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Fruits of a categorical approach to probability theory

Dedicated to Roman Frič, the man who suggested the design of a categorical approach to probability theory, and to the one who opened up the world of mathematics to me, to my dear friend and colleague.

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Abstract

Motivated by good reputation and simplicity of a categorical language and also its effectiveness to get to the heart of the matter, approximately two decades ago R. Frič and M. Papčo suggested to apply basic category theory apparatus on Kolmogorovian probability theory. Their idea resulted in the concept of *generalized probability*.

In this theory, the central position is occupied by the category **ID** whose objects are D-posets of fuzzy subsets of a universe and whose morphisms are sequentially continuous D-poset structure preserving maps.

A set X, partially ordered by a relation $\leq$, with the smallest element 0_X, the greatest element 1_X, and with a partial binary operation $\ominus$ such that $a \ominus b$ for $a, b \in X$ is defined if and only if $b \leq a$, is said to be a *D-poset* provided the following two axioms are fulfilled: $a \ominus 0_X = a$ for every $a \in X$; for $a, b, c \in X$, $c \leq b \leq a$ implies $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

Via a D-poset of indicator functions over a universe, sure, impossible and complementary events can be modelled and also a system of events determined by states can be obtained. If A, B are subsets of Ω and $\setminus$ denotes the standard set difference, then $A^c = \Omega \setminus A$, $A \cap B = A \setminus (\Omega \setminus B)$, $A \cup B = (A^c \cap B^c)^c$, thus the information about differences is sufficient to describe/reconstruct a σ-field of subsets of Ω. And so, Kolmogorovian probability domain i.e. a σ-algebra of events can be included in some ID-model.

All crucial notions in the generalized probability theory are presented either as objects or as morphisms of **ID**. Each specific probability theory corresponds to some particular subcategory of the reference category. Kolmogorovian, fuzzy and several other probabilities are distinct models of ID-theory while their relationships are described in categorical language terms. Generalized random events are specific ID-objects, probability measures (states), random variables and observables are specific ID-morphisms. Systems of events, serving as probability domains, are equipped by a partial order, partial operations and also a convergence, hence probability theory becomes the matter of algebra. Moreover, some of Kolmogorovian theory disadvantages are eliminated in ID-concept.

The aim of this paper is to survey relevant results obtained mostly by R. Frič, M. Papčo and followers who—throughout the last two decades—applied the categorical approach to probability theory. Due to exactly the ID-framework, some fundamental theorems were reformulated and proved to present the resulting probability in a more transparent way.

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Introduction

Regarding good experience in several different disciplines and in belief that basic category theory apparatus provides tools and language to reformulate and prove fundamental theorems and to present the resulting probability in a more transparent way, *a categorical approach to probability theory* was applied in a series of papers, mostly by R. Frič, M. Papčo and followers. The fruit of their research is called *generalized probability theory*. Some of Kolmogorovian probability theory disadvantages are eliminated in it.

This paper aim is to survey its crucial notions and constructions. The former are presented either as objects or as morphisms of the category **ID**. D-posets of fuzzy subsets of a universe are **ID**-objects and sequentially continuous D-homomorphisms are **ID**-morphisms. Each specific probability theory corresponds to some special subcategory of the reference category. Kolmogorovian, fuzzy and several other probabilities become special cases of the theory while their relationships are described in categorical language terms.

In each from particular probability theories, generalized random events are objects of the corresponding subcategory, and probability measures (or, equivalently, states), random variables and observables are its morphisms. Systems of events, serving as probability domains, carry additional mathematical equipment—a partial order, partial operations and also a convergence, hence probability theory becomes the matter of algebra, too.

Why categories in probability theory?

What should one impose a categorical machinery on probability theory for?

Short answer: Reasons could be the same as were when the first mathematician meditated on mathematics — *"for the love of truth or for curiosity or for the fun of the thing" (G. K. Chesterton in [10]).*

Little longer answer: Because a categorical approach was already crowned with success in another areas. And maybe it could be worth to improve classical Kolmogorovian probability theory. Especially, if a researcher is aware of some of its disadvantages.

"As category theory is still evolving, its functions are correspondingly developing, expanding and multiplying. At minimum, it is a powerful language, or conceptual framework, allowing us to see the universal components of a family of structures of a given kind, and how structures of different kinds are interrelated." Stanford Encyclopedia of Philosophy stands on the matter. J.-P. Marquis, the author of the quotation, even points out that *category theory has come to occupy a central position in contemporary mathematics and theoretical computer science, and is also applied to mathematical physics* (see [52]).

One more quotation, this time from *Has Progress in Mathematics Slowed Down?* by P. R. Halmos ([43]): *"Category theory started as (and continues to be) a convenient language in which to describe many phenomena, and for many of us that's all it is. For the fascinated specialist it is a subject of research that can discuss ... and other such towers of elaborate complications of great fascination—long may it wave."*

In [44], there was published the 5-pages long list of publications devoted to study of a category theory position in philosophy of mathematics and philosophy of science. They are sorted into four areas: *Does category theory break set theory? Does category theory provide an alternative foundation of mathematics? Does category theory help structuralism? Miscellaneous.*

So, why not to try luck also in probability theory case? It looks promising...

Kolmogorov probability theory and categories

Via probability theory, people are able to seize situations in which some sort of uncertainty — a random event occurs. The notion of Kolmogorov probability space is the fundamental and classical tool which is used for it. Despite its undoubtedly effectiveness it has several conceptual disadvantages. Applying *a categorical approach to probability theory* can dispose some of them, and at the same time, can open new perspective, too.

Some experience indicates probability theory professionals are harsh on applying categorical apparatus. From outsider point of view it looks they have own world which fully suffices for their mathematical life. Anything else is at least annoying or superfluous. *"Is there an introduction to probability theory from a structuralist/categorical perspective?"* is the question which opens a web-debate on MathOverflow (see [53]), and which is characteristic. Contributions from another fields of study are not very gladly accepted. Outsiders are not welcome.

The following text scope is to provide a survey of such outsiders' research results. Their investigation ambition was to show why and how elementary category theory can help with understanding and describing fundamental notions and constructions of the (classical) Kolmogorovian probability theory ([29], [34], [35]).

Referring to [39], the J. Goguen categorical approach manifesto first dogma is the following: *To each species of mathematical structure, there corresponds a* category *whose objects have that structure, and whose morphisms preserve it.* How does look the Kolmogorovian probability in this perspective?

It makes sense to summarize fundamental elements of the theory first.

The notion of a *probability space* is fundamental for the probability theory axiomatization suggested by A. N. Kolmogorov in *Grundbegriffe der Wahrscheinlichkeitsrechnung* ([48]). It is a triple $(\Omega, \mathbb{A}, P)$ such that Ω is a set, $\mathbb{A}$ is some family of its subsets equipped with set operations, and P is an additive normed measure on $\mathbb{A}$. Via Ω all possible (conceptual) random experiment outcomes are modelled. A family $\mathbb{A}$ is said to be a *σ-algebra* (also *σ-field*) of *events*. By an event $A \in \mathbb{A}$ there is possible to catch/describe some uncertainty phenomenon. It "consists" of random experiment outcomes which indicate that the event $A \in \mathbb{A}$ occurs. The set algebra of $\mathbb{A}$ gives the tool to manipulate with events or to use the logic of sentences connected with the random experiment in question. Each $A \in \mathbb{A}$ can be represented via its indicator (function) $\chi_A \colon \Omega \longrightarrow \{0, 1\}$, $\chi_A(\omega) = 1$ for $\omega \in A$ and $\chi_A(\omega) = 0$ otherwise. The σ-additive normed measure $P \colon \mathbb{A} \to [0, 1]$ is said to be a *probability*. The number $P(A)$ estimates how "big" is the event $A \in \mathbb{A}$ in proportion to the whole Ω and other events.

With respect to particular "good" properties of the real numbers structure, there is no surprise they are employed in classical probability theory, too. The most prominent example of a probability space is the triple $(R, \mathbb{B}_R, P_R)$ where R is the set of all real numbers, $\mathbb{B}_R$ is the σ-algebra of Borel measurable subsets of R, and P_R is a probability on $\mathbb{B}_R$. Moreover, to capture/evaluate/model randomness via real measuring scale, the notion of a random variable is used as the tool. A mapping $f \colon \Omega \to R$ is said to be a *random variable* if and only if f is *measurable* and *measure preserving*, i. e. the preimage $f^{\leftarrow}(B) = \{\omega \in \Omega \mid f(\omega) \in B\}$ of every Borel measurable subset B of R belongs to $\mathbb{A}$ and $P_R(B) = P(f^{\leftarrow}(B))$ for every $B \in \mathbb{B}_R$. In that case, the probability P_R is called a *(probability) distribution of a random variable f*.

It is worth mentioning that, despite its name, a random variable is neither random nor variable. Clearly, f is a mapping (function), not variable, and assignment $\omega \mapsto f(\omega)$ is not random. If something, the pair (R, F), where $F \colon R \to [0, 1]$ is a distribution function given by $F(r) = P\big(f^{\leftarrow}((-\infty, r))\big) = P\big(\{\omega \in \Omega \,|\, f(\omega) < r\}\big)$, $r \in R$, can be considered "randomisation of real variable".

But in probability theory and mathematical statistics, random variables as one of fundamental notions do not represent main study objects, they play more or less auxiliary role only (more in [31]). Concerning this field of study, probabilistic laws on R, R^n and R^R in language of distribution functions, characteristic functions and probability densities are in the spotlight.

Thus, from some point of view, a dual mapping $f^{\leftarrow}$ to random variable f plays more important role. It assigns some event from $\mathbb{A}$ to a Borel measurable set from $\mathbb{B}_R$. Clearly, $f^{\leftarrow}$ preserves Boolean structure $\mathbb{B}_R$, so it is a Boolean morphism which to a real event from $\mathbb{B}_R$ assigns an event from original (*sample*) space $(\Omega, \mathbb{A}, P)$. Moreover, $P_R = P \circ f^{\leftarrow}$, so it is a composition of the mapping $f^{\leftarrow}$ and the function P. Thus stochastic information about the "observed" event can be obtained by finding a corresponding "theoretical" event $f^{\leftarrow}(B) \in \mathbb{A}$, and by determining $P\big(f^{\leftarrow}(B)\big)$. Strangely enough, $f^{\leftarrow}$ does not have its own name in the classical probability theory.

Despite undoubted success of Kolmogorovian approach to probability based on the measure theory (see [51]), the notion of a probability space is suffering from several conceptual disadvantages (compare with [34]). Since $\mathbb{A}$ is a Boolean algebra, it does not enable to model situations on which there is necessary to apply multivalued logics or suitable fuzzy mathematics. A. Khrennikov in [47] even declares that *"in the opposite to geometry, probability theory was not transformed in an elastic formalism containing numerous probabilistic models which can be used for descriptions of different physical phenomena. Probability theory is still a rigid structure. This structure can be compared with the rigid Euclidean cub. Attempts to use the unique Kolmogorov model for describing all physical phenomena can be compared with attempts to represent all geometrical models by Euclidean cubs. However, geometric reality is not restricted to reality of cubs as well as probabilistic reality is not restricted to reality of Kolmogorov probability spaces."* One of the above cited paper goals is to demonstrate "pathological" behaviour of quantum probabilities as a consequence of the Kolmogorovian concept use. According to the author of [47], high level abstraction does not provide a possibility to control a connection between probabilities and *statistical ensembles* or *random sequences (collectives)*. This way, such monsters as Bell inequalities are products of formal manipulations with abstract Kolmogorovian probabilities.

Next weak point of the theory in question is that P as an additive mapping does not preserve Boolean operations on events. Finally, the codomain of P is the unit interval $[0, 1]$, but the codomain of a Boolean morphism $f^{\leftarrow}$ is σ-algebra $\mathbb{A}$, thus two important mappings of classical probability theory are of a different nature and there is not possible to compose them (see Figure 1). From the categorical point of view, both $\mathbb{A}$ and $[0, 1]$ should be objects of some suitable category. And, at the same time, all crucial mappings should be its morphisms which preserve objects' structure and there is possible to compose them in propitious case resulting in a morphism again. But this is not the Kolmogorovian probability case!

Above mentioned classical probability disadvantages can be overcome as it is usual in mathematics — fundamental notions are put into broader context while "good" properties are not lost. With respect to a categorical approach, events should be objects of a "desired" reference category, and both probabilistic measures and mappings $f^{\leftarrow}\colon \mathbb{B}_R \to \mathbb{A}$ should be its morphisms. And, because of successes in other fields, why do not expect some pleasant surprises and unexpected results?

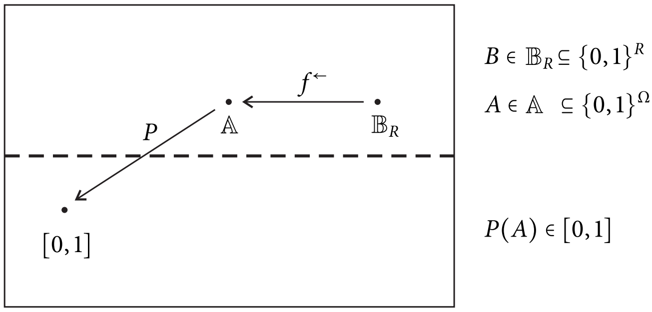

Figure 1. P and $f^{\leftarrow}$ — two important mappings in classical probability theory which are of different nature and do not satisfy categorical approach requirements

How

Tell me what objects and morphisms do you work with and I will tell you who you are.

Applying elementary category theory on the theory of probability resulted in the concept of the *generalized probability*. In the lead role, a reference category is casted. Its subcategories represent particular probability theories. In this way, Kolmogorovian, fuzzy and other probabilities live under the same roof of the generalized probability as its special cases, and their relationships are described in the categorical language. The impact of such approach is possibility to deal with universal constructions, classifications, universal properties and relations, etc. In each particular probability theory, generalized (random) events are objects of the corresponding subcategory of the reference category. Probabilistic measures (states), random variables and observables are becoming its morphisms. A family of events serving as a probability domain is enriched with additional mathematical "equipment": a partial order, partial algebraic operations, and a convergence, too. So, a probability becomes also of algebra matter.

Probability domains

In production of generalized probability domain, the construction of an *epireflection* is used (according to [1], [2]). Basically, it is a categorial embedding of a corresponding object $\mathcal{O}$ into its epireflection $e(\mathcal{O})$ such that "added value" of $\mathcal{O}$ is kept and $e(\mathcal{O})$ has "something more". The embedding is carried out via a functor e (epireflector). A mapping e assigns to each object $\mathcal{O}$ equipped with basic (i.e. object-constituting) properties exactly one object $e(\mathcal{O})$ from a subcategory consisting from objects carrying some special properties. At the same time, e maps each morphism $f\colon\ \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ into exactly one morphism $e_f\colon\ e(\mathcal{O}_1) \longrightarrow e(\mathcal{O}_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_1 & \xrightarrow{\quad f \quad} & \mathcal{O}_2 \\ {\scriptstyle e_{\mathcal{O}_1}}\big\downarrow & & \big\downarrow{\scriptstyle e_{\mathcal{O}_2}} \\ e(\mathcal{O}_1) & \xrightarrow[\quad e_f \quad]{} & e(\mathcal{O}_2) \end{array}$$

An example of an epireflection is the embedding of the set of rational numbers Q into the set of real numbers R. Another one is the embedding of a field of sets $\mathbf{A}_0$ into a generated σ-field $\mathbf{A} = \sigma(\mathbf{A}_0)$. (The first construction enables real analysis, the second one limit stochastics.)

According to [32] and [33] the algorithm of probability domain construction can be summarized into the following steps:

- Specifying a "collection of events $\mathcal{U}$".
- A choice of a "cogenerator C". Usually, it is some structured set corresponding to a random situation in question which enables "measurements". (There are often used two-element Boolean algebra $\{0,1\}$, unit interval $I = [0,1]$ equipped with Łukasiewicz MV-structure, D-poset, effect algebra, A-poset, suitable lattice, etc.).
- A selection of a collection X of "properties" which are measured via a cogenerator C such that X separates $\mathcal{U}$.
- Representation of each event $u \in \mathcal{U}$ by its "evaluation": $u \in \mathcal{U}$ is identified with $u_X \in C^X$, $u_X \equiv \{x(u);\ x \in X\}$ (see the following Figure 2).

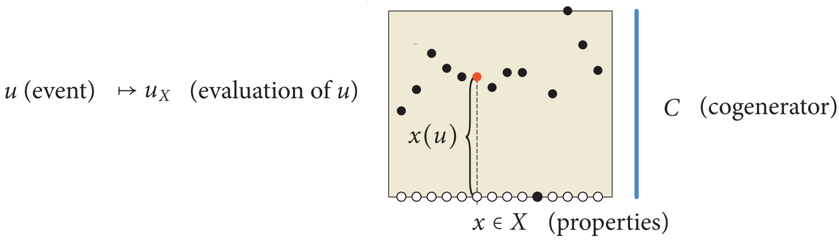

Figure 2. An event u is represented via its "evaluation" using a cogenerator C

- Forming minimal "subalgebra" D of the system C^X including $\{u_X;\ u \in \mathcal{U}\}$ (see the following diagram, Figure 3);

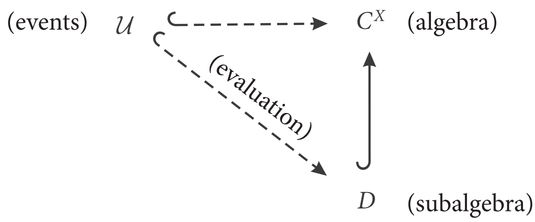

Figure 3. From a collection of events to their (categorical) algebra

- A subalgebra $D \subseteq C^X$ can serve as a *probability domain.*

Cogenerator role — D-poset casted

In the construction described above, the cogenerator plays crucial role. In a certain sense, it enforces the same behaviour (structure, algebra, logics) from a system of evaluations as it has itself. Each choice of a "reasonable" cogenerator "generates" a domain for some specific probabilistic model (see [32], [33], [26], [29]). For example, σ-algebra of subsets of Ω serving as a domain for Kolmogorovian (classical) probability can be viewed as a sequentially closed Boolean system of evaluations Ω using the cogenerator

$(\{0,1\}, \neg, \wedge, \vee)$. Other possibility is to hire the unit interval $[0,1]$ equipped with binary Łukasiewicz operations $\oplus$, $\odot$ and a unary operation *, defined as $a \oplus b = \min\{1, a+b\}$, $a \odot b = \max\{0, a+b-1\}$, $a^* = 1-a$ for $a, b \in [0,1]$. The resulting product is an MV-algebra of evaluations of a universe which is called a *bold algebra*. It serves as a domain of fuzzy probability (see [65], [56], [54]).

Nontraditional cogenerators lead to nontraditional theories. For instance, for a given natural number n, an n-dimensional simplex $S_n = \{(x_1, x_2, \ldots, x_n) \in I^n;\ \sum_{i=1}^{n} x_i \leq 1\}$ together with the pointwise difference can be considered a cogenerator. This assumption leads to *simplex probability theory* in which each generalized random event represents n "mutually competing factors" (cf. [26]). For a change, nonstandard unit interval as the cogenerator is behind occurrence of "infinitely small" events (see [61]).

In the *generalized probability theory* which is going to be described — and, yes, in the desired one which should heal categorical failings of Kolmogorovian approach — a *D-poset* takes up a central position as the cogenerator. D-posets were introduced by F. Kôpka and F. Chovanec in [50] (more can be found in [11]). Their motivation was to model some quantum phenomena. They are generalisations of Boolean algebras and MV-algebras. It is a simple structure – it does not take a lot of effort to comprehend it, and it enables easy manipulation. Via a D-poset, sure, impossible and complementary events can be modelled, and – as it will be shown further – also a system of events determined by states can be maintained.

A set X, partially ordered by a relation $\leq$, with the smallest element 0_X, the greatest element 1_X, and with a partial binary operation $\ominus$ such that $a \ominus b$ for $a, b \in X$ is defined if and only if $b \leq a$, is said to be a *D-poset* if and only if the following axioms are fulfilled:

(D1) $a \ominus 0_X = a$ for every $a \in X$.

(D2) For $a, b, c \in X$, $c \leq b \leq a$ implies $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

A mapping h from a D-poset X into a D-poset Y which preserves D-structure is said to be a *D-homomorphism*. So, D-homomorphism preserves a partial order, the image of the smallest respectively the greatest element is the smallest respectively the greatest element, and the image of the difference is the difference of images.

If Ω is a set, then the quintuple $(\mathcal{P}(\Omega), \subseteq, \emptyset, \Omega, \setminus)$ such that by the operation $\setminus$ is meant a standard set operation, is a D-poset. Since for $A, B \in \Omega$, equalities $A' = \Omega \setminus A$, $A \cap B = A \setminus (\Omega \setminus B)$, $A \cup B = (A' \cap B')'$ hold true, the language of this set D-poset is sufficient to describe/reconstruct a σ-field of subsets of Ω. If, in addition, it is sequentially closed with respect to the difference operation, one can study a σ-field of events from Ω as a probability domain in this language.

Let Ω be a set. Let $u, v \in \{0,1\}^\Omega$ and let $v \leq u$ if and only if $v(\omega) \leq u(\omega)$ for every $\omega \in \Omega$. Then a quintuple $(\{0,1\}^\Omega, \leq, 0_\Omega, 1_\Omega, \ominus)$ such that 0_Ω, 1_Ω are constant functions, and the operation $\ominus$ is for u, v, $v \leq u$, given by $(u \ominus v)(\omega) = u(\omega) - v(\omega)$ for every $\omega \in \Omega$, is another example of a D-poset. In fact, it is a D-poset of indicator functions defined on Ω. By identifying $A \subseteq \Omega$ and its indicator function $\chi_A \in \{0,1\}^\Omega$, every field $\mathbb{A}$ of subsets of Ω can be considered a D-poset: $\mathbb{A}$ is partially ordered ($\chi_B \leq \chi_A$ if and only if $B \subseteq A$), and if $B \subseteq A$, then the difference $\ominus$ is given by $\chi_A \ominus \chi_B = \chi_{A \setminus B}$. At the same time, it is the result of above mentioned categorial construction where the D-poset $(\{0,1\}, \leq, 0, 1, -)$ was applied as a cogenerator.

The unit interval with real numbers standard order, 0 as the smallest element, 1 as the greatest element, and with standard subtraction of a smaller number from a bigger one, thus the quintuple $([0,1], \leq, 0, 1, -)$ is a D-poset, too.

Finally, if Ω is a set, then $([0,1]^\Omega, \leq, 0_\Omega, 1_\Omega, \ominus)$ is a D-poset, where its structure is defined analogously to $(\{0,1\}^\Omega, \leq, 0_\Omega, 1_\Omega, \ominus)$. In the fuzzy mathematics language, it is a D-poset of all fuzzy subsets of the universe Ω. Another partial operations on fuzzy subsets systems can be found in [49].

Regarding categorial concept of the generalized probability theory, D-posets of random events evaluations with the use of $([0,1], \leq, 0, 1, -)$ as the cogenerator have leading position. In fuzzy mathematics they are called D-posets of fuzzy subsets of a universe. In accordance with [61], a system $(\mathcal{X} \subseteq I^X, \leq, 0_{\mathcal{X}}, 1_{\mathcal{X}}, -)$ is said to be a *D-poset of fuzzy subsets of a set* X if and only if, for every $x \in X$, $v \leq u$ holds true if and only if $v(x) \leq u(x)$, $0_{\mathcal{X}}$ is null function (the smallest element), $1_{\mathcal{X}}$ is unit function (the greatest element), and operation $-$ is given by $(u - v)(x) = u(x) - v(x)$ for every $x \in X$ such that $v \leq u$ holds for u, v.

If no confusion arises, a D-poset $(\mathcal{X} \subseteq I^X, \leq, 0_{\mathcal{X}}, 1_{\mathcal{X}}, -)$ will be abbreviated to $\mathcal{X}$.

In the following, a *reduced* system $(\mathcal{X} \subseteq I^X, \leq, 0_{\mathcal{X}}, 1_{\mathcal{X}}, -)$ will be always discussed, i.e. to every pair $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists u such that $u(x_1) \neq u(x_2)$. It is a technical assumption which allows to avoid certain pathologic behavior (detailed explanation can be found in [17]). Reduced D-posets of fuzzy subsets of some universe are called *ID-posets*.

Outline of ID-probability

Denote **ID** the category having (reduced) D-posets of fuzzy subsets as objects and having sequentially continuous D-homomorphisms as morphisms. If X is a universe, objects of **ID** are subobjects of the powers I^X. The description of the category **ID** construction can be found in [61], together with corresponding study of its several properties. The category **ID** is the one providing the framework within which the ambition of a categorical approach to probability theory is fulfilled: crucial notions of the theory are represented as ID-objects or ID-morphisms, or the game is played inside one of suitable subcategories of **ID**. Each particular probability theory (Kolmogorovian, fuzzy, IF-probability etc.) is described in some subcategory of **ID** environment. This enables to study relationships between theories, to give their classification, to search for a transition from one to another or for invariants, to embed one into another…

In that framework, particular probability domains are objects (ID-posets) of the corresponding subcategory of **ID**. *Generalized random events* are modelled by mappings from a universe X of *generalized elementary events* into a cogenerator (D-poset) C. Generalized random events constitute a *generalized random field* (object) equipped with suitable D-structure within C^X. Finally, each morphism from an object in question into C (actually, each projection from C^X into C) is treated as a *generalized probability measure* or a *state*. So, a generalized probability measure preserves relevant D-structure and its codomain is an object of **ID**. This way, the random events field structure is determined by generalized probability measures, thus it is initial.

It is well known that D-posets are equivalent to *effect algebras* which were introduced by D. J. Foulis a M. K. Benett in [15] to model effects in quantum mechanics. It is an algebraic structure with a partially defined binary operation called *orthosum* which is commutative and associative, and within which to every element from a universe there exists unique orthosupplement such that zero is orthosupplement of one. More on this topic can be found in [14].

Considering that above mentioned equivalence, i.e. partial orthosum can be one-to-one transpose onto partial difference and a partial order can be defined, there is possible to construct the category **IE** in the similar manner as **ID**. Its objects are IE-algebras and its morphisms are sequentially continuous EA-homomorphisms, i.e. effect algebra structure preserving maps. The procedure is described in [59]. Moreover, relationships between **ID** and **IE** are studied there, and some applications for probability theory are suggested, too. So, those who instead of D-posets prefer effects algebras can use the category **IE** for categorical improvement/upgrade of Kolmogorovian probability.

Reflecting G. Boole approach to logics, namely via "disjunctive unions" (see [6], [7]), R. Frič and V. Skřivánek introduced A-posets as another structure isomorphic to D-posets in [38]. A quintuple $(S, \leq, 0_S, 1_S, \oplus)$ with partial order $\leq$, the smallest element 0_S, the greatest element 1_S, and partial, commutative and associative binary operation $\oplus$ called *sum*, is said to be an *A-poset* if and only if to every $a \in S$ there exists exactly one $a^c \in S$ (its complement) such that $a \oplus a^c = 1_S$, and for $a, b, a_1, b_1 \in S$ the assumption that $a \oplus b$ is defined and $a_1 \leq a$, $b_1 \leq b$ implies the sum $a_1 \oplus b_1$ is defined and $a_1 \oplus b_1 \leq a \oplus b$ holds true. As one can expected, the trick may be repeated and there is possible to build the category **IA** whose objects are IA-posets and morphisms are sequentially continuous A-homomorphisms, i.e. A-structure preserving maps.

Obviously, categories **ID**, **IE** and **IA** are mutually isomorphic. According to [38], IA-approach provides more transparent relationship between "summing" and "logics", especially from interpretation point of view. Moreover, in contrast to ID-approach, the transition to bold algebras via pointwise maximums and minimums is more natural. For this sake, among other issues the ramification of elementary events is more acceptable.

Kolmogorovian classic in ID

How is the classical Kolmogorovian triple $(\Omega, \mathbb{A}, P)$ *presented in ID-probability?*

Denote **FS** the category whose objects are fields of fuzzy subsets of a universe and the morphisms are sequentially continuous Boolean homomorphisms. It is known that a field $\mathbb{A}$ of subsets of X is σ-field if and only if $\mathbb{A}$ is sequentially continuous in $\{0,1\}^X$. Denote **CFS** *full* subcategory of **FS** consisting from σ-fields. (A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is full if and only if every $\mathcal{A}$-morphism from $\mathcal{B}$-object into $\mathcal{B}$-object is also $\mathcal{B}$-morphism.)

Denote **FSD** the full subcategory of **ID** consisting from fields of subsets of a universe which are viewed as D-posets and denote **CFSD** its full subcategory consisting from σ-fields.

As it was already mentioned, from a set difference $A \setminus B$ defined for $B \subseteq A$ there is possible to reconstruct a union, an intersection and a complement of sets. Moreover, if h is D-homomorphism from a field of sets $\mathbb{A}$ into a field of sets $\mathbb{B}$, then h preserves set operations. This leads to one-to-one functorial correspondence between categories **FS** and **FSD**:

Theorem 1 ([19, Lemma 3.4], [61, Lemmas 2.2 and 2.3], [29, Proposition 2.1])**.**

(i) Categories **FS** *and* **FSD** *are isomorphic.*

(ii) Categories **CFS** *and* **CFSD** *are isomorphic.*

(iii) Let $\mathbb{A}$ *be a field of sets and* P *is a mapping from* $\mathbb{A}$ *into* $[0,1]$*. Then* P *is a probability measure if and only if* P *is a sequentially continuous D-homomorphism.*

Thus, as far as fields of sets are concerned, the difference structure is sufficiently rich for information and it bears all essential to reconstruct them. Classical random events are considered to be D-posets of crisp fuzzy subsets of a set of all random experiment outcomes. D-homomorphisms become Boolean homomorphisms or probability measures in case of sequentially continuous D-homomorphisms with the unit interval as a codomain.

Fuzzy probability in ID

In [65] Lotfi A. Zadeh suggested to extend a probability domain from a σ-field to a system of fuzzy subsets A of Euclidean n-dimensional space R^n such that a membership degree function $\mu_A: R^n \to [0,1]$ is Borel measurable. If P is a probability measure on Borel measurable sets, then a probability of an event A is given as the Lebesgue-Stieltjes integral $\int_{R^n} \mu_A(x)\,\mathrm{d}P$. From 1968, when the paper was published, fuzzification of a probability theory considerably developed. It is worth to mention several theory constituting papers by S. Gudder ([40], [42]) and by S. Bugajski ([8], [9]), as well as another significant contributions by D. Mundici, B. Riečan, T. Neubrunn ([63], [64]), A. Di Nola, M. Navara ([13], [56], [55], [57]) and R. Mesiar ([54]).

Fuzzy random events, fuzzy probability measures and fuzzy random variables form basic elements of a fuzzy probability theory. *Fuzzy random events* are modelled by measurable functions $\mathcal{M}(\mathbb{A})$ over a classical measurable space $(\Omega, \mathbb{A})$ into the unit interval $[0,1]$. Each *fuzzy probability measure* is a Lebesgue integral with respect to some probability measure on $\mathbb{A}$. A system $\mathcal{P}(\mathbb{A})$ of all probability measures on $\mathbb{A}$ represents *a set of all elementary fuzzy events* (compare with [8]).

How does the fuzzy probability act on ID-probability stage?

Let $\mathbb{A}$ be a σ-field of subsets of Ω, and let $\mathcal{M}(\mathbb{A})$ be a set of all measurable functions from $\mathbb{A}$ into the interval $I = [0,1]$. This assumption enables to take $\mathcal{M}(\mathbb{A})$ for a D-poset of fuzzy subsets of Ω (D-equipment is given pointwise). If Ω is a singleton, then I has the form of $\mathcal{M}(\mathbb{A})$.

Denote **BID** full subcategory of the category **ID** whose objects are MV-algebras of fuzzy subsets of a universe and denote **CBID** its full subcategory consisting from sequentially closed objects. Due to [22, Corollary 2.8], each object of **CBID** is a *Łukasiewicz clan (tribe)*. Moreover, let **CGBID** be the full subcategory of **ID** having objects in the form of $\mathcal{M}(\mathbb{A})$. Let $\mathcal{X} \subseteq I^\Omega$ be an MV-algebra of fuzzy subsets of Ω. If there exists a σ-field $\mathbb{A}$ of subsets of Ω such that $\mathcal{X} = \mathcal{M}(\mathbb{A})$, then $\mathcal{X}$ is said to be a *generated Łukasiewicz clan (tribe)*. It is known (see [22, Lemma 2.5] and [54, Theorem 5.1]) that $\mathcal{X}$ is a generated Łukasiewicz tribe if and only if it consists of all constant functions defined on Ω ranging in I and it is sequentially closed in I^Ω.

Theorem 2 ([29, Theorem 2.2]). *Let $\mathcal{X} \subseteq I^\Omega$ be a generated Łukasiewicz tribe and let $\mathbb{A}$ be a σ-field of subsets of Ω such that $\mathcal{X} = \mathcal{M}(\mathbb{A})$. Let h be a map from $\mathcal{X}$ into I. Then the following statements are equivalent:*

(i) The map h is sequentially continuous D-homomorphism;

(ii) There exists a probability measure P on $\mathbb{A}$ such that $h(u) = \int u\,\mathrm{d}P$ for every $u \in \mathcal{X}$.

Corollary 3. *Let $\mathcal{X} \subseteq I^\Omega$ is a generated Łukasiewicz tribe and let $\mathbb{A}$ is a σ-field of subsets of Ω such that $\mathcal{X} = \mathcal{M}(\mathbb{A})$. Then there exists one-to-one correspondence between probability measures on $\mathbb{A}$, Lebesgue measurable integrals on $\mathcal{M}(\mathbb{A})$ with respect to probabilities on $\mathbb{A}$ and sequentially continuous D-homomorphisms from $\mathcal{X}$ into I. The map h from Theorem 2 is actually a state on $\mathcal{X}$ and regarding Butnari-Klement theorem (see [64, Theorem 8.1.12]), states on Łukasiewicz tribes are integrals with respect to probability measures.*

Thus fuzzy random events and fuzzy probability measures as constituting notions of fuzzy probability theory can be expressed in language of D-posets of fuzzy subsets of a universe. Since the same holds true for Kolmogorovian probability theory fundamental notions, a natural question that arises is the following:

What is the categorical relationship between Kolmogorovian and fuzzy probability theory?

First, it is worth to point out that σ-fields as Kolmogorovian probability domains and generated Łukasiewicz tribes as fuzzy probability domains are sequentially closed with respect to limits of random events sequences.

Let $\mathcal{X} \subseteq I^\Omega$ be a D-poset of fuzzy subsets of Ω and let n be a natural number. A system $\mathcal{X}$ is said to be *divisible by* n if and only if to every $u \in \mathcal{X}$ there exists $v \in \mathcal{X}$ such that $u(\omega) = nv(\omega)$. A system $\mathcal{X}$ is said to be *divisible* if and only if it is divisible for every natural number n.

Theorem 4 ([29, Lemma 2.6])**.** *Let $\mathcal{X} \subseteq I^\Omega$ be an MV-algebra of fuzzy subsets of Ω. Then the following statements are equivalent:*

(i) A system $\mathcal{X}$ is divisible and sequentially closed in I^Ω;

(ii) A system $\mathcal{X}$ is generated Łukasiewicz tribe.

According to [33], some special Łukasiewicz tribes together with divisibility and lattice viewing on ID-posets enable a classification of probability domains as well as effective description of a transition from classical to fuzzy probability (not only categorical but topological, too).

Denote $\mathbb{N}$ the set of all natural numbers. Denote I_n an ID-poset of elements from the set $\{0, 1/n, 2/n, \dots, (n-1)/n, 1\} \subseteq I$, $n \in \mathbb{N}$. Denote I_Q an ID-poset of all rational numbers from the unit interval. ID-posets I_n, $n \in \mathbb{N}$, and I_Q are non-trivial subobjects of the ID-poset I. Except I_Q the remaining are at the same time sequentially closed in I.

Let $\mathbb{A}$ be a σ-field of subsets of a universe X. Denote $\mathcal{M}(\mathbb{A})$ the set of all measurable functions from X into I. Denote $\mathcal{M}_Q(\mathbb{A})$, respectively $\mathcal{M}_n(\mathbb{A})$, $n \in \mathbb{N}$, its subset consisting from all measurable functions with I_Q, respectively I_n, as a codomain. Obviously, both $\mathcal{M}_n(\mathbb{A})$ and $\mathcal{M}_Q(\mathbb{A})$ can be viewed as ID-posets and subobjects of I^X. And, apparently, $\mathcal{M}_1(\mathbb{A})$ and $\mathbb{A}$ can be identified. All but $\mathcal{M}_Q(\mathbb{A})$ are sequentially closed in I^X. Finally, denote $s(\mathbb{A})$ the set of all simple measurable functions over a σ-field $\mathbb{A}$, i.e. functions of the form $\sum_{i=1}^k a_i \chi_{A_i}$, where $a_i \in [0,1]$, $A_i \in \mathbb{A}$, A_i are mutually disjoint and cover X, $i \in \{1, 2, \dots, k\}$, $k \in \mathbb{N}$. It is easy to see that $\mathcal{M}_n(\mathbb{A}) \subseteq s(\mathbb{A})$, $n \in \mathbb{N}$. Concerning a divisibility, $\mathcal{M}_n(\mathbb{A})$ is divisible by a natural number n and systems $s(\mathbb{A})$, $\mathcal{M}_Q(\mathbb{A})$, $\mathcal{M}(\mathbb{A})$ are divisible.

As it was already mentioned above, there are situations in which is both profitable and natural to require that ID-posets are lattices. In that cases, lattice operations $\vee$, $\wedge$ are, for u, v from ID-poset $\mathcal{X} \subseteq I^X$, defined pointwise: $(u \vee v)(x) = u(x) \vee v(x) = \max\{u(x), v(x)\}$ and $(u \wedge v)(x) = u(x) \wedge v(x) = \min\{u(x), v(x)\}$ for every $x \in X$. An ID-poset $\mathcal{X} \subseteq I^X$ is said to be a *lattice ID-poset* if and only if $u \vee v \in \mathcal{X}$ and $u \wedge v \in \mathcal{X}$ for every $u, v \in \mathcal{X}$.

Latticeness imposed on ID-posets has far-reaching consequences in modelling quantum phenomena. It is closely related to the notion of compatibility. Elements u and v of a D-poset X with the difference $\ominus$ are said to be *compatible* if and only if there exist $c, d \in X$ such that $d \le u \le c$, $d \le v \le c$ and $c \ominus u = v \ominus d$.

Let $\mathcal{X} \subseteq I^X$ be a lattice ID-poset and let $c = u \vee v$ and $d = u \wedge v$ for $u, v \in \mathcal{X}$. Then $(u \vee v)(x) \ominus u(x) = (v)(x) \ominus (u \wedge v)(x)$ for every $x \in X$, hence each pair $u, v \in \mathcal{X}$ is compatible. A general theory of D-posets ([11]) guarantees that an ID-poset in which every two elements are compatible is an MV-algebra of fuzzy sets, i.e. a *bold algebra* (also $\mathrm{T_L}$-clan):

Theorem 5 ([33, Theorem 3.6]). *Let $\mathcal{X} \subseteq I^X$ be a lattice ID-poset. Then X is a bold algebra.*

Corollary 6 ([33, Corollary 3.7]). *Let $\mathcal{X} \subseteq I^X$ be an ID-poset. Then the following statements are equivalent:*

(i) A system $\mathcal{X}$ is a lattice ID-poset;

(ii) A system $\mathcal{X}$ is a bold algebra.

Corollary 7 ([33, Corollary 3.8]). *Let $\mathcal{X} \subseteq I^X$ be a sequentially closed ID-poset. Then the following statements are equivalent:*

(i) A system $\mathcal{X}$ is a lattice ID-poset;

(ii) A system $\mathcal{X}$ is a Łukasiewicz tribe.

Corollary 8 ([33, Corollary 3.10]). *Let $\mathcal{X} \subseteq I^X$ be a sequentially closed divisible ID-poset. Then the following statements are equivalent:*

(i) A system $\mathcal{X}$ is a lattice ID-poset;

(ii) There exists a σ-field A of subsets of X such that $\mathcal{X} = \mathcal{M}(\mathbb{A})$.

Summary: To sum up, **ID** is a suitable category in which traditional domains of probability can be characterized by natural properties. The category **ID** environment enables to model both the sure and the impossible event, and complementary events, too. The structure of events is determined by states. Closed ID-posets satisfy a natural requirement that the probability domains should be closed with respect to sequential limits. It enables limit stochastic. Lattice ID-posets are bold algebras. Closed lattice ID-posets are Łukasiewicz tribes. The transition from the classical random events represented by σ-fields to fuzzy random events represented by measurable functions is characterized by divisibility. Łukasiewicz tribes form a category in which both classical and fuzzy events live. The probability of an event can be calculated via an integral. From $\mathbb{A} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbb{A})$ it follows that the classical events (σ-fields of sets) are "minimal" and the fuzzy events (generated Łukasiewicz tribes) are "maximal" probability domains.

In [18], R. Frič introduced an epireflector $\mathbf{F}\colon$ **CFSD** $\to$ **CGBID** which sends each σ-field $\mathbb{A} \subseteq \{0,1\}^X$ to a system $\mathcal{M}(\mathbb{A}) \subseteq I^X$ of all measurable functions from X into I. It is so-called fuzzification functor which extends every $\mathbb{A}$ to $\mathcal{M}(\mathbb{A})$ and every morphism $h\colon \mathbb{A} \to \mathbb{B}$ from the category **CFSD** to exactly one morphism $h_e\colon \mathcal{M}(\mathbb{A}) \to \mathcal{M}(\mathbb{B})$ in the category **CGBID**. Since there exists a morphism $g\colon \mathcal{M}(\mathbb{A}) \to \mathcal{M}(\mathbb{B})$ such that for no morphism $h\colon \mathbb{A} \to \mathbb{B}$ the equality $h_e = g$ is fulfilled, the extension is essential. Moreover, due to Theorem 4 it is in certain sense minimal:

Theorem 9 ([29, Theorem 2.8]). **CGBID** *is the smallest of all full subcategories* **K** *of* **ID** *extending* **CFSD** *such that if $\mathcal{X} \subseteq I^\Omega$ is an object of* **K**, *then $\mathcal{X}$ is divisible and sequentially closed in I^Ω.*

So, the functor **F** assigns to each Kolmogorovian probability domain $\mathbb{A}$ its fuzzification $\mathcal{M}(\mathbb{A}) = \mathbf{F}(\mathbb{A})$, and to each Boolean homomorphism h from one probability domain to the another one (observable) its fuzzification $\mathbf{F}(h)$ which is D-homomorphism from one fuzzified probability domain into another fuzzified one.

Since fuzzification functor **F** assigns to "crisp" probability domains "fuzzy" ones and **CGBID** is not subcategory of **CFSD** (in general, they have no common object), to embed $\mathbb{A}$ into $\mathcal{M}(\mathbb{A})$ there is necessary some "bigger" category in which both of them are included. And, at the same time, there is needed a functor **E** such that **F** the restriction of **E**, i.e. $\mathbf{E}(\mathbb{A}) = \mathcal{M}(\mathbb{A})$ for every object $\mathbb{A}$ of the category **CFSD**. Such category and functor were introduced in [30]:

Denote **EID** the full subcategory of **ID** consisting of all objects in the form $\mathcal{M}_n(\mathbb{A})$, $n \in \mathbb{N}$, and $\mathcal{M}(\mathbb{A})$. The assignment $\mathcal{M}_n(\mathbb{A}) \mapsto \mathcal{M}(\mathbb{A})$ yields the epireflector **E** of **EID** into **CGBID**.

Theorem 10 ([30, Lemma 4.1.1]). *Let $\mathbb{A}$ and $\mathbb{B}$ be a σ-field of subsets of X and Y, respectively. Let h, g be sequentially continuous D-homomorphisms of $\mathcal{M}(\mathbb{A})$ into $\mathcal{M}(\mathbb{B})$ such that $h(\chi_A) = g(\chi_A)$ for each $A \in \mathbb{A}$. Then*

(i) $h(\chi_A/n) = g(\chi_A/n)$ for all $A \in \mathbb{A}$, $n \in \mathbb{N}$;

(ii) $h(u) = g(u)$ for all $u = \sum_{i=1}^{k} a_i\chi_{A_i} \in \mathcal{M}_n(\mathbb{A})$, $n \in \mathbb{N}$;

(iii) $h(u) = g(u)$ for all $u = \sum_{i=1}^{k} a_i\chi_{A_i} \in s(\mathbb{A})$;

(iv) $h(u) = g(u)$ for all $u \in \mathcal{M}(\mathbb{A})$.

Corollary 11 ([30, Corollary 4.1.2]). *Let $\mathbb{A}$ and $\mathbb{B}$ be a σ-field of subsets of X and Y, respectively. Let $\mathcal{O}(\mathbb{A})$ and $\mathcal{O}(\mathbb{B})$ be objects of the category* **EID** *and let h, g be sequentially continuous D-homomorphisms of $\mathcal{O}(\mathbb{A})$ into $\mathcal{O}(\mathbb{B})$. If $h(A) = g(A)$ for every $A \in \mathbb{A}$, then $h = g$.*

Corollary 12 ([30, Corollary 4.1.3]). *Let $\mathbb{A}$ be a σ-field of subsets of X and let $\mathcal{O}(\mathbb{A})$ be an object of the category* **EID**. *Let h be a sequentially continuous D-homomorphism of $\mathcal{O}(\mathbb{A})$ into I.*

(i) Then there exists a unique probability measure m on $\mathbb{A}$ such that for all $u \in \mathcal{O}(\mathbb{A})$ the equality $h(u) = \int u \,\mathrm{d}m$ holds true.

(ii) Let $\overline{h}(u) = \int u \,\mathrm{d}m$ for $u \in \mathcal{M}(\mathbb{A})$. Then $\overline{h}$ is the unique sequentially continuous D-homomorphism of $\mathcal{M}(\mathbb{A})$ into I such that $\overline{h}(u) = h(u)$ for all $u \in \mathcal{O}(\mathbb{A})$.

Theorem 13 ([30, Theorem 4.1.4]). *Let $\mathbb{A}$, respectively $\mathbb{B}$, be a σ-field of subsets of X, respectively Y. Let $\mathcal{O}(\mathbb{A})$ and $\mathcal{O}(\mathbb{B})$ are objects of the category* **EID** *and let h be a sequentially continuous D-homomorphism of $\mathcal{O}(\mathbb{A})$ into $\mathcal{O}(\mathbb{B})$. Then there exists a unique sequentially continuous D-homomorphism $\overline{h}$ of $\mathcal{M}(\mathbb{A})$ into $\mathcal{M}(\mathbb{B})$ such that $\overline{h}(u) = h(u)$ for all $u \in \mathcal{O}(\mathbb{A})$.*

Reflecting the previous facts, the functor **E** can be defined in the following way: $\mathbf{E}(\mathcal{O}(\mathbb{A})) = \mathcal{M}(\mathbb{A})$ for each object $\mathcal{O}(\mathbb{A})$ of the category **EID** and $\mathbf{E}(h) = \overline{h}$ for each morphism h of $\mathcal{O}(\mathbb{A})$ into $\mathcal{O}(\mathbb{B})$, where $\overline{h}$ is the unique morphism of $\mathcal{M}(\mathbb{A})$ into $\mathcal{M}(\mathbb{B})$ which is the extension of h.

Theorem 14 ([30, Lemma 4.1.5, Theorem 4.1.6]). **E** *is the epireflector of the category* **EID** *into the category* **CGBID**.

The epireflection from Theorem 14 can be enlarged on some "bigger" category in comparison with **CGBID**. Due to [33], the resulting category **CBID** is the full subcategory of **ID** consisting of Łukasiewicz tribes. The corresponding functor **E** is given as $\mathbf{E}(\mathcal{X}) = \mathcal{M}(\mathbb{A})$ for every Łukasiewicz tribe $\mathcal{X}$ such that $\mathbb{A} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbb{A})$, and $\mathbf{E}(h) = \overline{h}$ for every sequentially continuous D-homomorphism of a Łukasiewicz tribe $\mathcal{X}$,

$\mathbb{A} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbb{A})$, into a Łukasiewicz tribe $\mathcal{Y}$, $\mathbb{B} \subseteq \mathcal{Y} \subseteq \mathcal{M}(\mathbb{B})$. The map $\overline{h}$ is the unique sequentially continuous D-homomorphism of $\mathcal{M}(\mathbb{A})$ into $\mathcal{M}(\mathbb{B})$ such that $\overline{h}$ is the extension of h (se [33, Theorem 4.2]).

Theorem 15 ([33, Lemma 4.3, Corollary 4.4])**. E** *is an epireflector of the category* **CBID** *into the category* **CGBID**.

Further results, mostly concerning topological and algebraic aspects of categorical fuzzification of cogenerators as well as generalized probability domain constructions and generalized probability measures, can be found in [58], [45], [46], [23], [27], [24], [20].

Another probability theories in ID

In [32] a probability theory based on the category $\mathbf{S_nD}$ was suggested. Subsequently, it was elaborated in [29], [30], [26]. It is the category which is cogenerated by a simplex $S_n = \{(x_1, x_2, \ldots, x_n) \in I^n;\ \sum_{i=1}^{n} x_i \leq 1\}$ with partial pointwise order, partial pointwise difference and sequential convergence. Objects of $\mathbf{S_nD}$ are subobjects of S_n^X. In S_nD-probability theory, probability domains have n components. Each generalized random event is represented by a collection of n mutually competing rivals. The range of states is a simplex of n-tuples of possible "rewards" S_n such that their sum is a number from the unit interval $[0, 1]$. For $n = 1$ the S_nD-probability theory is reduced to the fuzzy probability theory. The choice $n = 2$ leads to the IF-probability (see [62], [38]) where IF-events are modelled by pairs (μ, ν) of fuzzy subsets $\mu, \nu \in [0, 1]^X$ of a universe X such that $\mu(x) + \nu(x) \leq 1$ for every $x \in X$ (see Figure 4). The system of events is pointwise partially ordered. (Following K. Atanassov approach in [3], on the contrary, $(\mu_1, \nu_1) \leq (\mu_2, \nu_2)$ if and only if $\mu_1 \leq \mu_2$ and $\nu_2 \leq \nu_1$, e.g. in [62]).

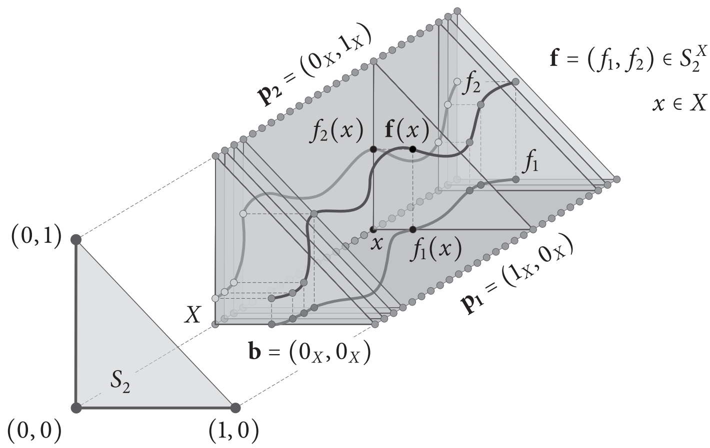

Figure 4. Construction of events in S_2D-probability

Random variables and observables in ID

The notion of a random variable enables to capture randomness via measuring instrument deflections. De facto, it is a special map f of a fictive (excluding finite number of exceptions) probability space $(\Omega, \mathbb{A}, P)$ into a probability space of real numbers $(R, \mathbb{B}_R, P_R)$ which preserves a structure of events and a probability measure. So, a random variable is a special measurable function: the preimage of a Borel measurable set is measurable, i.e. it belongs to a system of original space measurable sets. In general, there is no necessity to take into account only $(R, \mathbb{B}_R, P_R)$-cases. Vice versa one can study measurability between any measurable spaces. Let $(X, \mathbb{A})$ and $(Y, \mathbb{B})$ be measurable spaces. A map $f\colon\ X \to Y$ is said to be $(\mathbb{B}, \mathbb{A})$-*measurable* (abbreviated to *measurable*) if and only if to each set $B \in \mathbb{B}$ there exists $A \in \mathbb{A}$ such that $f^{\leftarrow}(B) = A$. In language of indicator functions, the measurability requirement looks as: A map $f\colon\ X \to Y$ is measurable if and only if for every $B \in \mathbb{B}$, the composition $\chi_B \circ f$ is an indicator function of some set from $\mathbb{A}$, i.e. $(\chi_B \circ f)(x) = \chi_{f^{\leftarrow}(B)}(x) = \chi_A(x)$ for every $x \in X$. Thus a measurability can be studied via a composition of suitable functions. This approach, at the same time, induces a map $f^{\leftarrow}\colon\ \mathbb{B} \to \mathbb{A}$ given by the assignment $\chi_B \mapsto \chi_{f^{\leftarrow}(B)}$ for each indicator function $\chi_B \in \mathbb{B}$. In quantum structures theory as well as in theoretical physics, it is called *observable*.

If f is a measurable map of $(X, \mathbb{A})$ into $(Y, \mathbb{B})$, then the corresponding induced map $f^{\leftarrow}$ is a sequentially continuous Boolean homomorphism. According to [16], the continuity is sequential with respect to the Hausdorff convergence of sequences of sets: $\{A_n\}$ converges to A if and only if $\limsup A_n = \liminf A_n$, or, equivalently with respect to pointwise convergence of indicator functions, $\chi_A = \lim_{n\to\infty} \chi_{A_n}$.

If f is a random variable of a set Ω of random experiment outcomes (elementary events) modelled by a Kolmogorovian probability space $(\Omega, \mathbb{A}, P)$ into the set of real numbers R, then it is an information channel through which P is "shifted" to a distribution P_f. More precisely, to P is assigned a probability measure on Borel measurable sets given by $P_f\big((-\infty, r)\big) = P\big(f^{\leftarrow}((-\infty, r))\big) = P\big(\{\omega \in \Omega \,|\, f(\omega) < r\}\big)$. Translated into the indicator functions language, for every Borel measurable set $B \subseteq R$ is fulfilled $P_f(\chi_B) = P(\chi_{f^{\leftarrow}(B)}) = (P \circ f^{\leftarrow})(B) = P(\chi_A)$ for some $A \in \mathbb{A}$, i.e. $P_f = P \circ f^{\leftarrow}$.

Let $(\Omega, \mathbb{A}, P)$ and $(\Xi, \mathbb{B}, Q)$ be classical probability spaces and let f be a measurable map of Ω into Ξ. A map f is said to be a *generalized random variable* if and only if $Q(B) = P(f^{\leftarrow}(B)) = (P \circ f^{\leftarrow})(B)$ for every $B \in \mathbb{B}$. Thus a generalized random variable f naturally pushes forward a probability P on $\mathbb{A}$ into a probability $P_f = Q$ on $\mathbb{B}$. This means that f defines a distribution map D_f, $D_f(P) = P_f$, of the set $\mathcal{P}(\mathbb{A})$ of all probability measures on $\mathbb{A}$ into the set $\mathcal{P}(\mathbb{B})$ of all probability measures on $\mathcal{P}(\mathbb{B})$.

Each elementary event ω of a classical probability space $(\Omega, \mathbb{A})$ and the corresponding degenerated point probability measure δ_ω can be identified via $\delta_\omega(A) = 1$ if $\omega \in \mathbb{A}$, otherwise $\delta_\omega(A) = 0$, $A \in \mathbb{A}$, $\omega \in \Omega$. This identification enables to describe a fuzzy random variable as a measurable map of the set $\mathcal{P}(\mathbb{A})$ of all probability measures on $\mathbb{A}$ into the set $\mathcal{P}(\mathbb{B})$ of all probability measures on $\mathcal{P}(\mathbb{B})$ over a classical probability space $(\Xi, \mathbb{B})$ (according to [8], see also in [37], [29]):

Let $(\Omega, \mathbb{A})$, $(\Xi, \mathbb{B})$ be measurable spaces. A map $T\colon\ \mathcal{P}(\mathbb{A}) \to \mathcal{P}(\mathbb{B})$ is said to be a *fuzzy random variable* if and only if, for each $B \in \mathbb{B}$, the assignment $\omega \mapsto \big(T(\delta_\omega)\big)(B)$ yields a measurable map of Ω into $[0, 1]$ such that

$$\big(T(P)\big)(B) = \int \big(T(\delta_\omega)\big)(B)\,\mathrm{d}P \tag{BG}$$

for each $P \in \mathcal{P}(\mathbb{A})$.

This definition corresponds with works of S. Bugajski ([8], [9]) and S. Gudder ([40], [42]) in which the notion is called a *statistical map*.

Statistical maps are equivalent to several other notions: observables, Markov (probability) kernels, dual statistical maps, statistical functions, effect-valued measures (see [25]).

If D_f is a distribution of a classical measurable map f of $(\Omega, \mathbb{A})$ into $(\Xi, \mathbb{B})$, then D_f is a fuzzy random variable. Actually, $\big(D_f(\delta_\omega)\big)(B) = \delta_\omega\big(f^{\leftarrow}(B)\big) = \chi_{f^{\leftarrow}(B)}(\omega)$, where

$\chi_{f^{\leftarrow}(B)}$ is an indicator function of $f^{\leftarrow}(B)$, and the equality (BG) is has the form

$$\big(D_f(P)\big)(B) = \int \chi_{f^{\leftarrow}(B)}\, \mathrm{d}P(\{\omega\}) = P\big(f^{\leftarrow}(B)\big).$$

In fact, this means that one can grasp a classical random event as some special case of a fuzzy random variable. So, this classical notion can be studied in "fuzzy depiction".

How are random variables and observables presented in ID-probability?

Let $\mathcal{X} \subseteq I^X$ be a D-poset of fuzzy subsets of a universe X. Then a pair $(X, \mathcal{X})$ is said to be an *ID-measurable space*. Let $(Y, \mathcal{Y})$ be another ID-measurable space and let f be a map of X into Y such that $u \circ f \in \mathcal{X}$ for every $u \in \mathcal{Y}$. Then f is said to be a $(\mathcal{Y}, \mathcal{X})$*-measurable map* (abbreviated to *measurable map*).

Define the category **MID** as follows: the objects are ID-measurable spaces and the morphisms are measurable maps. If $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ are fields of indicator functions over corresponding universes, then ID-measurability and the classical one can be identified. Moreover, via the assignment $u \mapsto u \circ f$, $u \in \mathcal{Y}$, a map f induces unique ID-homomorphism $f^{\lhd}$ of $\mathcal{Y}$ into $\mathcal{X}$. Such a map $f^{\lhd}$ is said to be a *fuzzy observable*.

Let $\delta_y(u) = u(y)$ for $y \in Y$ and $u \in \mathcal{Y}$. Obviously, such a map δ_y is an ID-homomorphism. Both $\mathcal{Y}$ and $(Y, \mathcal{Y})$ are said to be *sober* if and only if to each ID-homomorphism h of $\mathcal{Y}$ into I there exists $y \in Y$ such that $h = \delta_y$. It is known that if $\mathcal{Y}$ is sober, then to each ID-homomorphism h of $\mathcal{Y}$ into $\mathcal{X}$ there exists a measurable map f of $(X, \mathcal{X})$ into $(Y, \mathcal{Y})$ such that $h = f^{\lhd}$. It leads to a duality between ID-homomorphisms and ID-measurable maps. Its special case is the duality between observables and fuzzy random variables in the fuzzy (operational) probability theory. In [8], [9], applying functional analysis apparatus, the continuity of $f^{\lhd}$ and the duality in question were proven. The same was more transparently and straightforwardly shown via diagrams in [25] and [61]. Regarding [25, Corollary 3.3], to each statistical map (fuzzy random variable) there corresponds a dual statistical map (observable). The corresponding "category of fuzzy random variables" and the "category of fuzzy observables" are dually isomorphic.

Recall that in the category **CFSD** the D-homomorphisms are Boolean. It means that classical observable maps assign a crisp event to a crisp event. According to [18, Corollary 2.11], each sequentially continuous D-homomorphism of a σ-field $\mathbb{A} \subseteq \{0,1\}^{\Omega}$ into a σ-field $\mathbb{B} \subseteq \{0,1\}^{\Xi}$, i.e. a classical observable, can be extended to unique sequentially continuous D-homomorphism of a generated Łukasiewicz tribe $\mathcal{M}(\mathbb{A})$ into a generated Łukasiewicz tribe $\mathcal{M}(\mathbb{B})$, i.e. to a fuzzy observable. Despite the extension is unique, there exist fuzzy observables which are not extensions of any classical observables. Those maps assign a non-crisp event to a crisp one. It indicates genuine quantum quality of a studied phenomenon. The example can be found in [37, Example 3.1]. Quantum character of an effect also appears when a fuzzy random variable "transforms" a degenerated measure (a point) to a non-degenerated one (a spectrum). Another related information can be found in [8], [9], [40], [41], [29], [30], [34].

Let $\mathcal{X} \subseteq I^X$ be an ID-poset. Let X^* be a set of all ID-homomorphisms of $\mathcal{X}$ into I. If $x \in X$ and δ_x are identified, then X can be viewed as a subset of X^*. Denote u^* a set $\{u(x) \,|\, x \in X^*\} \subseteq I^{X^*}$ for $u \in \mathcal{X}$. Then the system $\mathcal{X}^* = \{u^* \in I^{X^*} \,|\, u \in \mathcal{X}\}$ can be considered as an ID-object, where $\mathcal{X}^*$ is sober and the assignment $u \mapsto u^*$ yields an ID-isomorphism of $\mathcal{X}$ into X^*.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be ID-measurable spaces, let f^* be a measurable map of $(X^*, \mathcal{X}^*)$ into $(Y^*, \mathcal{Y}^*)$ and let $(f^*)^{\lhd}$ be the dual ID-morphism of $\mathcal{Y}^*$ into $\mathcal{X}^*$ sending $u^* \in \mathcal{Y}^*$ to $u^* \circ f^* \in \mathcal{X}^*$. Finally, for $u \in \mathcal{Y}$, let $f^{\lhd}(u)$ be the restriction $(u^* \circ f^*) \upharpoonright X$ of the composition $u^* \circ f^* \in X^*$ to X. It is obvious that $f^{\triangleleft}$ is an ID-isomorphism of $\mathcal{X}$ into $\mathcal{Y}$ (see [61], [29]).

Probability spaces in ID-perspective

The study of elements which constitute a probability space in the categorical framework naturally results in research of properties of probability spaces themselves:

A triple $(\Omega, \mathcal{M}(\mathbb{A}), \int(\cdot)\,\mathrm{d}P)$ is said to be a *fuzzy probability space* if and only if $(\Omega, \mathbb{A}, P)$ is a classical Kolmogorovian probability space, $\mathcal{M}(\mathbb{A})$ is a D-poset of fuzzy subsets of Ω corresponding to $\mathbb{A}$, and $\int(\cdot)\,\mathrm{d}P$ is a probability integral with respect to P. Such definition is feasible because of the one-to-one correspondence between classical probability spaces and their fuzzifications.

Let $(\Omega, \mathcal{M}(\mathbb{A}), \int(\cdot)\,\mathrm{d}P)$ and $(\Xi, \mathcal{M}(\mathbb{B}), \int(\cdot)\,\mathrm{d}Q)$ be fuzzy probability spaces. A sequentially continuous D-homomorphism h of $\mathcal{M}(\mathbb{A})$ into $\mathcal{M}(\mathbb{B})$ is said to be *probability integral preserving* if and only if $\int v\,\mathrm{d}Q = \int h(v)\,\mathrm{d}P$ for every $v \in \mathcal{M}(\mathbb{B})$. In fact, it is an observable which is called *fuzzy observable*. If, moreover, $h(\chi_B) \in \mathbb{A}$ for all $\chi_B \in \mathbb{B}$, then h is said to be a *restricted fuzzy observable*.

Denote **CP** the category having classical probability spaces as objects and classical observables as morphisms. Denote **FP** the category having fuzzy probability spaces as objects and fuzzy observables as morphisms. How are the categories **CP** and **FP** related? According to extension theorems in [18] and [29], the following statement is true:

Theorem 16 ([58, Theorem 4.7])**.** *Let $(\Omega, \mathbb{A}, P)$ and $(\Xi, \mathbb{B}, Q)$ be classical probability spaces and let $(\Omega, \mathcal{M}(\mathbb{A}), \int(\cdot)\,\mathrm{d}P)$ and $(\Xi, \mathcal{M}(\mathbb{B}), \int(\cdot)\,\mathrm{d}Q)$ be the corresponding probability spaces. Let h_c be a classical observable of $\mathbb{B}$ into $\mathbb{A}$. Then there exists unique observable h such that $h_c(B) = h(B)$ for all $B \in \mathbb{B}$.*

Denote **RFP** the subcategory of **FP** having the fuzzy probability spaces as objects and the restricted fuzzy observables as morphisms. In consequence of Theorem 16 there exists an isomorphism between **CP** and **RFP**:

Theorem 17 ([58, Theorem 4.8])**.** *Categories* **CP** *and* **RFP** *are isomorphic.*

Thus there is a canonical isomorphism between the category **CP** representing classical probability theories and the subcategory **RFP** of the category **FP** representing fuzzy probability theories. The objects of the two categories are in a canonical one-to-one correspondence, but the fuzzy probability theory has "more" morphisms.

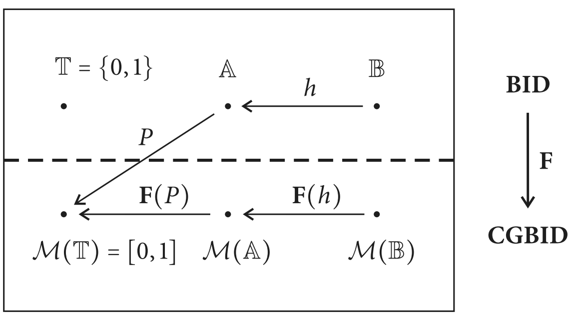

Figure 5. Epireflector **F** – fuzzification functor

Summary: Fuzzy probability theory is non-trivial categorical extension of classical Kolmogorovian probability theory. The embedding of the one into the another is carried out via a functor $\mathbf{F}$ (see Figure 5). In fuzzy probability theory there are phenomena which *sui generis* cannot be modeled within classical Kolmogorovian probability theory. Finally (cf. [34]),

(i) to each σ-field $\mathbb{A}$ of random events in Kolmogorovian probability theory there corresponds its fuzzified field of random events $\mathcal{M}(\mathbb{A})$, and the correspondence is one-to-one;

(ii) to each (generalized) observable $h\colon\ \mathbb{B} \to \mathbb{A}$ there corresponds its fuzzified observable $\mathbf{F}(h)\colon\ \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{A})$, and the correspondence between classical and fuzzy observables fails to be one-to-one;

(iii) each probability measure $P\colon\ \mathbb{A} \to \mathcal{M}(\mathbb{T}) = [0,1]$ is the restriction of the fuzzy observable $\mathbf{F}(P)\colon\ \mathcal{M}(\mathbb{A}) \to \mathcal{M}(\mathbb{T})$, $\mathbf{F}(P) = \int (\cdot)\,\mathrm{d}P$, where $\mathbb{T} = \{0,1\}$ is considered as a bold algebra; dual mappings to observables model transition from states on one generalized field of events to states on another field of events;

(iv) there is a fuzzy observable $h\colon\ \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{A})$ such that no observable in classical probability theory is its restriction; such a "genuine" fuzzy observable maps a crisp event to a genuine fuzzy event and it has *per se* quantum quality.

ID-probability — summing up definition

Definition 18 ([29, Definition 3.1])**.** Let $\mathcal{X} \subseteq I^X$ be an object of the category **ID**. Then $\mathcal{X}$ is said to be an *ID-probability domain* and each element of $\mathcal{X}$ is said to be an *ID-random event*. Let s be an ID-morphism of $\mathcal{X}$ into I. Then s is said to be an *ID-state*. Let $\mathcal{Y} \subseteq I^Y$ be another ID-probability domain and let h be an ID-morphism of $\mathcal{Y}$ into $\mathcal{X}$. Then h is said to be an *ID-observable*. Let f^* be a measurable map of $(X^*, \mathcal{X}^*)$ into $(Y^*, \mathcal{Y}^*)$. Then f^* is said to be an *ID-random variable* and the corresponding ID-morphism $f^{\triangleleft}$ sending $u \in \mathcal{Y}$ to $(u^* \circ f^*) \restriction X \in \mathcal{X}$ is said to be the *dual observable* to f^*.

Categorical bonuses

Because the ID-approach to probability theory is categorical, there is possible to exploit its *categorical* "added value" effectively in further research and applications. Short description of several selected results obtained in this way follows:

— Categorical constructions of the coproduct and the product of ID-measurable spaces together with characterization of their properties can be found in [59]. In [34], the transition from IF-probability to fuzzy probability is elaborated using coproducts and products of suitable objects of the category **ID**.

— Referring to the argument by B. Coecke in work on categories devoted to the practicing physicist ([12]) — *"to be able to conceive two systems as one whole and to consider the compound operations inherited from the operations on the individual systems, we pass from ordinary categories to a particular case of the 2-dimensional variant of categories, monoidal category"* — there was suggested a *tensor modification* of ID-theory in [34].

Fuzzy probability theory obtains "tensor" quality in the category **BIDT** environment.

First, it is necessary to introduce a "terminal object" $\top$. It has to be a "degenerated D-poset of fuzzy subsets of a universe" in which the smallest and the greatest elements coincide. The object $\top$ consists of only one element. Its structure is trivial. To each

D-poset X, there exists unique morphism of X into $\top$. The existence of such object is guaranteed by only formal change of D-poset definition: For one thing, it is not supposed that $0_X \neq 1_X$, for another, $[0,1]^{\emptyset} = \{\emptyset\} = \top$ is considered as a degenerated D-poset of fuzzy subsets for $X = \emptyset$. Analogously, $\top$ can be viewed as a bold algebra, too.

After this cosmetic definition change, the category **BIDT** can be defined as follows ([34]):

(i) The objects are ordered pairs $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X} \subseteq [0,1]^X$ is a bold algebra, and $\mathcal{Y} \subseteq [0,1]^Y$ is either a bold algebra or a degenerated D-poset $\top = [0,1]^{\emptyset}$, and operations are defined coordinatewise;

(ii) The morphisms are ordered pairs (f, g) of sequentially continuous D-homomorphisms and the composition is parallel: $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$ whenever the compositions $f_2 \circ f_1$, $g_2 \circ g_1$ are defined.

— The transition from one stochastic situation to another one can be given via a transformation of one probability space into another one. It naturally leads to a generalized random walk and Markov kernels with many applications. In discrete cases, these notions are studied applying ID-approach (examples are included, too) in [60], [36], [37].

— The transition from classical probability space to its "divisible" extension is outlined in [28] and [21]. It is characterized by the extension of two-valued Boolean logic on $\mathbb{A}$ to multivalued Łukasiewicz logic on $\mathcal{M}(\mathbb{A})$ and the divisibility of random events: for each random event $u \in \mathcal{M}(\mathbb{A})$ and each positive natural number n, the quotient u/n belongs to $\mathcal{M}(\mathbb{A})$ and $\int (u/n)\, \mathrm{d}P = (1/n) \int u \,\mathrm{d}P$.

Moreover, in [28] and [5] are studied joint random experiments and asymmetrical stochastic dependence/independence of one constituent experiment on the other one. And a canonical construction of conditional probability so that observables can be viewed as conditional probabilities is presented there, too.

— Due to [4], each probability measure induces an additive linear preorder on classical random events which helps making decisions. In the categorical framework, the author of [4] showed that probability integrals are exactly the additive sequentially continuous mappings $L: \mathcal{M}(\mathbb{A}) \to [0,1]$ preserving order, top and bottom elements, called *linearizations*.

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