiting Professorship at University of Johannesburg.

included. Most of the concepts and methods on graph labellings have their origin in the dissertation [29] of Rosa and in his famous 1967 paper [30], and in the 1980 paper by Graham and Sloane [14].

The Gallian electronic survey [11] does mention our paper [17] with Kurtulík but it does not mention our 2015 book [15], on whose methods and techniques the paper [17] is completely based. We thought that the reason might be that the book [15] was not easily accessible. Yet the Gallian survey [11] does not also mention the 2017 book [22] by López and Muntaner-Batle, which we think definitely deserves to be mentioned in this survey as it is described by Schulte [31] as follows: *This is not merely a curated presentation of classified labelings. The book contains open problems, an unpublished result from Erdös on graceful graphs, and some applications, including coding theory. A paper published from Joseph A. Gallian referenced here is "A guide to the graph labeling zoo" (Discrete Applied Mathematics 49, 1994, 213-229). This slim, focused atlas is not only a delineation of species that can be met in the menagerie, but a sketch of what can still be found in the wild.* We remark that the book [15] is mentioned in [22] as well as in [3] and [23]. Since the 2021 edition of the Gallian electronic survey [11] does mention sixteen papers by López and Muntaner-Batle, we wish here to point – for the future – the members of the team of this electronic survey to the two mentioned books [15] and [22] on graph labellings.

The study of graph labellings began with a problem on decompositions of the complete graph K_{2m+1} on $2m+1$ vertices into $2m+1$ subgraphs which are all isomorphic to a given tree with m edges. The problem was posed as a conjecture by Ringel [27] at a symposium on graph theory held in Smolenice, Slovakia, in 1963. Likely at the same symposium (cf. Rosa [29] and [30]) Kotzig conjectured a stronger statement saying that the decomposition of K_{2m+1} into $2m+1$ subgraphs which are all isomorphic to a given tree of size m can be made cyclically. This problem has become known as *Ringel-Kotzig Conjecture.*

In 1965 in his Slovak dissertation [29] and in 1967 in his famous paper [30], Rosa with an intention of providing an insight to the Ringel-Kotzig Conjecture, introduced eight new labellings of graphs, of which four have become quite well-known and have been widely studied. These four graph labellings – in a hierarchy from the strongest to the weakest – were α-, β-, σ- and ρ-labellings. It is important to note that Rosa proved that a cyclic decomposition of K_{2m+1} into subgraphs isomorphic to a given graph G of size m exists if and only if there exists a ρ-labelling of the graph G. This is of course true provided all trees have the stronger β-labellings. With the publication of the Rosa paper [30], the conjecture that *every tree has a β-labelling* has become famous. Then due to a paper by Golomb [13] in 1972, β-labellings have become widely known under the name *graceful labellings.* We recall that a tree with m edges has a graceful labelling (i.e. the Rosa β-labelling) if all its vertices can be assigned the labels $\{0, 1, \ldots, m\}$ such that the absolute values of the differences in vertex labels between adjacent vertices form the set $\{1, \ldots, m\}$.

Hence the famous *Graceful Tree Conjecture* saying that all trees are graceful, which follows from the Ringel-Kotzig Conjecture as its consequence, is by no doubt due to Rosa and his works [29] and [30], and it should be known as *Rosa Conjecture.* It is sometimes referred to as the *Ringel-Kotzig-Rosa Conjecture.* Two of these three men, Kotzig and Rosa come from Slovakia, so as Slovak authors we claim that to call the Graceful Tree Conjecture only as the Ringel-Kotzig Conjecture is not correct. Yet in the famous 'Dynamic Survey of Graph Labeling' by Gallian [11] we read right at the beginning of the section "Graceful and Harmonious Labelings" the following statement: *The Ringel-Kotzig* *conjecture (GTC) that all trees are graceful has been the focus of many papers.* The claim that the famous Graceful Tree Conjecture is the Ringel-Kotzig Conjecture (though it only follows from it) has been mistakenly appearing also in many other sources on the graph labellings. We believe that in the future the Graceful Tree Conjecture will be rightly referred to as the Rosa Conjecture or the Ringel-Kotzig-Rosa Conjecture. One of the partial aims of this survey has also been to point to this unintentional yet unfortunate misunderstanding in the literature on the graph labellings, which can be investigated and our claims can be verified by checking the sources we refer to. (Though we understand that the crucial Rosa dissertation [29], which is written in Slovak, is not easily accessible to experts outside Slovakia.)

Since 1967 almost all attempts to solve the Ringel-Kotzig Conjecture have focused on proving the Graceful Tree Conjecture. The graphs that can be gracefully labelled are known as *graceful* graphs. The Graceful Tree Conjecture is nowadays one of the most famous open problems in Graph Theory. Paul Erdös *et al.* in [10] also recognized the merit and the depth of the problem. It might be said that only a limited progress has been done on the conjecture over the last 55 years though a great number of research papers, various theses and surveys have been published on the problem. Not much is still known about the structure of the graceful graphs; one of the results, which we want to mention with respect to the structure the graceful graphs is due to Hrnčiar and Haviar [18] (father of the first author), and it says that all trees of diameter five are graceful. As far as we know nobody has proved yet that all trees of diameter six are graceful.

However, we point to the fact that in February 2020 three authors Montgomery, Pokrovskiy and Sudakov [24] announced a proof of the Ringel-Kotzig Conjecture for *large* trees, where the size of the tree is *comparable* with the size of the complete graph. In their proof they used a language of *rainbow subgraphs* (see also [25]), which describe the ρ-labellings; more details on their proof are in [21] and in a better accessible work [16]. It is also inevitable to point to a fact mentioned in [11, page 8], which says that in July of 2020 Gnang posted on arXiv a manuscript [12] with a proof of the Graceful Tree Conjecture. It is not known to us as whether the proof in this manuscript is correct or not.

Apart from Gallian's regularly updated electronic survey [11] we refer the interested reader to other surveys by Edwards and Howard [8], and Brankovic and Wanless [4]. We also refer to theses by Robeva [28] and Superdock [34], and the books by Acharya, Arumugarn and Rosa [1] and the mentioned 2017 book [22] by López and Muntaner-Batle.

Now we concentrate more on our book [15] on which this survey is mostly based. In Chapters 2 and 3 of [15] we presented three ways of how to represent a graph with a vertex labelling. These three ways were *graph chessboards* and *labelling relations* introduced by us, and *labelling sequences* introduced by Sheppard [32]. We present these concepts together with basic facts and results concerning them in Sections 2 and 3 of this survey paper.

The graph chessboards provide a nice visualization of graphs with vertex labellings, but together with the labelling sequences and the labelling relations they are also useful for finding patterns or encoding labelled graphs. For example, in the front cover of the book [15] we represent 57624 so-called *free-*$\Delta H^{(5)}$ *graphs*, which all are graceful, within a single graph chessboard (see also [15, Figure 5.11]). In [15, Chapter 5] we showed several methods of constructing new gracefully labelled graphs using chessboard representations or labelling sequences. In particular, we presented a new construction of a gracefully labelled graph consisting of m copies of a given gracefully labelled bipartite graph and such that the m copies are arbitrarily connected by edges. We showed that special cases of this *free* Δ *construction* of [15, Chapter 5] were the well-known Δ *construction* due to Stanton and Zarnke [33] and to Koh, Tan and Rogers [20], and the *generalized* Δ *construction* due to Burzio and Ferarese [5].

In Chapter 4 of [15], of which survey we present here in Section 4, we characterized, via their labelling sequences, labelling relations and graceful graph chessboards, eleven classes of graceful graphs. Among the five classes of trees we characterized in this way were *stars*, *double stars*, *paths*, *caterpillars* and *firecrackers*. Among the two classes of cycle-related graphs we characterized were *cycles* and *fan graphs*, among the two classes of product-related graphs were *ladders* and *simple chains*, and finally, we characterized the *complete graphs* and the *complete bipartite graphs* via their labelling sequences, labelling relations and graceful graph chessboards. Our characterizations of all these classes of labelled graphs via the labelling sequences in [15, Chapter 4] can be understood as a (partial) answer to a problem by Sheppard [32]. More precisely, Sheppard in [32] in the section called 'Unsolved problems' formulated two open problems: *Is there a property of labelling sequences which would indicate a graceful labelling of a tree? Are there classes of gracefully labelled graphs which could be characterized via labelling sequences?* We notice that we answered the first of these questions in Chapters 2 and 3 of [15], while the second one was (partially) answered in Chapter 4 of [15] and we present that answer here.

The important value of visualization provided by the chessboard representations of classes of graceful graphs leads to seeing better the pattern of the graceful labellings in already known graceful representations of simple graphs. This is partly demonstrated in Section 4 of this paper, where the graceful representations of most of the graphs in the eleven classes had been known previously. However, the visualization given by the chessboard representations can also lead to creating new classes of graceful graphs, which is here in Section 5 demonstrated on the very recent example of the classes of k-enriched fan graphs by the first author and Kurtulík [17]. More precisely, we introduced classes of k-enriched fan graphs kF_n for all integers $k, n \geq 2$ and we proved that these graphs are graceful. We also provided a characterizations of the k-enriched fan graphs kF_n among all simple graphs via Sheppard's labelling sequences as well as via the labelling relations and the graph chessboards. We closed [17] with an open problem concerning another infinite family of extended fan graphs and we present it also here at the end of Section 5.

2 Preliminaries

The basic concepts and facts that we need in this survey were presented in [15, Chapter 2]. (Originally, most of those definitions were taken from [30] or [4].)

As usually, the *order* of a graph G is the number of vertices in G and the *size* of a graph G is the number of edges in G.

Definition 2.1. ([15, Definition 1.2.1]) A *vertex labelling* f of a graph G is a mapping of its vertex set V_G into the set of non-negative integers (which are called *vertex labels*).

We sometimes use the term *labelling* to mean *vertex labelling*. We should notice that usually the labelling is one-to-one and this will be expressed by the concept *one-to-one labelling*. In a labelling f of a graph G we denote by $f(V_G)$ the set of all vertex labels and by $f(E_G)$ the set of all edge labels. By the *induced label of an edge uv* in the labelling f is meant the number $|f(u) - f(v)|$ provided $f(u), f(v)$ are the labels of the vertices u, v, respectively.

The four graph labellings α, β, σ and ρ mentioned before, which were introduced by Rosa in his Slovak dissertation [29], and then in his paper [30], form a certain hierarchy:

every labelling in the sequence

$$\alpha\textbf{-labelling} \to \beta\textbf{-labelling} \to \sigma\textbf{-labelling} \to \rho\textbf{-labelling}$$

is at the same time also the next labelling in the hierarchy. For example, every α-labelling is also β-labelling, σ-labelling and ρ-labelling, but a ρ-labelling need not be σ-labelling and a β-labelling need not be α-labelling. For this hierarchy of labellings we find it appropriate to use the concept *Rosa hierarchy.*

Definition 2.2. ([15, Definition 1.2.2]) Let G be a graph with m edges and let f be its one-to-one labelling. Then f is called an *α-labelling* (or a *balanced* labelling) if

1. $f(V_G) \subseteq \{0, 1, \ldots, m\}$,
2. $f(E_G) = \{1, 2, \ldots, m\}$, and
3. there exists some $r \in \{0, 1, \ldots, m\}$ such that for every edge $uv \in E_G$ either $u \leq r < v$ or $v \leq r < u$.

Definition 2.3. ([15, Definition 1.2.3]) Let G be a graph with m edges and let f be its one-to-one labelling. Then f is called a *β-labelling* (or a *graceful* labelling) if

1. $f(V_G) \subseteq \{0, 1, \ldots, m\}$, and
2. $f(E_G) = \{1, 2, \ldots, m\}$.

Definition 2.4. ([15, Definition 1.2.4]) Let G be a graph with m edges and let f be its one-to-one labelling. Then f is called a *σ-labelling* if

1. $f(V_G) \subseteq \{0, 1, \ldots, 2m\}$, and
2. $f(E_G) = \{1, 2, \ldots, m\}$.

Definition 2.5. ([15, Definition 1.2.5]) Let G be a graph with m edges and let f be its one-to-one labelling. Then f is called a *ρ-labelling* if

1. $f(V_G) \subseteq \{0, 1, \ldots, 2m\}$, and
2. $f(E_G) = \{x_1, x_2, \ldots x_m\}$, where $x_i = i$ or $x_i = 2m + 1 - i$, for all $i \in \{1, 2, \ldots, m\}$.

The Ringel-Kotzig Conjecture ([27], [29], [30]) says:

Conjecture 2.6. (**Ringel-Kotzig Conjecture**): For any tree of size m the complete graph K_{2m+1} has a cyclic decomposition into $2m + 1$ copies of the given tree.

We already mentioned that Rosa showed in [29] and [30] that the Ringel-Kotzig Conjecture is equivalent to the existence of a ρ-labelling of every tree.

The famous Graceful Tree Conjecture, also known as the Ringel-Kotzig-Rosa Conjecture, which is some sources wrongly called the Ringel-Kotzig Conjecture instead of the Rosa Conjecture, can be stated very simply:

Conjecture 2.7. (**Graceful Tree Conjecture** [29],[30],[13]) All trees are graceful.

The next part of this section is devoted to complete graph decompositions based on Rosa's dissertation [29] and his paper [30], and on a survey paper [9] by El-Zanati and Vanden Eynden.

Definition 2.8. ([15, Definition 1.3.1]) Let K be a graph and G be its subgraph. A *G-decomposition* of K is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that the edge sets of the graphs G_i form a partition of the edge set of K. The elements of Δ are called *G-blocks*.

For a complete graph K_n with n vertices and the vertex set $\mathbb{Z}_n$, by *clicking* we mean increasing the vertex labels of a graph by one (cf. [9]). The number $\min\{|i-j|, n-|i-j|\}$ is said to be the *length* of an edge ij in K_n. It follows that if the elements of the vertex set V_{K_n} are placed in the plane in order as vertices of a regular n-gon, the length of edge ij means the shortest distance around the polygon between i and j. The edge ij is said to be a *wrap-around edge* provided the length of ij is $n - |i-j|$ (cf. [9]). It is worth to notice that clicking an edge does not change its length.

Definition 2.9. ([15, Definition 1.3.2]) A G-decomposition Δ is called *cyclic* if the following holds: if Δ contains a graph H, then it contains also the graph H' obtained by clicking H.

The relationship between the decomposition of a complete graph and the graceful labelling of a graph can be well illustrated on the following example:

Example 2.10. ([15, Pages 19-20]) We consider a gracefully labelled graph G of size $m = 7$ (cf. the left part of Figure 1 below).

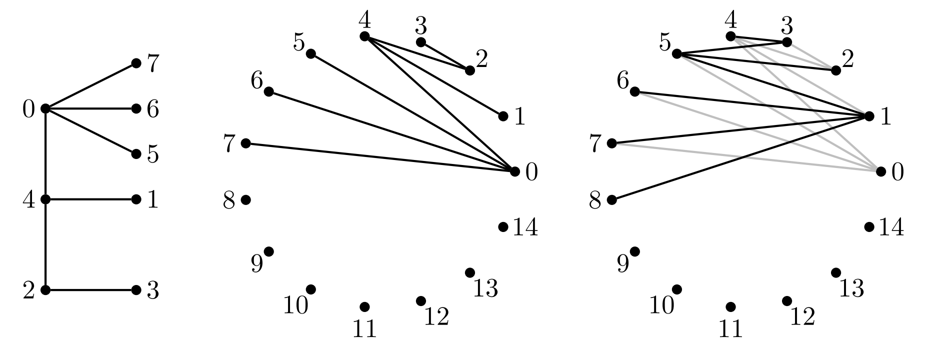


Figure 1. Cyclic decomposition of K_{15} induced by a gracefully-labelled graph [15, Figure 1.4]

To illustrate how this labelling induces a cyclic decomposition of K_{15}, we firstly place the 15 vertices onto a circle in the plane and label them $0, 1, 2, \ldots, 14$. Then we add the edges connecting the vertices $0, 1, \ldots, 7$ according to the edges of G (cf. the middle part of Figure 1). As the graph G is gracefully labelled, it follows that the edges have different lengths. The embedded copy of G forms the first block of the cyclic decomposition of the complete graph K_{15}. Then by subsequently clicking the first embedded graph (cf. the right part of Figure 1) one can get the other blocks. It can be seen that after $2m$ clickings every edge of K_{15} belongs to exactly one block. Hence we get a cyclic decomposition of the complete graph K_{15}, which is induced by the graceful graph G. And it is the graceful labelling of the graph G, which guarantees that all edges of G have different lengths when embedded to K_{15}. Finally, we notice that if any edge is replaced by an edge of the same length, then the new graph will again induce a cyclic decomposition of K_{15}. For instance, in the previous example we can replace the edge $\{1, 4\}$ by any edge of length 3 or by any (wrap-around) edge for which the absolute difference of its end-vertices is $15 - 3 = 12$, for instance the edge $\{1, 13\}$ (cf. Figure 2).

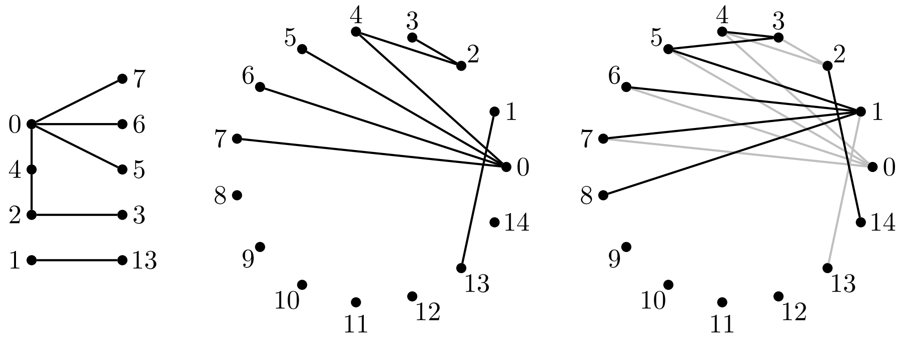


Figure 2. Cyclic decomposition of a complete graph induced by a ρ-labelled graph

We can replace an edge of length i in a given gracefully labelled graph G of size m by another edge of length i or a (wrap-around) edge of length $2m+1-i$ in order to induce again a cyclic decomposition of the complete graph K_{2m+1}. This then corresponds to the definition of ρ-labelling as a generalization of β-labelling.

To explain the illustrated connection between ρ-labellings and cyclic decompositions of complete graphs, we present the following result from [30].

Theorem 2.11. ([30, Theorem 7], [15, Theorem 1.3.3]) A cyclic decomposition of the complete graph K_{2m+1} into subgraphs isomorphic to a given graph G with m edges exists if and only if there exists a ρ-labelling of the graph G.

Hence a cyclic decomposition of the complete graph K_{2m+1} into subgraphs isomorphic to a given graph G of size m exists provided there exists any of the four labellings of G from the Rosa hierarchy: α-, β-, σ- or ρ.

Remark 2.12. ([29], [15, Remark 1.3.10]) A decomposition of K_n into triangles is obviously equivalent to a *Steiner triple system* of order n. It is a system of three-element subsets of an n-element set such that each pair of elements is precisely in one subset (cf., for example, [6]). In the complete graph K_n the triples are represented by graphs isomorphic to C_3 and the pairs by edges. Rosa in [29, Theorem 2.4] presents his own proof of the result of Peltesohn [26] that a cyclic decomposition of the complete graph K_n into triangles exists if and only if $n \equiv 1$ or 3 mod 6 and $n \neq 9$.

3 Graph chessboards, labelling sequences and labelling relations

We start this section with describing an old idea of the first author from the early 1990s that says that every labelled graph of order n can be visualized by a *chessboard*, i.e. a table with n rows and n columns, in which every edge uv is represented by a pair of dots with coordinates $[u,v]$ and $[v,u]$. Of course, one can also obtain such a graph chessboard by taking an adjacency matrix of a graph and by placing dots to the cells corresponding to "ones" in the matrix and by leaving the cells corresponding to "zeros" in the matrix empty.

While in Chapter 2 of [15] we described several types of graph chessboards, namely a simple chessboard, double chessboard, dual chessboard, M-chessboard and twin chessboard, in this survey we focus only on the concept of the simple chessboard.

Remark 3.1. ([15, Remark 2.1.1]) We note that while polishing the text of the book [15] for submission, we had discovered that an idea of a *GL-matrix*, similar to our simple chessboard, was presented already in 1990 in the paper [35], though the authors used it there only as a tool to deal with a generalization of Bodendiek's conjecture about graceful graphs and not in the directions presented in [15] and in this survey. The paper [3] (submitted in July 2014 essentially at the time when [15] was submitted) independently uses as its main tool the concept of a *graceful triangle* which again is similar to our idea of a simple chessboard; [3] overlaps with our text in the use of this tool to give other proofs to Sheppard's formulas, which are presented here in Theorems 3.3 and 3.4.

In the following part by G we mean a graph whose vertices are labelled by distinct numbers from the set $\{0, 1, 2, \ldots, n-1\}$. Consider a *chessboard of size* n, i.e. table with n rows (numbered $0, 1, \ldots, n-1$ from the top to the bottom) and n columns (numbered $0, 1, \ldots, n-1$ from the left to the right) as in Figure 3. By the cell with coordinates $[i, j]$ we mean the cell in the i-th row and the j-th column of the table. By the *r-th diagonal* (or the *diagonal with value* r) we mean the set of all cells with coordinates $[i, j]$ where $i - j = r$ and $i \geq j$. Then the 0-th diagonal is said to be the *main diagonal* and the other r-th diagonals (diagonals with values r) where $r \neq 0$ are said to be *associate diagonals*.

To a labelled graph G we now assign such a table in the way that for every edge $uv \in E_G$ we place a pair of dots in the pair of cells with coordinates $[u, v]$ and $[v, u]$. This table is said to be a *simple chessboard* of the graph G (see Figure 3). In [15, Chapter 2] we denoted the dots for the edge $uv \in E_G$ by $d_{u,v}$ and $d_{v,u}$ to emphasize their coordinates.

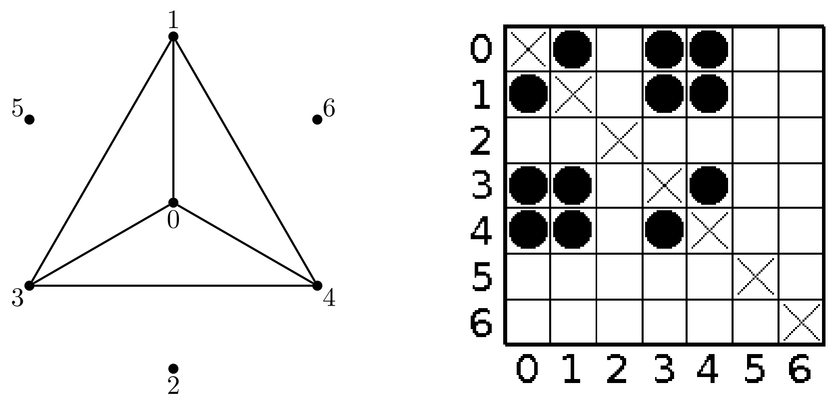


Figure 3. A labelled graph G and its simple graph chessboard [15, Figure 2.1]

It is obvious that our simple chessboards are symmetric about the main diagonal and we sometimes say that the dot $d_{v,u}$ is *symmetric* to the dot $d_{u,v}$.

We notice that on the simple chessboard of a graph (exactly as on its adjacency matrix) one can easily see some properties, for instance the size of the graph or degrees of the vertices. In [15, Chapter 2] we illustrate how to see some other properties of a graph in its simple chessboard, for instance, to see if a given graph is a tree.

Since the dots on the r-th associate diagonal correspond exactly to the edges with the induced label r, we say that a simple chessboard is *graceful* if there is exactly one dot on each of its associate diagonals. This exactly corresponds to the fact that the graph labelling is graceful. For instance, in Figure 4 we see a gracefully labelled graph G and the simple chessboard of G, which therefore is graceful.

Formally, we formulated the above fact in the following proposition.

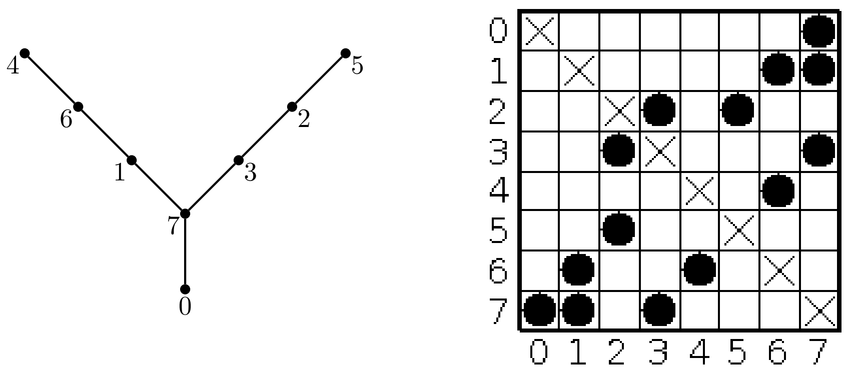


Figure 4. A gracefully labelled graph G and its graceful simple chessboard [15, Figure 2.2]

Proposition 3.2. ([15, Proposition 2.2.3]) A graph G is gracefully labelled if and only if its assigned simple chessboard is graceful.

Then one can use this result and the graceful simple chessboards to calculate the number of all gracefully labelled graphs of a given size. In [15, Chapter 2] we notice that the following result was originally proved by Sheppard in [32].

Theorem 3.3. ([32], [15, Theorem 2.2.4]) There are exactly $m!$ gracefully labelled graphs with m edges, vertices labelled with selected numbers from the set $\{0, 1, 2, \ldots, m\}$ and without isolated vertices.

In [32] Sheppard also counted the number of α-labelled graphs of size m. In [15] we presented the same result by our methods. (We notice that we only considered graphs without isolated vertices.)

Theorem 3.4. ([32], [15, Theorem 2.7.1]) The number of all α-labelled graphs of size m is

$$2\sum_{i=0}^{k}(i!)^2.(i+1)^{m-2i}, \qquad \text{if } m \text{ is even,}$$

$$2\sum_{i=0}^{k}(i!)^2.(i+1)^{m-2i} - (k!)^2.(k+1), \qquad \text{if } m \text{ is odd,}$$

where $k = \left\lfloor \frac{m-1}{2} \right\rfloor$.

Remark 3.5. ([15, Remark 2.7.2]) When comparing our formula above with that presented in [3], we notice that the equivalence of both formulas can be seen after substituting $k = \left\lfloor \frac{m-1}{2} \right\rfloor$ into our formula. There is a slight difference in the case of odd m that while we count the square two times and then we subtract it once, in [3] they do not count it and then add it.

Finally, by methods developed in Chapter 2 of [15], we were able to calculate the numbers of σ- and ρ-labelled graphs of size m.

Theorem 3.6. ([15, Theorem 2.7.3]) There are exactly $\frac{(2m)!}{m!}$ σ-labelled graphs of size m.

Theorem 3.7. ([15, Theorem 2.7.4]) There are exactly $(2m+1)^m$ ρ-labelled graphs of size m.

Then we calculated and displayed the numbers of graphs of sizes at most nine labelled by these four Rosa labellings in Table 1.

m	α-labelled	β-labelled	σ-labelled	ρ-labelled
1	1	1	2	3
2	2	2	12	25
3	4	6	120	343
4	10	24	1 680	6 561
5	30	120	30 240	161 051
6	106	720	665 280	4 826 809
7	426	5 040	17 297 280	170 859 375
8	1 930	40 320	518 918 400	6 975 757 441
9	9 690	362 880	17 643 225 600	322 687 697 779

Table 1. Number of labelled graphs of given size up to 9 [15, Table 2.1]

In Chapter 2 of [15] we also introduced the concepts of a *labelling distance* of trees and of a *graceful index* of a tree.

Consider a tree T of size m with vertices $v_0, v_1, v_2, \dots, v_m$ and consider bijective maps α, β from the set $\{v_0, v_1, v_2, \dots, v_m\}$ to the set $\{0, 1, 2, \dots, m\}$. Since these maps are labellings of the tree T, we denoted the obtained labelled graphs by T^α and T^β, respectively. Of course, each such map can be understood as a permutation of $\{0, 1, 2, \dots, m\}$. Because every permutation is a composition of finitely many transpositions (cyclic permutations of length two), we introduced in [15] the following concept:

Definition 3.8. ([15, Definition 2.8.1]) Let α, β be labellings of the vertices of a tree T with numbers $\{0, 1, \dots, m\}$. The minimal number of transpositions of vertex labels needed to change T^α to T^β is called a *labelling distance* of T^α and T^β and it is denoted by $\mathcal{D}(T^\alpha, T^\beta)$.

Of course, the labelling distance is a metric with non-negative integer values and we introduced then a *labelling distance* of a labelled tree.

Definition 3.9. ([15, Definition 2.8.2]) Let T^α be a labelled tree and let

$$\mathcal{G} = \{T^\gamma \mid \gamma \text{ is graceful}\}.$$

Then the minimal element of the set $\{\mathcal{D}(T^\alpha, T^\gamma) \mid T^\gamma \in \mathcal{G}\}$ will be called a *graceful distance* of the labelled tree T^α and denoted $\mathcal{G}(T^\alpha)$. If the set $\mathcal{G}$ is empty, we define $\mathcal{G}(T^\alpha) = \infty$.

The graceful distance of a labelled tree T^α can be understood as a measure of how far the labelling α is from a graceful labelling. Hence the graceful distance of any gracefully labelled tree is 0 and the graceful distance of any graceful tree (i.e. tree which can be gracefully labelled) possessing a non-graceful labelling α is some natural number. We denoted the graceful distance of a non-graceful tree by ∞.

It is interesting that the Graceful Tree Conjecture can then be formulated as follows:

Conjecture 3.10. ([15, Conjecture 2.8.3]) The graceful distance of all labelled trees is different from ∞.

Further, we defined a graceful index of a tree as follows:

Definition 3.11. ([15, Definition 2.8.4]) Let T be a tree of size m and let

$$\mathcal{L} = \{T^\alpha \mid \alpha \text{ is a labelling of T}\}$$

be the set of all its vertex labellings. Then $\mathcal{I}(T) = \max\{\mathcal{G}(T^\alpha) \mid T^\alpha \in \mathcal{L}\}$ will be called a *graceful index* of the tree T. In case $\mathcal{G}(T^\alpha) = \infty$ for some $T^\alpha \in \mathcal{L}$, we understand that $\mathcal{I}(T) = \infty$.

We noticed that the graceful index of a tree is well-defined since each graceful distance $\mathcal{G}(T^\alpha)$ (i.e. the minimal number of transpositions needed to change T^α to a gracefully labelled tree) is well-defined. Further, we remarked that the graceful index $\mathcal{I}(T)$ of a tree T can be understood as the maximal number of transpositions needed to reach a graceful labelling of T from an arbitrarily chosen labelling T^α. Then of course the Graceful Tree Conjecture claims that $\mathcal{I}(T) \neq \infty$ for every tree T.

After introducing the graceful index of a tree, one can make the following two easy observations:

Proposition 3.12. ([15, Proposition 2.8.5]) *Let T be a graceful tree of size m. Then $\mathcal{I}(T) \leq m$.*

Proposition 3.13. ([15, Proposition 2.8.6]) *If T is a star, then $\mathcal{I}(T) = 1$.*

We ended Chapter 2 of [15] by illustrating decompositions of complete graphs on the corresponding chessboards. Of course, a simple chessboard corresponding to a complete labelled graph has a dot in each cell (see Figure 5).

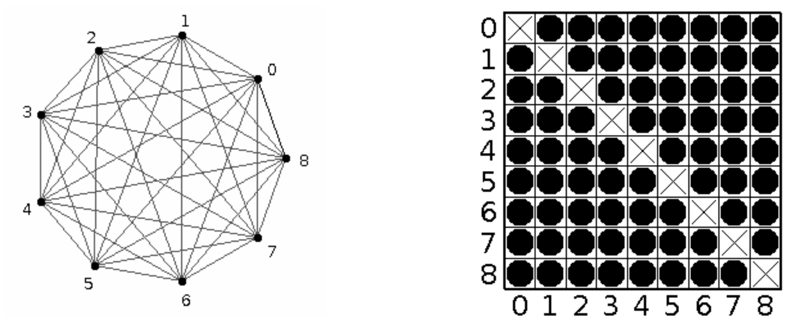


Figure 5. The complete graph K_9 and its simple graph chessboard [15, Figure 2.14]

We pointed out that an edge decomposition of a complete graph can be seen in its simple chessboard as a partition of all the dots on associate diagonals into blocks of dots. Because a decomposition of the given complete graph into triangles (i.e. graphs isomorphic to the cycles of length 3) corresponds to a Steiner triple system (cf. [29], p. 20 and [7]), one can find a Steiner triple system by using our simple chessboard. We only need to find a partition of the set of all dots under the main diagonal into blocks of triples with the coordinates (a, b), (b, c), (a, c). In Figure 6 each triple corresponds to a set of three dots in the chessboard denoted by different letters, where the triples are A: $\{0, 1, 2\}$, B: $\{3, 4, 5\}$, C: $\{6, 7, 8\}$, D: $\{0, 3, 6\}$, E: $\{1, 4, 7\}$, F: $\{2, 5, 8\}$, G: $\{0, 4, 8\}$, H: $\{1, 5, 6\}$, I: $\{2, 3, 7\}$, J: $\{0, 5, 7\}$, K: $\{1, 3, 8\}$, L: $\{2, 4, 6\}$:

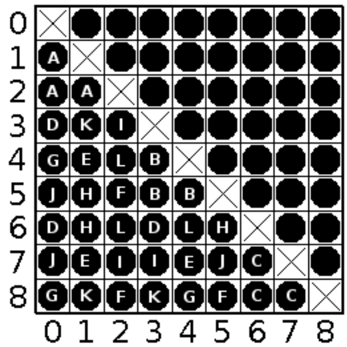


Figure 6. Chessboard visualization of a decomposition of the graph K_9 [15, Figure 2.15]

If a decomposition of a complete graph contains with each graph G also the graph obtained by clicking G, then it was said to be cyclic. We also recall that *clicking* an edge means increasing the labels of both end-vertices of the edge by one (modulo n), where n is the order of the given complete graph. Since in the simple chessboard each edge is represented by a dot, clicking of an edge corresponds to moving its dot inside the diagonal one cell towards bottom-right. We illustrated this in Figure 7. There the dot in the middle chessboard was obtained by clicking the dot in the left chessboard, and provided we click it once more, we jump into a complement diagonal (see the right chessboard in Figure 7).

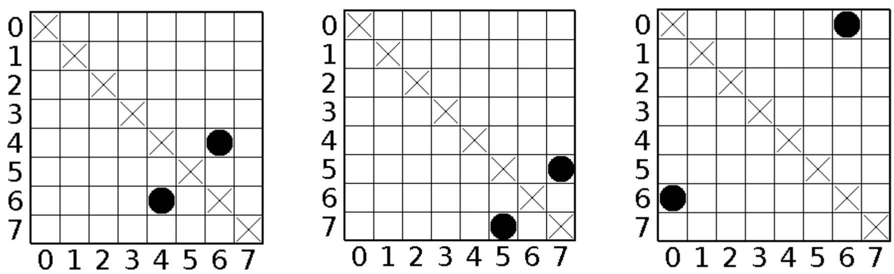


Figure 7. Clicking of a graph [15, Figure 2.16]

Then, step-by-step, each dot during the $2m$ subsequent clickings, takes place in all dots in its associate diagonal, so the simple chessboard has a dot in each of its cells. Hence one can see that we indeed generated a cyclic decomposition of the given complete graph.

We conclude this part by saying that this representation of clicking provided a very nice insight into Rosa's result presented both in [29] and [30] saying that a cyclic decomposition of the complete graph K_{2m+1} into subgraphs isomorphic to a given graph G of size m exists if and only if there exists a ρ-labelling of the graph G.

The concept of a *labelling sequence* was introduced by Sheppard in his paper [32]. The idea is that each gracefully labelled graph can be represented by a sequence of non-negative integers and the algebraic properties of the labelling sequence correspond to the properties of the gracefully labelled graph.

Definition 3.14. ([32], [15, Definition 3.1.1]) For every positive integer m, the sequence of non-negative integers $(j_1, j_2, \ldots, j_m)$, denoted (j_i), is a *labelling sequence* if

$$0 \leq j_i \leq m - i \quad \text{for all } i \in \{1, 2, \ldots, m\}. \tag{LS}$$

In [32] Sheppard showed that there is a unique correspondence between gracefully labelled graphs (without isolated vertices) and labelling sequences. Hence the labelling sequences can also (exactly as simple chessboards) be understood as a tool to encode graceful labellings of graphs. The Sheppard correspondence was described as follows:

Theorem 3.15. ([32], [15, Theorem 3.1.2]) There is a one-to-one correspondence between graphs with m edges having a graceful labelling f and between labelling sequences (j_i) of m terms (entries). The correspondence is given by

$$j_i = \min\{f(u), f(v)\}, \quad i \in \{1, 2, \ldots, m\},$$

where u, v are the end-vertices of the edge labelled i.

Since graceful simple chessboards are simple chessboards corresponding to gracefully labelled graphs, one can state the following:

Proposition 3.16. ([15, Proposition 3.1.3]) There is a one-to-one correspondence between all graceful simple chessboards and all labelling sequences.

The correspondence between the graceful simple chessboards and the labelling sequences was illustrated on the following example:

Example 3.17. ([15, 3.1.4]) The graceful simple chessboard in Figure 8 corresponds to the labelling sequence (2, 4, 2, 3, 1, 1, 0). The labelling sequence is the sequence of the bottom coordinates of dots on the associate diagonals.

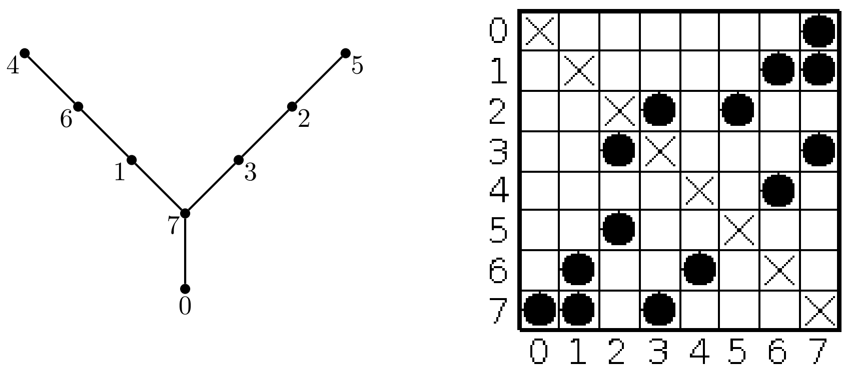


Figure 8. A gracefully labelled graph and the graceful simple chessboard corresponding to its labelling sequence (2, 4, 2, 3, 1, 1, 0) [15, Figure 3.1]

We were able to characterize the labelling sequences corresponding to the α-labellings of graphs.

Definition 3.18. ([15, Definition 3.1.5]) A labelling sequence corresponding to an α-labelling is called a *balanced* labelling sequence.

Corollary 3.19. ([15, Corollary 3.1.6]) labelling sequence (j_i) is balanced if and only if

$$j_1 + 1 - i \leq j_i \leq j_1 \quad \text{for all } i \in \{1, 2, \ldots, m\}.$$

The third important tool to study gracefully labelled graphs, after the simple chessboards and the labelling sequences, is that of a *labelling relation*.

Definition 3.20. ([15, Definition 3.5.1]) Let $L = (j_1, j_2, \ldots, j_m)$ be a labelling sequence. Then the relation $A(L) = \{[j_i, j_i + i], i \in \{1, 2, \ldots, m\}\}$ is called a *labelling relation* assigned to the labelling sequence L.

1	2	3	...	m
j_1	j_2	j_3	...	j_m
$j_1 + 1$	$j_2 + 2$	$j_3 + 3$	...	$j_m + m$

Figure 9. Displaying the labelling relation in a table [15, Figure 3.3]

To visualize a labelling relation, in [15] we used a *labelling table* (see Figure 9). It can be seen that its heading contains the numbers $1, 2, \ldots, m$, then the first row contains the numbers from the labelling sequence and the second row contains the sums of numbers from the heading and the first row. It is obvious that the pairs from first and second row in each column are then the elements of the labelling relation (and thus also the edges of the graph).

Example 3.21. ([15, Example 3.5.2]) In Figure 10 we can see the labelling table assigned to the labelling sequence (2, 4, 2, 3, 1, 1, 0).

1	2	3	4	5	6	7
2	4	2	3	1	1	0
3	6	5	7	6	7	7

Figure 10. An example of the labelling table [15, Figure 3.4]

In [15] we showed how the labelling relations are related with paths in a given graph:

Theorem 3.22. ([15, Theorem 3.5.3]) Let G be a graph of diameter d and $A(L)$ be the labelling relation corresponding to some labelling sequence L of G. Then the relation

$$R = \left(A(L) \cup A(L)^{-1}\right)^{\lceil \frac{d}{2} \rceil + 1} \tag{RL}$$

is the relation containing exactly the pairs of vertices v_i, v_j such that there is a path between v_i and v_j.

Then we provided yet another answer to the Sheppard's Problem as we were able to characterize labelling sequences of gracefully labelled trees via labelling relations using (RL) above:

Theorem 3.23. ([15, Theorem 3.5.4]) A sequence $(j_1, j_2, \ldots, j_m)$ is a labelling sequence of a gracefully labelled tree G of order $m + 1$ if and only if

$$\{0, 1, \ldots, m\}^2 \subseteq R,$$

where R is the relation from (RL).

We mentioned that Sheppard asked which classes of gracefully labelled graphs could be characterized via the labelling sequences. In [15] we introduced two similar problems:

Problem 3.24. ([15, Problem 3.5.5]) Are there classes of gracefully labelled graphs which could be characterized via chessboard representations?

Problem 3.25. ([15, Problem 3.5.6]) Are there classes of gracefully labelled graphs which could be characterized via labelling relations?

4 Chessboard representations of classes of graceful graphs

In this section we present a survey on chessboard representations of classes of graceful graphs taken from [15, Chapter 4]. More precisely, we present not only characterisations of eleven classes of gracefully labelled graphs via their graceful simple chessboards, but also via the labelling sequences and labelling relations. In fact, in [15, Chapter 4] we usually firstly characterized gracefully labelled graphs in those eleven classes via their labelling sequences, and only then as a corollary came a complete characterization also by the corresponding graceful simple chessboard and labelling relation. The characterizations via the labelling sequences can be understood as a (partial) answer to the problem of Sheppard mentioned earlier.

4.1 Stars

By the *star* S_n is meant a tree of order $n+1$, which is isomorphic to the complete bipartite graph $K_{1,n}$, thus one vertex – it is called the *central vertex* of the star – has degree n and all the other vertices have degree 1 (see Figure 11).

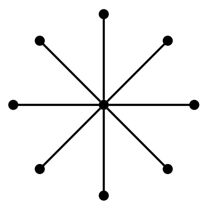


Figure 11. Example of a star [15, Figure 4.1]

In [15] we firstly characterized stars (with a graceful labelling) via their labelling sequences.

Theorem 4.1. ([15, Theorem 4.1.1]) A sequence $(j_1, j_2, \ldots, j_m)$ of non-negative integers is a labelling sequence of a star G (with a graceful labelling) if and only if either

a) $j_i = 0$, for every $i \in \{1, 2, \ldots, m\}$, or (LSS1)

b) $j_i = m - i$, for every $i \in \{1, 2, \ldots, m\}$. (LSS2)

Now as a corollary of the previous theorem comes a complete characterization of the stars via their labelling sequences, labelling relations and graceful simple chessboards.

Corollary 4.2. ([15, Corollary 4.1.2]) Let G be a graph of size m that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a star,

(2) $\mathcal{L} = \{(0, 0, 0, \dots, 0), (m-1, m-2, m-3, \dots, 0)\}$,

(3) for all $L \in \mathcal{L}$ the labelling relation $A(L)$ equals either $\{[0, i] \mid i \in \{1, 2, \dots, m\}\}$ or $\{[i, m] \mid i \in \{0, 1, \dots, m-1\}\}$,

(4) all graceful chessboards of G look like one of the chessboards in Figure 12.

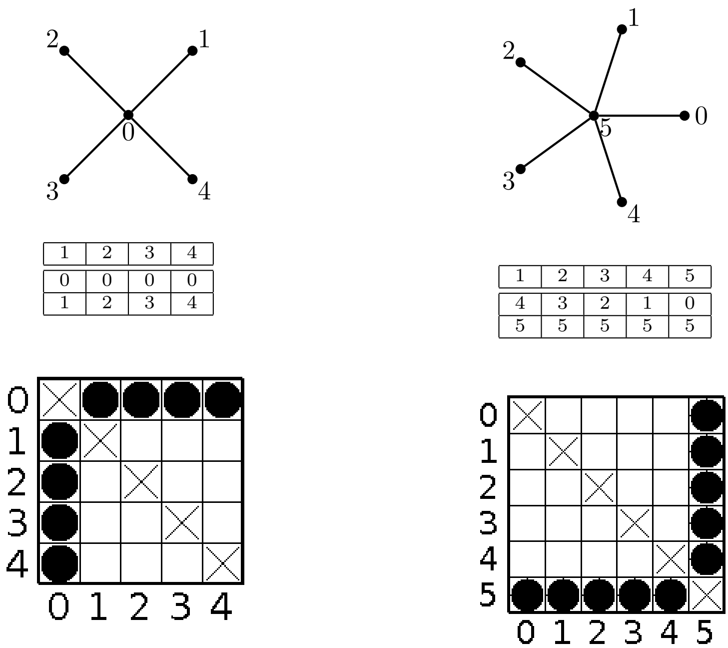


Figure 12. Representations of graceful labellings of stars [15, Figure 4.2]

Example 4.3. ([15, Example 4.1.3]) Sequences (0, 0, 0, 0) and (4, 3, 2, 1, 0) are labelling sequences of stars. Corresponding labelling relations, chessboards and graph diagrams are in Figure 12. Graceful chessboard of a star has all the dots (under the main diagonal) either in the first column or in the last row.

4.2 Double Stars

By a *double star* is meant a graph obtained by concatenating any number of paths of length 2 in their end-vertices (see Figure 13). The common vertex of all the paths is called a *central vertex* of a double star.

Again, in [15] we firstly characterized double stars via their labelling sequences.

Theorem 4.4. ([15, Theorem 4.2.1]) Let G be a graph of size m. Then G is a double star if and only if there exists a labelling sequence $(j_1, j_2, \dots, j_m)$ of G such that

$$j_i = \begin{cases} 0, & \text{if i is even,} \\ \frac{m-i+1}{2}, & \text{if i is odd.} \end{cases} \qquad (LSDS)$$

This also yields that double stars are graceful graphs.

An example of a graceful labelling of a double star can be seen in Figure 14.

As a consequence of the previous theorem now comes a full characterization of the stars via their labelling sequences, labelling relations and simple graph chessboards.

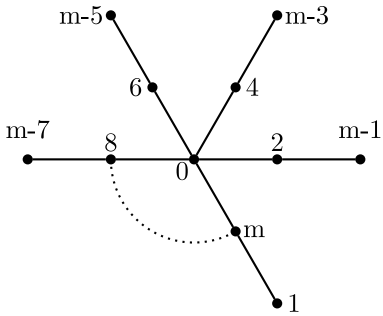
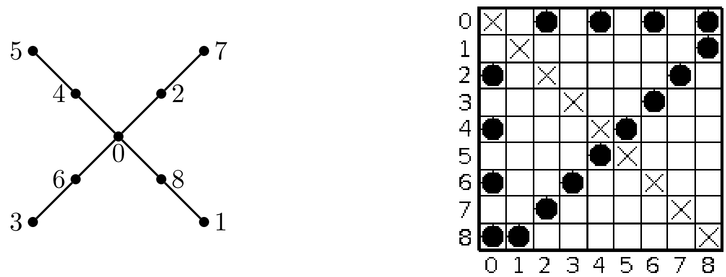


Figure 13. Graceful labelling of a double star [17, Figure 4.3]



1	2	3	4	5	6	7	8
4	0	3	0	2	0	1	0
5	2	6	4	7	6	8	8

Figure 14. Representations of a graceful labelling of a double star [15, Figure 4.4]

Corollary 4.5. ([15, Corollary 4.2.2]) Let G be a graph of size m that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a double star,

(2) there exists a labelling sequence $L \in \mathcal{L}$ which satisfies (LSDS),

(3) for some $L \in \mathcal{L}$ the labelling relation is

$$A(L) = \left\{[0, 2i] \mid i \in \left\{1, 2, \dots, \frac{m}{2}\right\}\right\} \cup \left\{[i, m+1-i] \mid i \in \left\{1, 2, \dots, \frac{m}{2}\right\}\right\},$$

(4) some chessboard of G looks like the chessboard in Figure 14.

Example 4.6. ([15, Example 4.2.3]) Sequences (3, 0, 2, 0, 1, 0) and (4, 0, 3, 0, 2, 0, 1, 0) are labelling sequences of double stars, the latter is displayed in Figure 14 with its corresponding graph diagram, labelling relation and graph chessboard.

4.3 Paths

There is no need to introduce the concept of a *path* but we introduce a special class of labelling sequences related to them.

Definition 4.7. ([15, Definition 4.3.1]) Let $(j_1, j_2, \ldots, j_m)$ be a labelling sequence. We shall call it the 2-*linear labelling sequence* of length m if

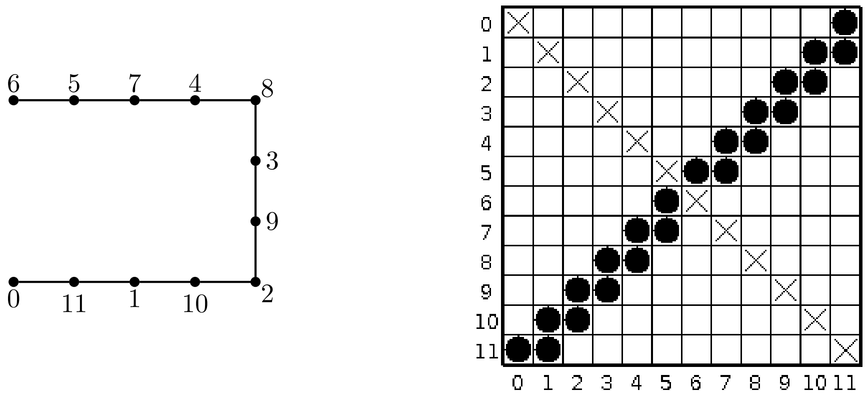
$$j_i = \left\lfloor \frac{m-i+1}{2} \right\rfloor \quad \text{for all } i \in \{1, 2, \ldots, m\}. \qquad (LSP)$$

Example 4.8. ([15, Example 4.3.2]) Sequences (5, 4, 4, 3, 3, 2, 2, 1, 1, 0) and (5, 5, 4, 4, 3, 3, 2, 2, 1, 1, 0) are 2-linear labelling sequences of lengths 10 and 11, respectively.

We characterized paths via the 2-linear labelling sequences as follows:

Theorem 4.9. ([15, Theorem 4.3.3]) Let G be a graph of size m. Then G is a path if and only if there exists a 2-linear labelling sequence of length m of G. Moreover, the end-vertices of this path are labelled 0 and $\lceil \frac{m}{2} \rceil$.

An example of a graceful labelling of a path can be seen in Figure 15.



1	2	3	4	5	6	7	8	9	10	11
5	5	4	4	3	3	2	2	1	1	0
6	7	7	8	8	9	9	10	10	11	11

Figure 15. Representations of a graceful labelling of a path [15, Figure 4.5]

As a corollary we characterized paths via their labelling sequences, labelling relations and chessboards:

Corollary 4.10. ([15, Corollary 4.3.4]) Let G be a graph of size m that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a path,

(2) there exists a labelling sequence $L \in \mathcal{L}$ which satisfies (LSP),

(3) for some $L \in \mathcal{L}$ the labelling relation

$$A(L) = \{[u, v] \in \{0, 1, \ldots, m\}^2 \mid u < v \wedge m \leq u + v \leq m + 1\},$$

(4) some chessboard of G looks like the chessboard in Figure 15.

4.4 Caterpillars

By a *caterpillar* is meant a tree such that after the removal of its leaves (as the vertices of degree one) one obtains a path (see Figure 16).

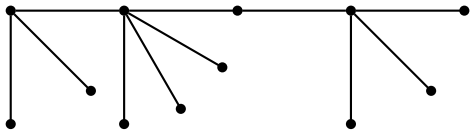


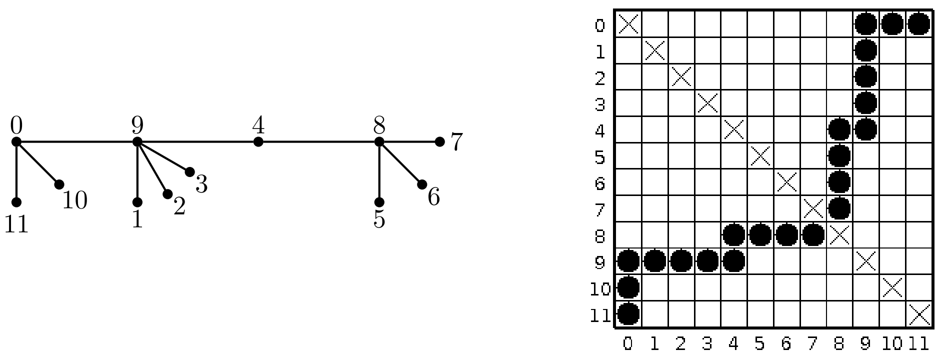
Figure 16. Example of a caterpillar [15, Figure 4.6]

We showed in [15] that every caterpillar can be represented by a non-increasing labelling sequence.

Theorem 4.11. ([15, Theorem 4.4.1]) Graph G of size m is a caterpillar if and only if there exists a labelling sequence $(j_1, j_2, \dots, j_m)$ of G such that

$$0 \leq j_i - j_{i+1} \leq 1 \quad \text{for every } i \in \{1, 2, \dots, m-1\}. \qquad (LSC)$$

An example of a graceful labelling of a caterpillar can be seen in Figure 17.



1	2	3	4	5	6	7	8	9	10	11
7	6	5	4	4	3	2	1	0	0	0
8	8	8	8	9	9	9	9	9	10	11

Figure 17. Representations of a graceful labelling of a caterpillar

As a consequence of the previous theorem we again fully characterized caterpillars via their labelling sequences, labelling relations and chessboards.

Corollary 4.12. ([15, Corollary 4.4.2]) Let G be a graph of size m that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a caterpillar,

(2) there exists a labelling sequence $L \in \mathcal{L}$ which satisfies (LSC),

(3) for some $L \in \mathcal{L}$ the labelling relation has a property that if $[u, v] \in A(L)$ and $[u, v] \neq [0, m]$, then there exists a pair $[u', v'] \in A(L)$ such that $v' - u' = v - u + 1$ and either $u = u'$ or $v = v'$,

(4) some chessboard of G looks like the chessboard in Figure 17.

Example 4.13. ([15, Example 4.4.3]) The sequence (7, 6, 5, 4, 4, 3, 2, 1, 0, 0, 0) is a labelling sequence of a caterpillar. Corresponding labelling relation, chessboard and graph diagram are in Figure 17. Chessboard of a caterpillar has its dots (under the main diagonal) arranged in a continuous non-decreasing zigzag line starting at the bottom left and heading towards the main diagonal.

4.5 Firecrackers

By a *firecracker* is meant a graph that one gets by connecting multiple isomorphic copies of a given star to the vertices of a path (as it can be seen in Figure 18). The following characterization of firecrackers via their labelling sequences can be easily derived using graph chessboard (see again Figure 18).

Theorem 4.14. ([15, Theorem 4.5.1]) Let G be a graph of size $m = kn - 1$ for some $k, n \in \mathbb{N}$. Then G is a firecracker containing k stars S_{n-1} if and only if there exists a labelling sequence $(j_1, j_2, \dots, j_m)$ of G such that

$$j_i = \begin{cases} n\lfloor \frac{kn-i}{2n} \rfloor, & \text{if } i \equiv 0 \pmod{n}, \\ n\lceil \frac{kn-i}{2n} \rceil - 1 - (i \bmod 2n), & \text{if } i \equiv 1, 2, 3, \dots, n-1 \pmod{2n}, \\ n\lceil \frac{kn-i}{2n} \rceil - 1, & \text{if } i \equiv n+1, \dots, 2n-1 \pmod{2n}. \end{cases}$$

As a consequence we fully characterized firecrackers via their labelling sequences, labelling relations and graph chessboards.

Corollary 4.15. ([15, Corollary 4.5.2]) Let G be a graph of size $m = kn - 1$ that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a firecracker,

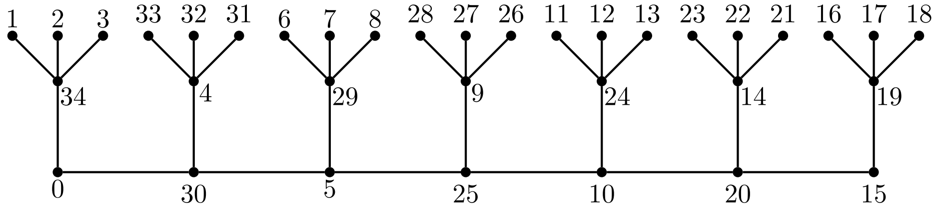
(2) there exists a labelling sequence $L \in \mathcal{L}$ which satisfies (LSFC),

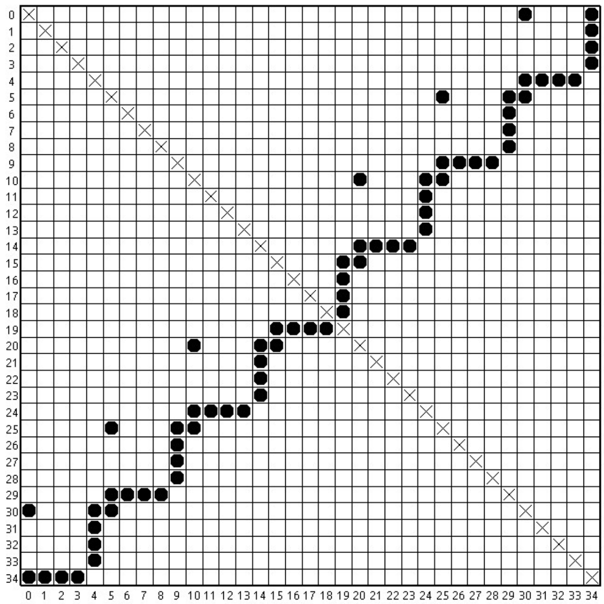
(3) for some $L \in \mathcal{L}$ the labelling relation is

$$A(L) = \left\{ \left[n\lfloor \frac{kn-i}{2n} \rfloor, n\lfloor \frac{kn-i}{2n} \rfloor + i \right] \mid i \equiv 0 \pmod{n} \right\} \cup$$
$$\left\{ \left[n\lceil \frac{kn-i}{2n} \rceil - 1 - (i \bmod 2n), n\lceil \frac{kn+i}{2n} \rceil - 1 \right] \mid i \equiv 1, \dots, n-1 \pmod{2n} \right\} \cup$$
$$\left\{ \left[n\lceil \frac{kn-i}{2n} \rceil - 1, n\lceil \frac{kn-i}{2n} \rceil - 1 + i \right] \mid i \equiv n+1, \dots, 2n-1 \pmod{2n} \right\},$$

(4) some chessboard of G looks like the chessboard in Figure 18.

Example 4.16. ([15, Example 4.5.3]) The sequence (18, 17, 16, 15, 15, 14, 14, 14, 14, 10, 13, 12, 11, 10, 10, 9, 9, 9, 9, 5, 8, 7, 6, 5, 5, 4, 4, 4, 4, 0, 3, 2, 1, 0) is a labelling sequence of a firecracker. Corresponding labelling relation, chessboard and graph diagram are in Figure 18.





1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
18	17	16	15	15	14	14	14	14	10	13	12	11	10	10	9	9
19	19	19	19	20	20	21	22	23	20	24	24	24	24	25	25	26

18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
9	9	5	8	7	6	5	5	4	4	4	4	0	3	2	1	0
27	28	25	29	29	29	29	30	30	31	32	33	30	34	34	34	34

Figure 18. Representations of a graceful labelling of a firecracker [15, Figure 4.8]

4.6 Cycles

Again, there is no need to define the concept of a *cycle* but we introduce an operation on labelling sequences which we call a *shift*.

Definition 4.17. ([15, Definition 4.6.1])

Let $L = (j_1, j_2, \ldots, j_m)$ be a labelling sequence of length m. Let k and c be given non-negative numbers such that $1 \leq k \leq m$ and $0 \leq c \leq (m - k)$. Now let $L' =$

$(j'_1, j'_2, \dots, j'_{m+1})$ be an assigned labelling sequence of length $m+1$ such that

$$j'_i = \begin{cases} j_i, & \text{if } i < k \\ c, & \text{if } i = k \\ j_{i-1}, & \text{if } i > k. \end{cases}$$

Then we say that the labelling sequence L' was obtained by a c-shift at k.

Example 4.18. ([15, Example 4.6.2]) Labelling sequence (5, 1, 4, 3, 2, 1, 0) was obtained by a 1-shift at 2 from labelling sequence (5, 4, 3, 2, 1, 0).

In this subsection we stepped outside the class of trees and the situation is a bit different than previously. We firstly notice that L' defined above is indeed a labelling sequence (and hence it must represent a gracefully labelled graph), but the graphs corresponding to the sequences L and L' are not isomorphic (they have different sizes). In spite of this in some cases after removal of one edge from the obtained graph with L' we get the former graph with L. This fact is the content of the following lemma:

Lemma 4.19. ([15, Lemma 4.6.3])
Let $L = (j_1, j_2, \dots, j_m)$ be a balanced labelling sequence. Let L' be a labelling sequence obtained by a c-shift at k from L, for given non-negative numbers c and k. Let G, G' be the graphs corresponding to labelling sequences L, L', respectively. If we have that $j_a + a < j_b + b$ for every a, b such that $1 \leq a < k \leq b \leq m$, then the graph G is isomorphic to the graph $G' - \{c, c+k\}$ with the isomorphism

$$f(x) = \begin{cases} x+1, & \text{if } x \in \{j_i + i \mid i \in [k, m]\} \\ x, & \text{otherwise.} \end{cases}$$

We note that Rosa [30] showed that every cycle of length n such that $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ can be gracefully labelled. In [15] we showed that every cycle whose length satisfies the above condition has a labelling sequence, which can be obtained by a certain shift from some 2-linear labelling sequence.

Theorem 4.20. ([15, Theorem 4.6.4]) Let n be a non-negative number such that $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ and let $k = \lfloor \frac{n}{2} \rfloor$. Graph G is a cycle of length n if and only if it has a labelling sequence which can be obtained by a 0-shift at k from a 2-linear labelling sequence of length $n-1$.

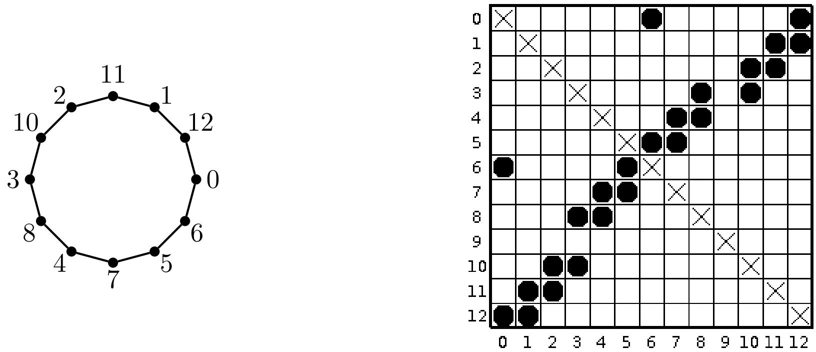
As a consequence of the previous theorem we obtained a full characterization of the graceful cycles of length m via their labelling sequences, labelling relations and simple graph chessboards.

Corollary 4.21. ([15, Corollary 4.6.5]) Let G be a graph of size m that can be gracefully labelled, where $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$, and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a cycle of length m,

(2) there exists a labelling sequence $L \in \mathcal{L}$ which can be obtained by a 0-shift at $\lfloor \frac{m}{2} \rfloor$ from the 2-linear labelling sequence of length $m-1$,

(3) for some $L \in \mathcal{L}$ the labelling relation $A(L) = \{[u, v] \in \{0, 1, \dots, m\}^2 \mid m-1 \leq u+v \leq m \wedge 0 < v-u < \lfloor \frac{m}{2} \rfloor\} \cup \{[u, v] \in \{0, 1, \dots, m\}^2 \mid m \leq u+v \leq m+1 \wedge \lfloor \frac{m}{2} \rfloor < v-u < m\} \cup \{\{0, 1, \dots, \lfloor \frac{m}{2} \rfloor\}\}$,



1	2	3	4	5	6	7	8	9	10	11	12
5	5	4	4	3	0	3	2	2	1	1	0
6	7	7	8	8	6	10	10	11	11	12	12

Figure 19. Representations of a graceful labelling of a cycle [15, Figure 4.9]

(4) some chessboard of G looks like the chessboard in Figure 19.

Example 4.22. ([15, Example 4.6.6]) Sequence (3, 2, 0, 2, 1, 1, 0) is a labelling sequence of the cycle of length 7 and it was obtained by performing a 0-shift at 3 from the 2-linear labelling sequence (3, 2, 2, 1, 1, 0) of length 6.

4.7 Fan Graphs

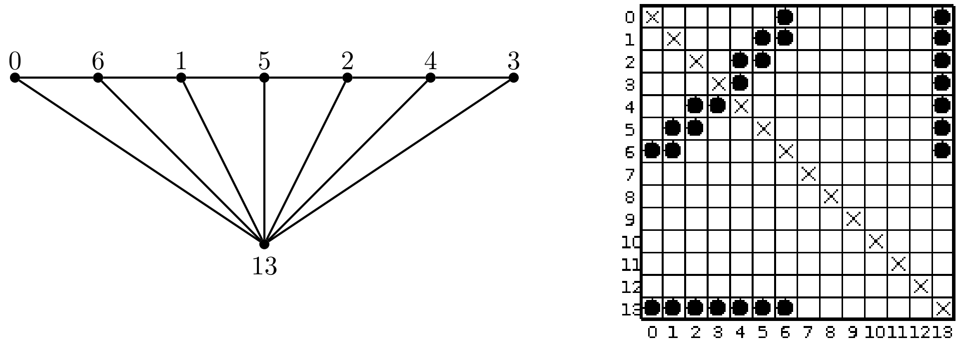
Because in the last section of our survey paper we deal with some extensions of fan graphs, for which we needed to modify slightly the previously used definition of a fan graph from [15], here we apply already this modified (and perhaps more natural) definition. Hence by the *fan graph* F_n we shall mean a join of a path P_n and a single vertex K_1 (see Figure 20 with the representations of the fan graph F_7). We require to consider $n \geq 2$ in order to have the shape of "a fan". Hence the fan graph F_n has order $n+1$ and size $2n-1$. (We notice that in [15, Section 4.4.7] we considered the fan graphs that are F_{n+1} in our present notation. Hence in the terminology of [15], in Figure 20 we would have the fan graph F_6 with considering $n = 6$. This is the reason why for our graphs G studied in [15, Section 4.4.7] we considered their size to be $m = 2n + 1$.)

The following theorem from [15] presented in the newly adopted terminology characterized fan graphs via their labelling sequences.

Theorem 4.23. ([15, Theorem 4.7.1]) Let G be a graph of size $m = 2n - 1$ for some positive natural number n. Then G is the fan graph F_n if and only if there exists a labelling sequence $(j_1, j_2, \ldots, j_m)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{n-i}{2} \right\rfloor, & \text{if } i \leq n-1, \\ m - i, & \text{if } i > n-1. \end{cases} \qquad (LSFN)$$

As a consequence we present – again, in the newly adopted terminology – our characterization from [15] of the fan graphs via their labelling sequences, labelling relations and graph chessboards.



1	2	3	4	5	6	7	8	9	10	11	12	13
3	2	2	1	1	0	6	5	4	3	2	1	0
4	4	5	5	6	6	13	13	13	13	13	13	13

Figure 20. Representations of a graceful labelling of the fan graph F_7 [15, Figure 4.10]

Corollary 4.24. ([15, Corollary 4.7.2]) Let n be a positive natural number. Let G be a graph of size $m = 2n - 1$ that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is the fan graph F_n,

(2) there exists a labelling sequence $L \in \mathcal{L}$ which satisfies (LSFN),

(3) for some $L \in \mathcal{L}$ the labelling relation is

$$A(L) = \left\{[i, n-1-i] \mid i \in \left\{0, 1, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor \right\}\right\} \cup$$

$$\cup \left\{[i, n-i] \mid i \in \left\{1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor \right\}\right\} \cup \{[i, m] \mid i \in \{0, 1, \dots, n-1\}\},$$

(4) some chessboard of G looks like the chessboard in Figure 20.

Example 4.25. ([15, Example 4.7.3]) Sequence (3, 2, 2, 1, 1, 0, 6, 5, 4, 3, 2, 1, 0) is a labelling sequence of the fan graph F_7 (see Figure 20) and (1, 1, 0, 3, 2, 1, 0) is a labelling sequence of the fan graph F_4.

4.8 Ladders

By a *ladder* is meant a Cartesian product of two paths, one of which is of length 1.

In [15] we firstly characterized the ladders via their labelling sequences.

Theorem 4.26. ([15, Theorem 4.8.1]) Let G be a graph of size $m \equiv 1 \pmod 3$. Then G is a ladder if and only if there exists a labelling sequence $(j_1, j_2, \dots, j_m)$ of G such that

$$j_i = 2\left\lfloor \frac{m-i+1}{3} \right\rfloor \quad \text{for all } i \in \{1, 2, \dots, m\}. \qquad (LSL)$$

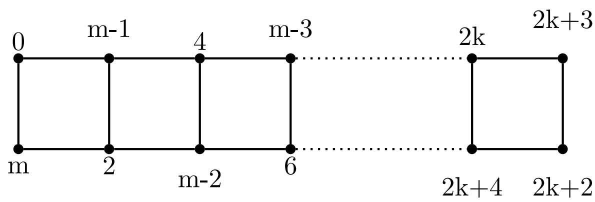


Figure 21. Graceful labelling of a ladder with $k = \frac{m-4}{3}$ for even m [15, Figure 4.11]

An example of a ladder and its graceful labelling is shown in Figure 21.

As a consequence we characterized ladders via their labelling sequences, labelling relations and chessboards.

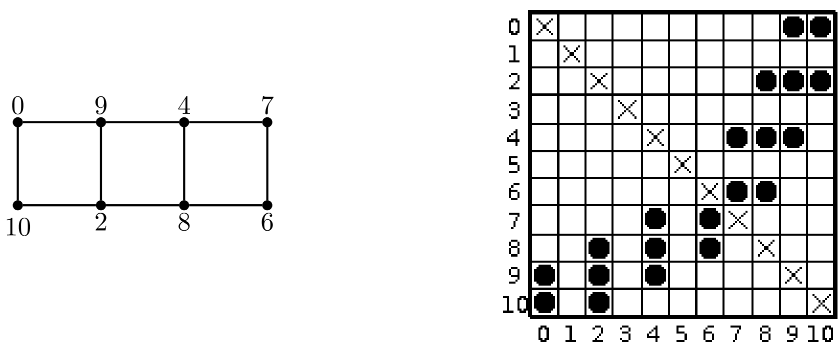
Corollary 4.27. ([15, Corollary 4.8.2]) Let G be a graph of size $m \equiv 1 \pmod 3$ that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a ladder,

(2) there exists a labelling sequence $L \in \mathcal{L}$ which satisfies (LSL),

(3) for some $L \in \mathcal{L}$ the labelling relation $A(L)$ consists of ordered pairs of form $[2i, m-i+j]$ for $i \in \{0, 1, 2, \dots, \lfloor \frac{m}{3} \rfloor\}$, where $j \in \{-1, 0\}$ if $i = 0$ and $j \in \{0, 1\}$ if $i = \lfloor \frac{m}{3} \rfloor$ and $j \in \{-1, 0, 1\}$ otherwise,

(4) some chessboard of G looks like the chessboard in Figure 22.



1	2	3	4	5	6	7	8	9	10
6	6	4	4	4	2	2	2	0	0
7	8	7	8	9	8	9	10	9	10

Figure 22. Representations of a gracefully labelled ladder [15, Figure 4.12]

Example 4.28. ([15, Example 4.8.3]) The sequence (6, 6, 4, 4, 4, 2, 2, 2, 0, 0) is a labelling sequence of a ladder. Corresponding graph diagram, labelling relation and graph chessboard are in Figure 22. Hence a ladder can always be gracefully labelled so that its chessboard has the dots arranged in columns of three dots while the first and the last columns consist of two dots.

4.9 Simple Chains

By a *simple chain* is meant a graph obtained by joining a finite set of the graphs C_4 in the way that in a finite sequence of the graphs C_4 each two neighbouring graphs C_4 are glued together by identifying two vertices, one coming from each of them; within all of the graphs C_4, no two adjacent vertices are used for glueing (see Figure 23). In the first theorem we characterized simple chains via the labelling sequences.

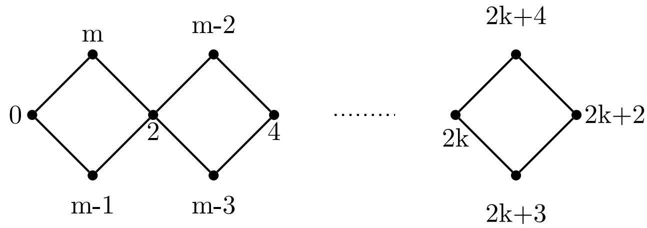
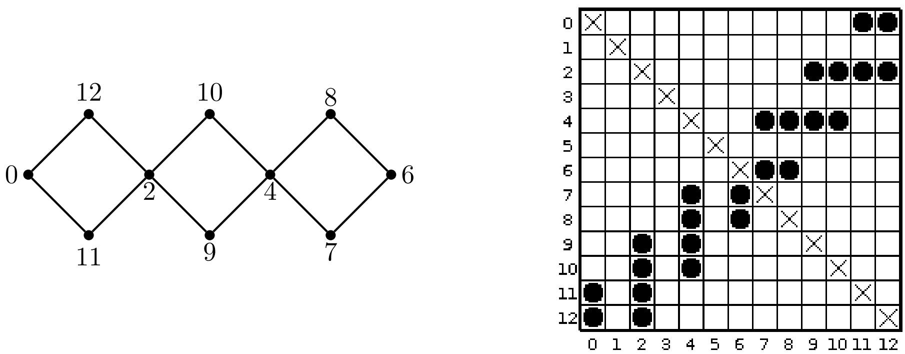


Figure 23. Graceful labelling of a simple chain [15, Figure 4.13]

Theorem 4.29. ([15, Theorem 4.9.1]) Let G be a graph of size $m \equiv 0 \pmod 4$. Then G is a simple chain if and only if there exists a labelling sequence $(j_1, j_2, \ldots, j_m)$ of G such that

$$j_i = 2\left\lfloor \frac{m-i+2}{4} \right\rfloor \text{ for all } i \in \{1, 2, \ldots, m\}. \qquad (LSSC)$$

An example of a graceful labelling of a simple chain and its representations can be seen in Figure 24.



1	2	3	4	5	6	7	8	9	10	11	12
6	6	4	4	4	4	2	2	2	2	0	0
7	8	7	8	9	10	9	10	11	11	11	12

Figure 24. Representations of a graceful labelling of a simple chain [15, Figure 4.14]

As a consequence we presented a full characterization of the simple chains via their labelling sequences, labelling relations and graph chessboards.

Corollary 4.30. ([15, Corollary 4.9.2]) Let G be a graph of size $m \equiv 0 \pmod 4$ that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a simple chain,

(2) there exists a labelling sequence $L \in \mathcal{L}$ which satisfies (LSSC),

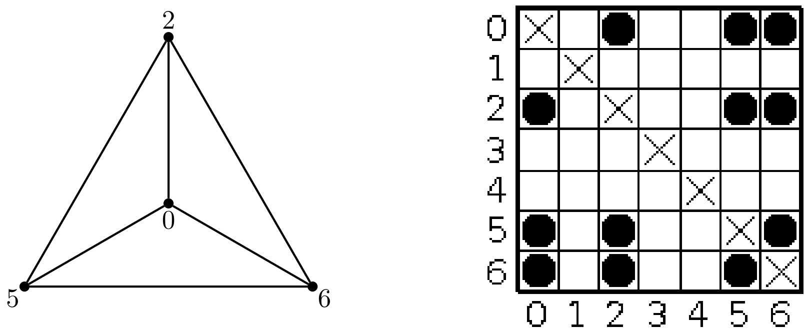
(3) for some $L \in \mathcal{L}$ the labelling relation $A(L)$ consists precisely of ordered pairs $[2i, m-2i+j]$ for $i \in \{0, 1, 2, \ldots, \frac{m}{4}\}$, where $j \in \{-1, 0\}$ if $i = 0$ and $j \in \{1, 2\}$ if $i = \frac{m}{4}$ and $j \in \{-1, 0, 1, 2\}$ otherwise,

(4) some chessboard of G looks like the chessboard in Figure 24.

Example 4.31. ([15, Example 4.9.3]) The sequence (6, 6, 4, 4, 4, 4, 2, 2, 2, 2, 0, 0) is a labelling sequence of a simple chain. Corresponding labelling relation, chessboard and graph diagram are in Figure 24. Hence a simple chain can always be gracefully labelled so that its chessboard has the dots arranged in columns of four dots while the first and the last columns consist of two dots.

4.10 Complete Graphs

For the complete graphs K_n the situation concerning the gracefulness is quite simple.



1	2	3	4	5	6
5	0	2	2	0	0
6	2	5	6	5	6

Figure 25. Representations of a graceful labelling of the complete graph K_4 [15, Figure 4.15]

It is known that a complete graph K_n is graceful if and only if $n \leq 4$. Below is the list of all possible labelling sequences of gracefully labelled complete graphs.

- Labelling sequence of K_2: (0).
- Labelling sequences of K_3: (0, 1, 0), (2, 0, 0).
- Labelling sequences of K_4: (0, 4, 1, 0, 1, 0), (5, 0, 2, 2, 0, 0).

Example 4.32. ([15, Example 4.10.1]) The sequence (5, 0, 2, 2, 0, 0) is a labelling sequence of the complete graph K_4. Corresponding labelling relation, chessboard and graph diagram are in Figure 25.

4.11 Complete Bipartite Graphs

By a *bipartite graph* is meant a graph whose vertex set can be decomposable into two disjoint sets A, B such that no two vertices within the same set are adjacent. And by the *complete bipartite graph* $K_{p,q}$ is meant a bipartite graph in which every vertex from A is adjacent to every vertex from B and such that $|A| = p$, $|B| = q$.

Similarly to the 2-linear labelling sequence from Definition 4.7 we defined the 1-linear labelling sequence:

Definition 4.33. ([15, Definition 4.11.1]) The labelling sequence

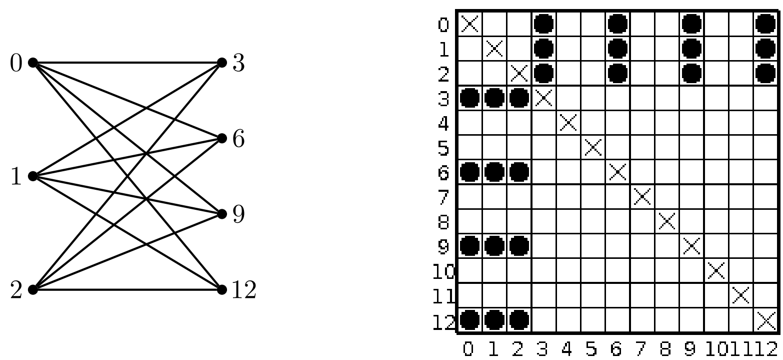
$$(k-1, k-2, k-3, \dots, 2, 1, 0)$$

of length k will be called the 1-*linear labelling sequence*.

Our first theorem characterized the complete bipartite graphs $K_{p,q}$ via the labelling sequences.

Theorem 4.34. ([15, Theorem 4.11.2]) Let G be a graph of size $m = p + q$ for some $p, q \in \mathbb{N}$. Then G is a complete bipartite graph $K_{p,q}$ if and only if there exists a labelling sequence $(j_1, j_2, \dots, j_m)$ of G made of q copies of 1-linear labelling sequence of length p.

An example of a graceful labelling of a complete bipartite graph can be seen in Figure 26.



1	2	3	4	5	6	7	8	9	10	11	12
2	1	0	2	1	0	2	1	0	2	1	0
3	3	3	6	6	6	9	9	9	12	12	12

Figure 26. Representations of a graceful labelling of the complete bipartite graph $K_{3,4}$. [15, Figure 4.16]

We finally characterized complete bipartite graphs via their labelling sequences, labelling relations and chessboards.

Corollary 4.35. ([15, Corollary 4.11.3]) Let G be a graph of size $m = p+q$ that can be gracefully labelled and let $\mathcal{L}$ be the set of all its labelling sequences. Then the following are equivalent:

(1) G is a complete bipartite graph $K_{p,q}$,

(2) there exists a labelling sequence $(j_1, j_2, \dots, j_m) \in \mathcal{L}$ which consists of q copies of 1-linear labelling sequence of length p,

(3) for some $L \in \mathcal{L}$ the labelling relation $A(L)$ consists precisely of ordered pairs $[x, py]$, where $x \in \{0, 1, \dots, p-1\}$ and $y \in \{1, 2, \dots, q\}$,

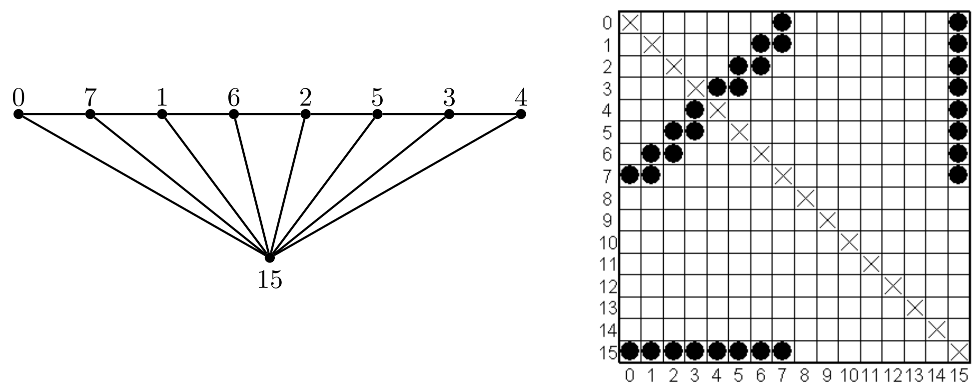
(4) some chessboard of G looks like the chessboard in Figure 26.

Example 4.36. ([15, Example 4.11.4]) The sequence (2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0) is a labelling sequence of the graph $K_{3,4}$. Corresponding labelling relation, chessboard and graph diagram are in Figure 26. Hence a complete bipartite graph can always be gracefully labelled so that its chessboard consists of dots arranged in vertical blocks of the same width, all beginning in the first column.

5 New results on k-enriched fan graphs

In this section we demonstrate on a very recent example of the classes of k-enriched fan graphs how chessboard representations and their careful investigation can lead to creating new classes of graceful graphs. This section of our survey is based on the paper [17] by the first author and Kurtulík.

As we already mentioned in the Subsection 4.7 of the previous section, in our new terminology that is slightly modified compared to [15, Subsection 4.7], by *fan graph* F_n we mean a join of the path P_n and a single vertex, i.e. the complete graph K_1. The "bottom part" of F_n obtained from the graph by excluding the edges of the main path P_n is formed by the star S_n.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	3	2	2	1	1	0	7	6	5	4	3	2	1	0
4	5	5	6	6	7	7	15	15	15	15	15	15	15	15

Figure 27. Representations of the gracefully labelled fan graph F_8 [17, Figure 6]

We wish to have $n \geq 2$ in order to have the shape of "a fan". The fan graph F_n has order $n+1$ and size $2n-1$. For an illustration, in Figure 27 we see a gracefully labelled fan graph F_8 and its corresponding simple chessboard and labelling relation.

The characterizations of the fan graphs via their labelling sequences, labelling relations and simple chessboards, taken from [15] and presented in the slightly modified terminology, were given in the Subsection 4.7 of the previous section. Hence we go immediately to new results presented in [17] where, by extending the concept of the fan graph F_n, we added an infinite family of classes of graceful graphs to the list of known simple graceful graphs. More precisely, we introduced classes of k-enriched fan graphs kF_n for all integers $k, n \geq 2$ and we proved that these graphs are graceful. We also presented characterizations of the k-enriched fan graphs kF_n among all simple graphs via Sheppard's labelling sequences as well as via labelling relations and graph chessboards. It was the visualization provided by our chessboard representations that enabled us to find certain "canonical" graceful labellings of these classes of graphs and that has led to our characterizations.

5.1 Double and Triple Fan Graphs and Their Descriptions

In [17] we meant by a *double fan graph* DF_n the fan graph F_n with extra n pendant vertices attached to the main path P_n. Double fan graphs DF_n can thus be understood such that one connects n paths P_2 to a given fan graph F_n in the way that one vertex of each of the paths P_2 is identified with one vertex of the main path P_n of the fan graph. If one connects in this way n paths P_3 to a fan graph F_n, then the resulting graph is called a *triple fan graph* and denoted by TF_n.

To illustrate the new concepts, in Figure 28 one can see the double fan graph DF_5 on the left side and the triple fan graph TF_4 on the right side.

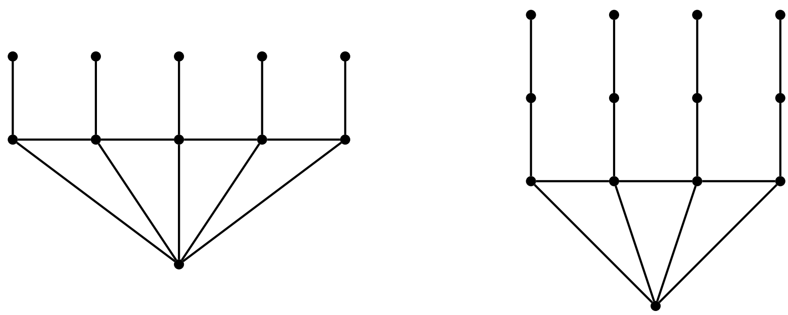
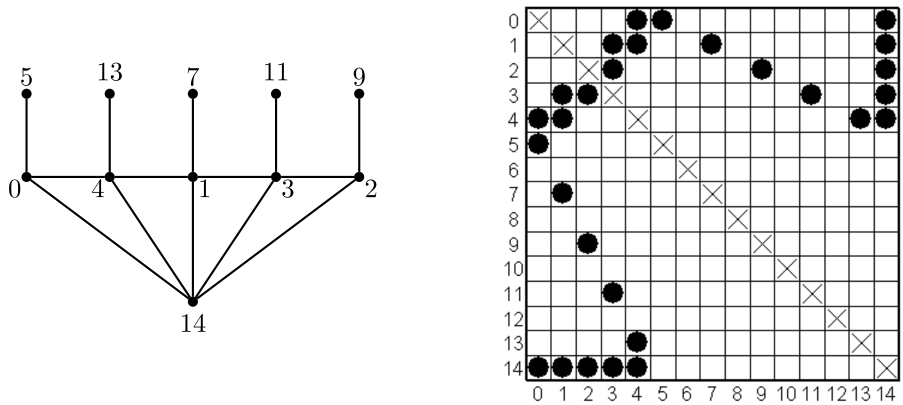


Figure 28. The double fan graph DF_5 (left) and the triple fan graph TF_4 (right) [17, Figure 7]

As we explained in detail in Section 4 of [17], one can divide vertices of DF_n and TF_n into these groups: (i) the *root vertex*, forming together with the n adjacent edges the "bottom" star, (ii) the "middle" vertices of the main path P_n, and (iii) the "upper" pendant vertices, which together with the "middle" vertices form n paths P_2 resp. P_3 connected to the main path. That is why double fan graphs DF_n and triple fan graphs TF_n can be naturally understood as certain extensions of the fan graphs F_n by adding in the "upper part" n paths P_2 (in case of the double fan graphs) resp. n paths P_3 (in case of the triple fan graphs). We therefore refer to the "bottom", "middle" and "upper parts" to assist in our proofs, though of course these parts are not pairwise disjoint and our naming of these parts refers only to our chosen visualization.

Example 5.1. ([17, Example 10]) The sequence $(2, 1, 1, 0, 0, 1, 2, 3, 4, 4, 3, 2, 1, 0)$ is a labelling sequence of a gracefully labelled double fan graph DF_5 of size 14 and order 11. The corresponding graph chessboard and labelling table are in Figure 29. Because the dots in the graph chessboard representing the edges of the mentioned "bottom", "middle" and "upper parts" of the double fan graph look respectively like the "bottom part", the "head" and the "neck" of a swan, we referred to such chessboards as *swan chessboards*.



1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1	1	0	0	1	2	3	4	4	3	2	1	0
3	3	4	4	5	7	9	11	13	14	14	14	14	14

Figure 29. Representations of the gracefully labelled double fan graph DF_5 [17, Figure 8]

In [17] we presented complete characterizations of the double fan graphs DF_n and the triple fan graphs TF_n by their simple chessboards, labelling sequences and labelling relations. Within these results we proved there existed specific "canonical" graceful labellings of these graphs, hence our double resp. triple fan graphs DF_n resp. TF_n can be considered as new additions to the list of graceful graphs.

Theorem 5.2. ([17, Theorem 11]) Let G be a graph of size $m = 3n - 1$ for some $n \in \mathbb{N} - \{1\}$. Then the following are equivalent:

(1) G is the double fan graph DF_n.

(2) There is a graceful labelling of G with a swan chessboard.

(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{n-i}{2} \right\rfloor, & \text{if } i < n, \\ i - n, & \text{if } n \leq i < 2n, \\ m - i, & \text{if } i \geq 2n. \end{cases} \qquad (LSDFG)$$

(4) There exists a labelling sequence L of G with the labelling relation

$$\begin{aligned} A(L) = &\{[u, v] \mid u, v \in \{0, 1, \ldots, n-1\},\ u < v,\ n - 1 \leq u + v \leq n\} \cup \\ &\{[u, n + 2u] \mid u \in \{0, 1, \ldots, n-1\}\} \cup \\ &\{[n - 1 - u, m] \mid u \in \{0, 1, \ldots, n-1\}\}. \end{aligned}$$

We showed that a double fan graph can be gracefully labelled such that its graph chessboard has the "head", the "neck" and the "bottom part" of a certain "swan". Then we went on in [17] to deal with a description of triple fan graphs.

Example 5.3. ([17, Example 12]) The sequence

$$(2, 2, 1, 1, 0, 6, 7, 8, 9, 10, 11, 0, 1, 2, 3, 4, 5, 5, 4, 3, 2, 1, 0)$$

is a labelling sequence of a triple fan graph of size 23 and order 19. The corresponding graph diagram, graph chessboard and labelling table are in Figure 30.

Notice that while the swan chessboard of a double fan graph had one "neck" connecting the "head" and the "bottom part" of the "swan", the simple chessboard of a triple fan graph has 3 blocks of dots that look like a "swan without head", then "the head separated from the neck", and one extra "separated neck". Hence we referred to such simple chessboards representing the triple fan graphs as *3-part swan chessboards.*

The characterisations of triple fan graphs TF_n that we gave in [17] could be proven analogously to the methods used for the double fan graphs DF_n, and are also covered by the general case solved in the last subsection.

Theorem 5.4. ([17, Theorem 13]) Let G be a graph of size $m = 4n - 1$ for some $n \in \mathbb{N} - \{1\}$. Then the following are equivalent:

(1) G is the triple fan graph TF_n.

(2) There is a graceful labelling of G with a 3-part swan chessboard.

(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

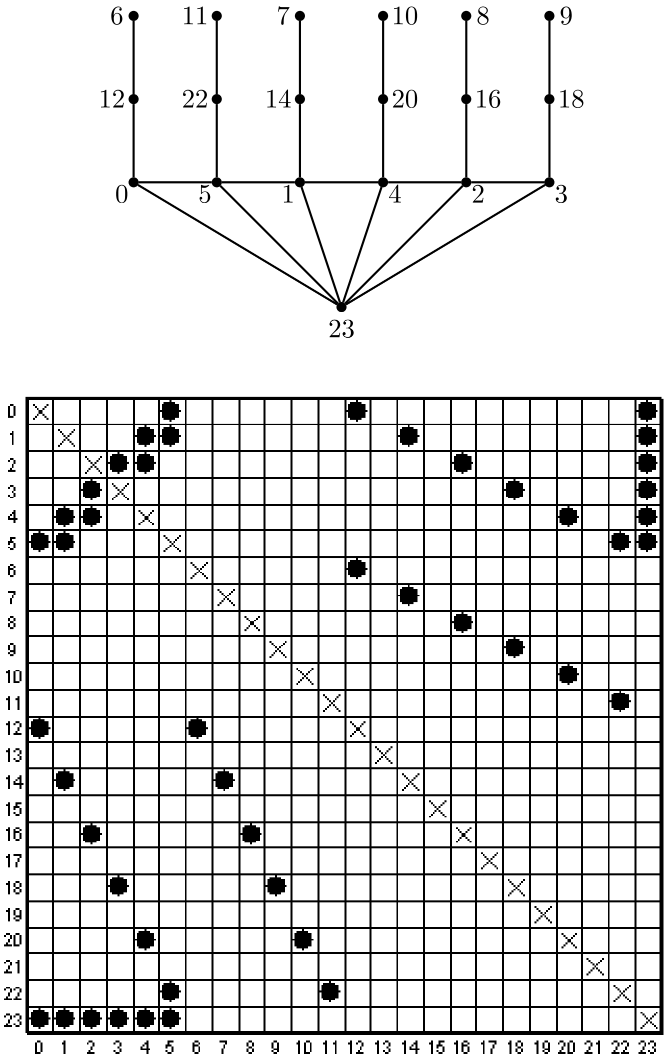
$$j_i = \begin{cases} \left\lfloor \frac{n-i}{2} \right\rfloor, & \text{if } i < n, \\ i, & \text{if } n \leq i < 2n, \\ i - 2n, & \text{if } 2n \leq i < 3n, \\ m - i, & \text{if } i \geq 3n. \end{cases} \tag{LSTFG}$$

(4) There exists a labelling sequence L of G with the labelling relation

$$\begin{aligned} A(L) = &\{[u, v] \mid u, v \in \{0, 1, \ldots, n-1\},\ u < v,\ n - 1 \leq u + v \leq n\} \cup \\ &\{[u, 2u] \mid u \in \{n, n+1, \ldots, 2n-1\}\} \cup \\ &\{[u, 2n + 2u] \mid u \in \{0, 1, \ldots, n-1\}\} \cup \\ &\{[n - 1 - u, m] \mid u \in \{0, 1, \ldots, n-1\}\}. \end{aligned}$$

5.2 General case: the k-enriched fan graphs kF_n and their descriptions

Now we explain how the representations via the graph chessboards assisted us in [17] to solve the general case. We naturally assumed, based on the descriptions of the double resp. triple fan graphs DF_n resp. TF_n that the more "separated necks" we have in the graph chessboard, the longer the paths in the "upper part" of the graph will be. However, our chessboard representations and the work with Graph processor, which was introduced in [19] and much used also in [15], had led us to a surprising discovery that the "necks" in the graph chessboard in fact do *not* represent in the corresponding graph the stars but the paths. Therefore we decided to use for these graphs a new concept *k-enriched fan graphs kF_n*, where the number k represents the order of the stars S_k in the "upper part" of the graph and the number n represents the order of the "middle" path P_n. The formal definintion of this new concept is below.



1	2	3	4	5	6	7	8	9	10	11
2	2	1	1	0	6	7	8	9	10	11
3	4	4	5	5	12	14	16	18	20	22

12	13	14	15	16	17	18	19	20	21	22	23
0	1	2	3	4	5	5	4	3	2	1	0
12	14	16	18	20	22	23	23	23	23	23	23

Figure 30. Representations of the gracefully labelled triple fan graph TF_6 [17, Figure 9]

Definition 5.5. ([17, Definition 14]) The k-enriched fan graph kF_n, for fixed integers $k, n \geq 2$, is the graph of size $(k+1)n - 1$ obtained by connecting n copies of the star S_k of order k to the fan graph F_n such that one vertex of each copy of the star S_k is identified with one vertex of the main path P_n of F_n.

We observed that here the stars S_k were connected to the main path P_n of the fan graph F_n exactly as previously – in the case of the double fan graphs DF_n resp. triple fan graphs TF_n – the paths P_2 (which actually are the stars S_2) resp. P_3 (which actually are the stars S_3) were connected to the main path P_n of the graph F_n.

Example 5.6. ([17, Example 15]) In Figure 31 we see a gracefully labelled 4-enriched fan graph $4F_6$ obtained by connecting 6 copies of stars S_4 of order 4 to the fan graph F_6 as described above. The corresponding simple chessboard and labelling table are also in Figure 31. The labelling sequence of this graph is

$$(2, 2, 1, 1, 0, 12, 13, 14, 15, 16, 17, 6, 7, 8, 9, 10, 11, 0, 1, 2, 3, 4, 5, 5, 4, 3, 2, 1, 0).$$

We notice that we can divide the vertices of the k-enriched fan graph kF_n with any $k \geq 2$ similarly as in the case of double and triple fan graphs. Hence the graph consists of the "bottom part" (the root vertex and the adjacent edges), the "middle part" (the main path P_n) and the "upper part" (the disjoint union of n stars S_k). The corresponding simple chessboard of the k-enriched fan graph kF_n for $k \geq 3$ has k blocks of dots that form a "swan without head", then "the separated head", and $k - 2$ extra "separated necks". Hence we referred to such simple chessboards representing the k-enriched fan graphs kF_n as *k-part swan chessboards.*

Theorem 5.7. ([17, Theorem 16]) Let G be a graph of size $m = (k + 1)n - 1$ for some fixed integers $k, n \geq 2$. Then the following are equivalent:

(1) G is the k-enriched fan graph kF_n.

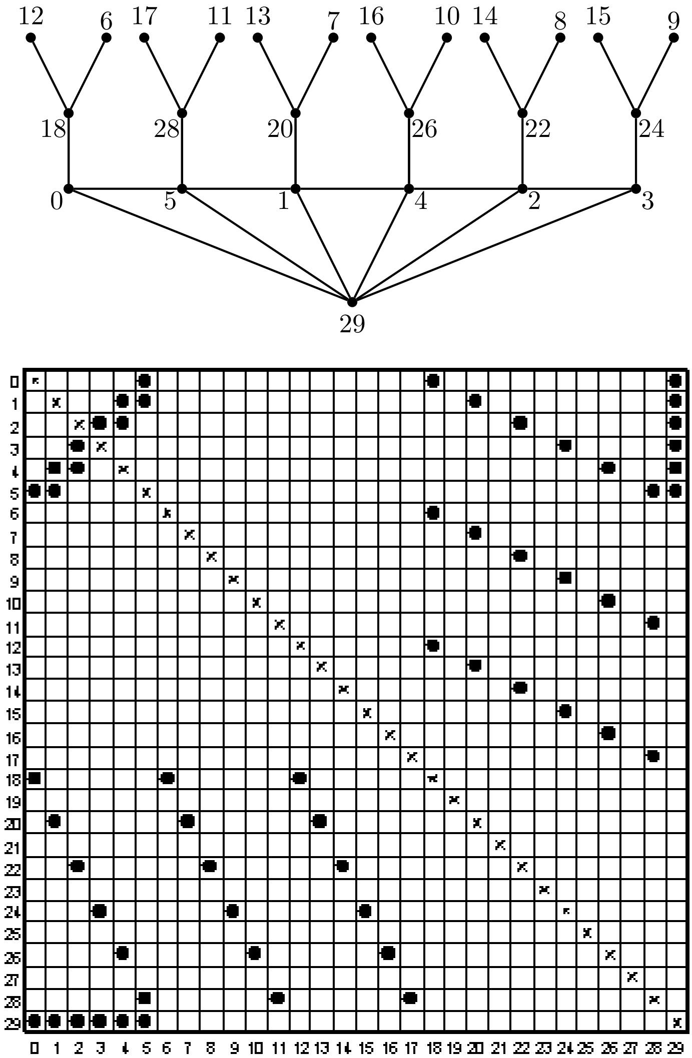
(2) There is a graceful labelling of G with a k-part swan chessboard.

(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} \left\lfloor \frac{n-i}{2} \right\rfloor, & \text{if } i < n, \\ i - n + (k-2)n, & \text{if } n \leq i < 2n, \\ i - n + (k-4)n, & \text{if } 2n \leq i < 3n, \\ \vdots & \vdots \\ i - n + (k - 2(k-1))n, & \text{if } (k-1)n \leq i < kn, \\ m - i, & \text{if } i \geq kn. \end{cases} \qquad (LSKFG)$$

(4) There exists a labelling sequence L of G with the labelling relation $A(L)$ of the form

$$\begin{aligned}
&\{[\left\lfloor \frac{n-i}{2} \right\rfloor, \left\lfloor \frac{n-i}{2} \right\rfloor + i] \mid i < n\} \cup \\
&\{[i - n + (k-2)n, 2i - n + (k-2)n] \mid n \leq i < 2n\} \cup \\
&\{[i - n + (k-4)n, 2i - n + (k-4)n] \mid 2n \leq i < 3n\} \cup \\
&\vdots \\
&\{[i - n + (k - 2(k-1))n, 2i - n + (k - 2(k-1))n] \mid (k-1)n \leq i < kn\} \cup \\
&\{[m - i, m] \mid i \geq kn\}.
\end{aligned}$$



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	1	1	0	12	13	14	15	16	17	6	7	8	9
3	4	4	5	5	18	20	22	24	26	28	18	20	22	24

16	17	18	19	20	21	22	23	24	25	26	27	28	29
10	11	0	1	2	3	4	5	5	4	3	2	1	0
26	28	18	20	22	24	26	28	29	29	29	29	29	29

Figure 31. Representations of the gracefully labelled 4-enriched fan graph $4F_6$ [17, Figure 10]

5.3 Conclusion

In our conclusion of [17] we pointed out that studies of other extended fan graphs can be found in the literature and we illustrated some of them in a figure (see Figure 32). In this figure the first two graphs were taken from the paper [2]. We remarked in [17] that the first graph from [2] was named there as a double fan graph, but it was different from our double fan graph. It in fact consisted of two fan graphs having a common main path. The second graph in the figure was obtained by adding some edges to a vertex of the main path.

As we pointed out in [17], one of the possibilities for other extensions of fan graphs would be adding *paths* P_k of the same length – instead of the *stars* S_k as we did in the paper [17] – to the main path P_n of the fan graph F_n. It is evident that these new graphs and the k-enriched fan graphs kF_n from [17] would be the same in the cases $k = 2$ and $k = 3$. We proposed to denote these *extended fan graphs* by P_kF_n while our k-enriched fan graphs kF_n in this more universal notation would be denoted then by S_kF_n. The extended fan graphs can look like the third graph in Figure 32, that is, such a graph would be in the proposed notation denoted by P_4F_3 because the paths P_4 are connected to the main path of the fan graph F_3. We concluded our paper [17] with the following open problem:

Problem 5.8. Are the extended fan graphs P_kF_n (obtained by connecting in the described way n paths P_k to the main path P_n of the fan graph F_n) graceful? And if so, is there a characterization of them via a certain "canonical" graph chessboard and the corresponding labelling sequence and the labelling relation like the characterizations of the graceful graphs presented in this paper?

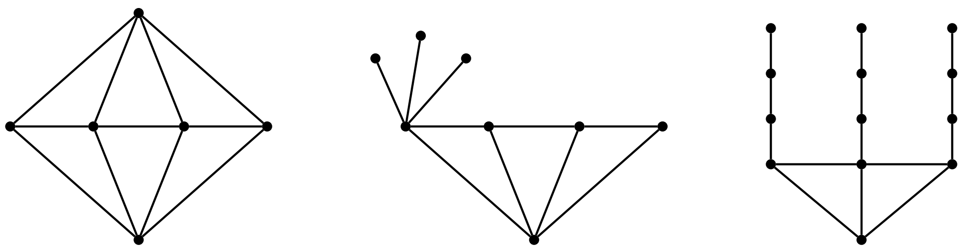


Figure 32. Other extended fan graphs [17, Figure 11]

Acknowledgements

The authors would like to thank the editor Ján Karabáš for his helpful comments improving the presentation of the paper.

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