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Representations and dualities for bounded lattices

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Abstract

We provide a survey of different approaches to dual representations and dualities for bounded lattices. The case of distributive lattices is particularly well understood, with the duality theorems of both Stone (1937) and Priestley (1970) providing powerful tools for the study of distributive lattices. For the case of general bounded lattices, the theory is not as straightforward as in the distributive case. One of the main problems is that if the underlying set of the dual space uses maximal filters or ideals of the lattice, then the duals of lattice homomorphisms are no longer functions. There is variation amongst the different approaches as to whether they use one or two sets in the dual objects, whether or not they have additional non-topological structure, and also whether the construction uses a form of the Axiom of Choice. We compare the many different approaches that have been used since the late 1970s. We also point out the links between dual representations of lattices and the construction of the canonical extension—a lattice completion that has applications for non-classical logics.

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1 Introduction

Duality theory for distributive lattices was pioneered by Stone [48] in 1937. He showed that the category of distributive lattices and lattice homomorphisms is dually equivalent to the category of spectral spaces and spectral maps. Priestley's duality [42], rather than using a purely topological dual space, equips the dual space with a partial order. This paved the way for the later development of the theory of natural dualities for algebras [6] and structures [13]. In addition to having a partial order, the topology of a Priestley space is Hausdorff, making it similar to Stone duality for Boolean algebras [47]. We recall the basic definitions and construction of these two dualities for distributive lattices in Section 2.1.

While the distributive case provides dual representations that are quite natural and easy to apply, representation and duality theorems for arbitrary bounded lattices are more complicated.

There are two main approaches to duality for lattices with respect to the elements of the dual space. The first approach generalises Stone and Priestley duality by using *maximal* sets from the lattice as elements of the dual. Exactly what is meant by maximal sets will be explained in detail in each case. Examples of this include Urquhart [49] (Section 3), Hartung [30, 31] (Section 5) and Ploščica [41] (Section 6). By using maximality in the construction of the dual space, some of these representations will restrict to Stone or Priestley duality when the lattices considered are distributive.

However, such representations encounter difficulties when trying to dually represent lattice homomorphisms. The dual of a lattice homomorphism that is not surjective cannot be represented by a function and so relations must be used. Hartung [31] uses pairs of relations as the morphisms between dual spaces. Moreover, the composition of such pairs of relations is not the usual relational composition (see Section 5.3).

Allwein and Hartonas [2] (Section 4) fix the problem of dual representations of homomorphisms by using an enlarged dual space (i.e. without only maximal objects). Hartonas and Dunn [26] (Section 7) and Hartonas [27] (Section 8) also do not use any kind of maximality in their chosen lattice subsets that form the elements of the dual structure.

In Section 9, the duality of Moshier and Jipsen [37] takes an approach similar to that of Stone's duality for distributive lattices. The idea is to use a dual space that is purely topological. However, the underlying set of their dual space consists of all filters of the lattice **L** and hence their duality does not restrict to Stone's duality when the lattice is distributive. The duality of Celani and González [5] (Section 10) is similar in spirit to the Moshier–Jipsen duality in that the dual spaces are purely topological, but their choice of filters for the underlying set is different. The last duality that we cover is that of Gehrke and van Gool [22] (Section 11). It is quite different to the other constructions, in two ways. Firstly, their route to the dual spaces goes via so-called *adjoint pairs of distributive lattices* (see Definition 76). Secondly, their construction is designed to dually represent only admissible homomorphisms (see Definition 75).

For each of the representations or dualities in this survey, we present a sketch of how the dual spaces are constructed and also, if described, how the morphisms in the dual categories are defined. We hope that this will provide a useful reference for a reader wanting to work with dual representations or dualities.

Below we summarise the different representations and dualities covered in this survey. We briefly note the features of the dual spaces in terms of the following characteristics: how many sets are used; whether there is any additional structure (relations etc.); the dual representation of lattice homomorphisms (if any); whether the representation restricts to Stone or Priestley duality for the case of distributive lattices.

- Urquhart 1978 [49] (Section 3): one set with topology; two quasi-orders; only surjective homomorphisms can be dually characterised; restricts to Priestley duality.
- Allwein and Hartonas 1993 [2] (Section 4): one set with topology; two quasi-orders; lattice homomorphisms are dually represented as structure-preserving continuous functions; does not restrict to either Stone or Priestley duality.
- Hartung 1992 & 1993 [30, 31] (Section 5): two sets (each with topology); binary relation between the two sets; does not restrict to either Stone or Priestley duality.
- Ploščica 1995 [41] (Section 6): one set with topology; reflexive binary relation; no dual characterisation of homomorphisms is attempted; restricts to Priestley duality.
- Hartonas and Dunn 1997 [26] (Section 7): two sets (each with topology); binary relation between sets; duals of homomorphisms are pairs of functions; does not restrict to Stone or Priestley duality.
- Hartonas 1997 [27] (Section 8): one set with topology; a closure operator; dual of a lattice homomorphism is a single function; does not restrict to Stone or Priestley duality.

- Moshier and Jipsen 2014 [37] (Section 9): one set with topology; no additional structure; duals of lattice homomorphisms are functions; does not restrict to either Stone or Priestley duality.
- Celani and González 2020 [5] (Section 10): one set with topology; no additional structure; homomorphisms are dually represented by a single binary relation; restricts to Stone duality.
- Gehrke and van Gool 2014 [22] (Section 11): two sets (both with topology); binary relation between the sets; duals of admissible homomorphisms are pairs of functions; restricts to Priestley duality.

In general, whether the representation can restrict to Priestley or Stone duality will depend on whether a constructive or non-constructive approach is used for the underlying set of the dual space. For more about the constructive vs. non-constructive approaches, see the comments by Hartonas [27, Section 3.2] and Moshier and Jipsen [37, p.124].

Throughout this article we will be concerned with bounded lattices. In general, many of the results could be adapted to also work for lattices without a top or bottom element. For example, if one considers unbounded distributive lattices then the dual spaces will be so-called pointed Priestley spaces. We will write $\mathbf{L} = \langle L; \wedge, \vee, 0, 1\rangle$ for a bounded lattice and $\boldsymbol{\mathcal{L}}$ for the category of bounded lattices with bounded lattice homomorphisms. Whenever we write just $\mathbf{L}$ for the algebra, it should be assumed that the underlying set is L.

Maximal disjoint filter-ideal pairs were first used by Urquhart [49] in 1978 and similar objects have been used as the underlying set in two other representations that we include in this survey. Therefore, we devote Section 2.2 to exploring some properties of maximal disjoint filter-ideal pairs.

Using notation that is both consistent and meaningful has been a challenge. Wherever possible we aim to use the notation from the original papers so that the interested reader will have an easier transition should they choose to seek out the original source. We have however made small adaptations in order to make the notation within this article more consistent. In addition, since each section in essence presents a similar but different dual space of a bounded lattice, the same notation might occur in different sections but have a different meaning. This is unavoidable, as there are a limited number of fonts in which one can write a capital X. In general, $\mathcal{X}$ will denote a topological space with underlying set X and possibly some additional structure. Categories will be denoted by bold script letters and functors will be denoted by blackboard capitals.

For any order- or lattice-theoretic notions not defined here, we refer the reader to the book by Davey and Priestley [14].

2 Preliminaries

Although most readers are likely to be familiar with Priestley and Stone duality for distributive lattices, we find it useful to provide a brief outline of the constructions. This short summary partly serves to introduce notation, but also to make it easier for the reader to see the relationship between these two dualities and the various lattice representations and dualities that follow.

2.1 Dualities for distributive lattices

Priestley duality appeared in the seminal paper in 1970 [42]. Further properties of ordered topological spaces and applications of the duality appeared in a second paper in 1972 [43]. The presentation here is taken from the book by Davey and Priestley [14, Chapter 11] where the interested reader can find further details. As this is the first presentation of a duality, we will include more steps in the process than will be done for the representations and dualities presented in later sections.

Let $\mathbf{L}$ be a bounded distributive lattice and let $X = \mathsf{Idl_P}(\mathbf{L})$ its set of prime ideals. This set will be ordered by inclusion, and the goal is to equip that partially ordered set with a topology that will allow us to recover the lattice as the set of clopen down-sets. For $a \in L$, let $X_a = \{\, I \in \mathsf{Idl_P}(\mathbf{L}) \mid a \notin I \,\}$ and

$$S = \{\, X_b \mid b \in L \,\} \cup \{\, X \backslash X_c \mid c \in L \,\}.$$

Let τ be the topology for which S is a subbase and define $\mathbb{P}(\mathbf{L}) := (\mathsf{Idl_P}(\mathbf{L}), \subseteq, \tau)$.

Definition 1. Let $\mathcal{X} = (X, \leqslant, \tau)$ be a partially ordered set with a topology. We say that $\mathcal{X}$ is *totally order-disconnected* if whenever $y \nleqslant x$, there exists a clopen down-set U such that $x \in U$ and $y \notin U$. A compact totally order-disconnected space is known as a *Priestley space.*

Given a distributive lattice $\mathbf{L}$, the ordered topological space $\mathbb{P}(\mathbf{L})$ is a Priestley space.

Theorem 2 ([42])**.** *Let $\mathbf{L}$ be a bounded distributive lattice. The map*

$$\eta \colon a \mapsto X_a, \quad X_a = \{\, I \in \mathsf{Idl_P}(\mathbf{L}) \mid a \notin I \,\}$$

is an isomorphism from $\mathbf{L}$ onto the lattice of clopen down-sets of the dual space $\mathbb{P}(\mathbf{L}) = (\mathsf{Idl_P}(\mathbf{L}), \subseteq, \tau)$.

Two Priestley spaces are *order homeomorphic* if there exists a map between them which is both an order isomorphism and a homeomorphism.

Theorem 3 ([42])**.** *Let $\mathcal{X} = (X, \leqslant, \tau)$ be a Priestley space and let $\mathbb{L}(\mathcal{X})$ be the lattice of clopen down-sets of $\mathcal{X}$. Then $\mathcal{X}$ is order homeomorphic to $\mathbb{P}(\mathbb{L}(\mathcal{X}))$.*

In other sources, the presentation of Priestley duality might use prime filters rather than prime ideals. We find it appropriate to quote the following from the survey article by Priestley [44]: "The research literature concerning Priestley duality and its applications is divided roughly equally between these alternatives—a source of minor irritation."

To enable us to describe Stone's duality for distributive lattices, we first need some topological definitions. The *specialization order* on a topological space $\mathcal{X} = (X, \tau)$ is defined as follows:

$$x \sqsubseteq y \quad :\Longleftrightarrow \quad (\forall U \in \tau)(x \in U \Rightarrow y \in U) \quad \Longleftrightarrow \quad x \in \overline{\{y\}}.$$

This relation is always reflexive and transitive. Indeed, for any $x \in X$ we have ${\downarrow}x = \overline{\{x\}}$. The space $\mathcal{X}$ is T_0 if and only if $\sqsubseteq$ is a partial order.

Sober spaces are used by Stone [48] and will also be used in Section 9 and Section 11. Denote by $\mathcal{N}(x)$ the neighbourhood of open sets containing x. Then the space $\mathcal{X} = (X, \tau)$ is *sober* if the map $x \mapsto \mathcal{N}(x)$ is a bijection between X and the collection of completely prime filters in the lattice of opens. An equivalent definition can be given via irreducible sets. A set is called *irreducible* if $A \subseteq B \cup C$ implies $A \subseteq B$ or $A \subseteq C$. A space is sober if every closed irreducible set is of the form ${\downarrow}x$ for a unique point x.

Definition 4. A *spectral space* is a topological space $\mathcal{X} = (X, \tau)$ that is sober and in which the compact open sets form a base that is closed under finite intersections. A *spectral function* $f : \mathcal{X} \to \mathcal{Y}$ is a continuous function f such that for any compact open $U \subseteq Y$, $f^{-1}(U)$ is compact open in $\mathcal{X}$. Let $\mathcal{Spec}$ be the category of spectral spaces and spectral functions.

A significant difference between Priestley and Stone duality is that the topology of the Priestley dual of a lattice is T_2, whereas that of the Stone dual is not. This difference between T_2 topologies that are constructed in a 'two-sided' way and non-T_2 'one-sided' topologies will be also be seen in the representations of bounded lattices.

Let $\mathbf{L}$ be a bounded distributive lattice and, as before, consider the set $X = \mathsf{Idl}_\mathsf{P}(\mathbf{L})$. Now let σ be the topology generated by $\{ X_a \mid a \in X \}$. Notice how the topology here is one-sided, as opposed to the Priestley space topology above. Then $\mathbb{S}(\mathbf{L}) = (\mathsf{Idl}_\mathsf{P}(\mathbf{L}), \sigma)$ will be the Stone dual space of $\mathbf{L}$.

Given a spectral space $\mathscr{X} = (X, \sigma)$, the compact open subsets of X (ordered by inclusion) will form a distributive lattice. We denote this distributive lattice by $\mathbb{K}(\mathscr{X})$. For any distributive lattice $\mathbf{L}$ we have $\mathbf{L} \cong \mathbb{S}(\mathbb{K}(\mathbf{L}))$.

The definitions of $\mathbb{S}$ and $\mathbb{K}$ are extended in order to define functors $\mathbb{S}$ and $\mathbb{K}$ as follows. For $\mathbf{L} \in \mathcal{DL}$ let $\mathbb{S}(\mathbf{L}) = (\mathsf{Idl}_\mathsf{P}(\mathbf{L}), \sigma)$ and for $\mathbf{X} \in \mathcal{Spec}$, let $\mathbb{K}(\mathbf{X}) = (\{ Y \subseteq X \mid Y \text{ a compact open} \}, \cap, \cup, \emptyset, X)$. Similar to Priestley duality, for a bounded distributive lattice homomorphism $f : \mathbf{L}_1 \to \mathbf{L}_2$, let $\mathbb{S}(f) = f^{-1}$ and for $\varphi : \mathbf{X}_1 \to \mathbf{X}_2$, let $\mathbb{K}(\varphi) = \varphi^{-1}$.

Theorem 5 ([48])**.** *The functors $\mathbb{S}$ and $\mathbb{K}$ define a dual equivalence of categories between $\mathcal{DL}$ and $\mathcal{Spec}$.*

The link between Priestley and Stone duality (and explicit maps taking a Priestley space to a spectral space and vice versa) can be found in Chapter 6 of the forthcoming book by Gehrke and van Gool [23].

2.2 Maximal disjoint filter-ideal pairs

Any distributive lattice $\mathbf{L}$ has the property that for $x, y \in L$, if $x \nleq y$ then there exists a prime ideal I containing y but not x. Moreover, $L \backslash I = F$ will be a prime filter. However, when we do not require the lattice to be distributive, this no longer holds. In fact, the non-distributive (but modular) lattice M_3 has no prime ideals. However, a more general separation principle does exist. The definition below was first used by Urquhart [49].

Definition 6 ([49, Section 3])**.** Let $\mathbf{L}$ be a lattice. Then $\langle F, I \rangle$ is a *disjoint filter-ideal pair* of $\mathbf{L}$ if F is a filter of $\mathbf{L}$ and I is an ideal of $\mathbf{L}$ such that $F \cap I = \varnothing$. We say that a disjoint filter-ideal pair $\langle F, I \rangle$ is maximal if there is no disjoint filter-ideal pair $\langle G, J \rangle \neq \langle F, I \rangle$ such that $F \subseteq G$ and $I \subseteq J$. A maximal disjoint filter-ideal pair $\langle F, I \rangle$ of L is *total in* L if $F \cup I = L$.

We will often write MDFIP as an abbreviation for maximal disjoint filter-ideal pair. The proof of the following result makes use of Zorn's Lemma.

Lemma 7 ([49, Lemma 3])**.** *Let $\mathbf{L}$ be a lattice with $F \in \mathsf{Filt}(\mathbf{L})$ and $I \in \mathsf{Idl}(\mathbf{L})$ and $F \cap I = \emptyset$. There there exists $G \in \mathsf{Filt}(\mathbf{L})$ with $F \subseteq G$ and there exists $J \in \mathsf{Idl}(\mathbf{L})$ with $I \subseteq J$ such that $\langle G, J \rangle$ is an MDFIP.*

The above lemma can now be applied to the situation when $x \nleq y$. Clearly $\langle \uparrow x, \downarrow y \rangle$ will be a disjoint filter-ideal pair. Using Lemma 7 this can be extended to a maximal disjoint filter-ideal pair $\langle F, I \rangle$ such that $x \in F$, $y \in I$ and $F \cap I = \emptyset$.

MDFIPs play an important role in the representation theorem of Urquhart, and related notions are used by both Hartung (Section 5) and Ploščica (Section 6). Therefore, below we collect some useful facts about MDFIPs.

For finite lattices every filter is the up-set of a unique element and every ideal is the down-set of a unique element, so we can represent every disjoint filter-ideal pair $\langle F, I \rangle$ by an ordered pair $\langle \uparrow x, \downarrow y \rangle$ where $x = \bigwedge F$, $y = \bigvee I$ and $x \nleq y$.

For a lattice $\mathbf{L}$ we denote by $\mathsf{J}(\mathbf{L})$ the set of join-irreducible elements of $\mathbf{L}$ and by $\mathsf{M}(\mathbf{L})$ the meet-irreducible elements of $\mathbf{L}$. (In Section 12 we will also refer to *completely join-irreducibles*, denoted $J^{\infty}(\mathbf{L})$, and *completely meet-irreducibles*, denoted $M^{\infty}(\mathbf{L})$.) The fact below is an observation by Urquhart.

Proposition 8 ([49, p. 52])**.** *Let $\mathbf{L}$ be a finite lattice. If $\langle F, I\rangle$ is a maximal disjoint ideal-filter pair of $\mathbf{L}$ then $\bigwedge F$ is join-irreducible and $\bigvee I$ is meet-irreducible.*

Some of the equivalences of Proposition 9 below come from the paper by Gaskill and Nation [19, p. 353]. This result reveals some important features of MDFIPs.

Proposition 9 ([11, Proposition 2.3])**.** *Let $\mathbf{L}$ be a finite lattice and $\langle F, I\rangle$ be a maximal disjoint filter-ideal pair of $\mathbf{L}$. Then the following are equivalent:*

(i) *$\bigwedge F$ is join-prime;*

(ii) *$\bigvee I$ is meet-prime;*

(iii) *$F \cup I = L$;*

(iv) *F is a prime filter;*

(v) *I is a prime ideal.*

The equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) *hold even when $\mathbf{L}$ is not finite.*

3 Representation via doubly-ordered spaces

In 1978, Urquhart [49] provided a representation of arbitrary lattices via so-called *L-spaces*. The goal was to generalise Priestley duality for distributive lattices [42]. In particular, this meant that the underlying set of the dual space of $\mathbf{L}$ would be the MDFIPs of $\mathbf{L}$.

Definition 10. A triple $(X, \leqslant_1, \leqslant_2)$ is a *doubly-ordered set* if $\leqslant_1$ and $\leqslant_2$ are quasi-orders and $x \leqslant_1 y$ and $x \leqslant_2 y$ implies that $x = y$. A *doubly-ordered space* is a structure of the form $\mathcal{X} = (X, \leqslant_1, \leqslant_2, \tau)$ where $(X, \leqslant_1, \leqslant_2)$ is a doubly-ordered set and τ is a topology on X.

Urquhart defines the following operations on subsets of a doubly-ordered space, but of course they can be defined on a doubly-ordered set (see Section 12 for an application of this in the untopologised setting). Urquhart used the term *stable set* for what we call below an ℓ-stable set. The terms for the two types of stability defined below were first used by Allwein and Hartonas [2].

Definition 11. Let $(X, \leqslant_1, \leqslant_2)$ be a doubly-ordered set. For $Y \subseteq X$, define

$$\ell(Y) = \{\, x \in X \mid (\forall y \in Y)(x \nleqslant_1 y)\,\},$$
$$r(Y) = \{\, x \in X \mid (\forall y \in Y)(x \nleqslant_2 y)\,\}.$$

A subset $Y \subseteq X$ is called *ℓ-stable* if $\ell r(Y) = Y$ and *r-stable* if $r\ell(Y) = Y$.

If $(X, \leqslant_1, \leqslant_2)$ is a doubly-ordered set, then both ℓr and $r\ell$ define closure operators on X. Hence the collection of ℓ-stable subsets of a doubly-ordered set will form a complete lattice. On its own, this set-up would provide, for any lattice $\mathbf{L}$, a complete lattice into which $\mathbf{L}$ could be embedded. It will be the addition of a compact topology that will allow us to obtain a representation of the lattice itself.

Definition 12. If Y is a subset of a doubly-ordered space $\mathcal{X} = (X, \leqslant_1, \leqslant_2, \tau)$, then Y is *doubly closed* if both Y and $r(Y)$ are closed. A doubly-ordered space $\mathcal{X}$ is *doubly disconnected* if the following two implications hold for any $x, y \in X$:

$$x \nleqslant_1 y \implies (\exists Y \subseteq X)(Y = \ell r(Y) \text{ and } Y \text{ doubly closed with } x \in Y, y \notin Y)$$
$$x \nleqslant_2 y \implies (\exists Y \subseteq X)(Y = \ell r(Y) \text{ and } Y \text{ doubly closed with } x \in r(Y), y \notin r(Y))$$

Definition 13. A doubly-ordered space $\mathcal{S} = (X, \leqslant_1, \leqslant_2, \tau)$ is an *L-space* if

(i) $\mathcal{S}$ is a doubly-disconnected compact space,

(ii) whenever Y and Z are doubly-closed stable subsets of X, then both $r(Y \cap Z)$ and $\ell(r(Y) \cap r(Z))$ are closed in $\mathcal{S}$,

(iii) the family

$$\{ X \backslash Y \mid Y \text{ a doubly-closed stable set} \} \cup \{ X \backslash r(Y) \mid Y \text{ a doubly-closed stable set} \}$$

forms a subbase for τ.

Given an L-space as defined above, the family of doubly-closed ℓ-stable sets will form a lattice. For two doubly-closed ℓ-stable subsets Y and Z, we get

$$Y \wedge Z = Y \cap Z \quad \text{and} \quad Y \vee Z = \ell(r(Y) \cap r(Z)) = \ell r(Y \cup Z).$$

For an L-space $\mathcal{S} = (X, \leqslant_1, \leqslant_2, \tau)$, the dual lattice of $\mathcal{S}$ is given by:

$$\mathbb{L}(\mathcal{S}) = (\{ Y \subseteq X \mid Y = \ell r(Y), X \backslash Y \in \tau, X \backslash r(Y) \in \tau \}, \wedge, \vee, \emptyset, X).$$

Given a lattice $\mathbf{L}$, the dual space of $\mathbf{L}$ is formed by taking the maximal disjoint filter-ideal pairs of $\mathbf{L}$, denoted $X_{\mathbf{L}}$, as the underlying set, and equipping the set with two quasi-orders and a topology. We formalise this below.

Definition 14. Let $\mathbf{L}$ be a lattice and $X_{\mathbf{L}}$ the set of MDFIPs of $\mathbf{L}$. For $\langle F, I \rangle$ and $\langle G, J \rangle$ in $X_{\mathbf{L}}$ we define $\langle F, I \rangle \leqslant_1 \langle G, J \rangle$ iff $F \subseteq G$, and $\langle F, I \rangle \leqslant_2 \langle G, J \rangle$ iff $I \subseteq J$. For $a \in L$, let $u(a) = \{ \langle F, I \rangle \in X_{\mathbf{L}} \mid a \in F \}$. Define a topology τ on $X_{\mathbf{L}}$ by letting the family

$$\{ X_{\mathbf{L}} \backslash u(a) \mid a \in L \} \cup \{ X_{\mathbf{L}} \backslash r(u(a)) \mid a \in L \}$$

form a subbase for τ. The dual space of $\mathbf{L}$ is given by $\mathbb{S}(L) = (X_{\mathbf{L}}, \leqslant_1, \leqslant_2, \tau)$.

As in the case of Priestley duality, in the lemma below the proof of the compactness of the space is shown using Alexander's Subbase Lemma.

Lemma 15 ([49, Lemma 6])**.** *Let* $\mathbf{L}$ *be a lattice. Then* $\mathbb{S}(\mathbf{L})$ *is an L-space.*

Before giving the statement of Urquhart's representation, we note that a homeomorphism between between two doubly-ordered spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ is a function that is a topological homeomorphism and also an isomorphism with respect to $\leqslant_1$ and $\leqslant_2$.

Theorem 16 ([49, Theorems 1 and 2])**.** *For* $\mathbf{L}$ *a bounded lattice and* $a \in L$*, let* $u(a) = \{ \langle F, I \rangle \in X_{\mathbf{L}} \mid a \in F \}$*. Then* $\mathbf{L} \cong \mathbb{L}(\mathbb{S}(\mathbf{L}))$ *via the isomorphism* $a \mapsto u(a)$*.*

If $\mathcal{S}$ *is an L-space, then* $\mathcal{S}$ *is order-homeomorphic to* $\mathbb{S}(\mathbb{L}(\mathcal{S}))$*.*

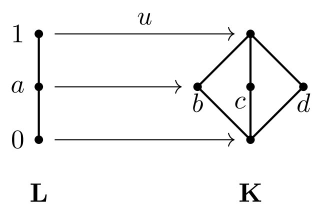

Figure 1. A non-surjective homomorphism whose codomain is a non-distributive lattice.

The order-homeomorphism in the above theorem is given by

$$x \mapsto \langle \{Y \in \mathbb{L}(\mathcal{S}) \mid x \in Y\}, \{Y \in \mathbb{L}(\mathcal{S}) \mid x \in r(Y)\}\rangle.$$

In Section 4 we will describe the difficulties with representing homomorphisms in Urquhart's setting. Before we do this, we note that Urquhart finds a number of different applications for this representation. Firstly, it is used to study congruences of lattices and representations of lattices of finite length (where topology is not required). In addition, the dual space of the lattices of bracketings of a finite set (i.e. the lattices T_n, as described by Friedman and Tamari [17]) are calculated. The final section of [49] considers congruence lattices ($\mathsf{Con}(\mathbf{L})$) and automorphism groups ($\mathsf{Aut}(\mathbf{L})$) of lattices. That section culminates in the theorem below.

Theorem 17 ([49, Theorem 13])**.** *Let* $\mathbf{D}$ *be a finite non-trivial distributive lattice and* $\mathbf{G}$ *a finite non-trivial group. Then there exists a finite lattice* $\mathbf{L}$ *such that* $\mathsf{Con}(\mathbf{L}) \cong \mathbf{D}$ *and* $\mathsf{Aut}(\mathbf{L}) \cong \mathbf{G}$.

Not only was the representation theorem applied to provide proofs for results such as Theorem 17, but the representation has subsequently been used to obtain Kripke-style semantics for non-distributive logics. Section 13 contains references to some applications of Urquhart's representation to non-classical logics.

4 Modifying Urquhart's representation to obtain a duality

Consider the following situation regarding a non-surjective homomorphism between two lattices, one of which is non-distributive. Let $\mathbf{L}$ be the three-element chain with $0 < a < 1$ and let $\mathbf{K}$ be M_3, the five-element modular (but non-distributive) lattice with atoms b, c and d. These two lattices and the non-surjective homomorphism u are illustrated in Figure 1.

Clearly $\langle {\uparrow}c, {\downarrow}d\rangle$ is an MDFIP of $\mathbf{K}$. If we then consider $u^{-1}(\langle {\uparrow}c, {\downarrow}d\rangle)$, we get the disjoint filter-ideal pair $\langle u^{-1}({\uparrow}c), u^{-1}({\downarrow}d)\rangle = \langle \{1\}, \{0\}\rangle$, which is *not* an MDFIP of $\mathbf{L}$. This example illustrates how the approach of dualising homomorphisms between lattices to morphisms between their dual spaces by taking their inverse maps will not always work for non-surjective lattice homomorphisms. This observation was made by Urquhart [49, p. 51], after which he does not further explore expanding his representation to a duality between categories. Urquhart's representation would however extend to a duality between bounded lattices with surjective bounded lattice homomorphisms and L-spaces with order homeomorphisms.

The work of Allwein and Hartonas [2] was aimed at remedying this problem. This paper is not as well known as it might be, probably due to the fact that it appeared as a technical report and not in a journal.

The approach of Allwein and Hartonas proceeded as follows: instead of taking the *maximal* disjoint filter-ideal pairs as the elements of the dual space of a lattice, they take *all* disjoint filter-ideal pairs of **L**. As with Urquhart, this set is then equipped with two quasi-orders and a topology. A full duality theorem is then achieved. However, it comes at a price. For the finite lattice M_3 (**K** in Figure 1), the dual space becomes considerably larger with this new approach. While the set of MDFIPs of M_3 has six elements, the set of all disjoint filter-ideal pairs will have 13 elements. Further, the definition of the dual spaces is more complicated than those we have just seen in Section 3.

We note that the definition of a doubly-ordered set, the maps ℓ and r, as well as the definition of ℓ-stable and r-stable subsets of a doubly-ordered set remain the same as in Section 3.

For a bounded lattice **L**, we denote by $Y_{\mathbf{L}}$ the set of all disjoint filter-ideal pairs. The quasi-orders $\leqslant_1$ and $\leqslant_2$ are defined for elements of $Y_{\mathbf{L}}$ as set containment on the filters and set containment on the ideals, respectively.

An important difference between using MDFIPs and disjoint filter-ideal pairs is seen in the following lemma. The result is clear when remembering that an arbitrary intersection of filters (ideals) is again a filter (ideal).

Lemma 18. *Consider* $(Y_{\mathbf{L}}, \leqslant_1, \leqslant_2)$ *for a bounded lattice* **L**. *For* $y, z \in Y_{\mathbf{L}}$, *define* $y \sim_1 z$ *iff* $y \leqslant_1 z$ *and* $z \leqslant_1 y$ *(and similarly for* $\sim_2$*). Then each of the partially ordered sets* $(Y_{\mathbf{L}})/\sim_1$ *and* $(Y_{\mathbf{L}})/\sim_2$ *are complete meet semilattices.*

In general, a doubly-ordered set $(X, \leqslant_1, \leqslant_2)$ is called a *complete doubly-ordered set* if both $X/\sim_1$ and $X/\sim_2$ are complete meet semilattices. This extra feature of certain doubly-ordered sets is often used by Allwein and Hartonas in [2].

Definition 19 ([2, Definition 2.3])**.** Let $(X, \leqslant_1, \leqslant_2)$ be a complete doubly-ordered set. If $A \subseteq X$ is a $\leqslant_1$ up-set then it is $\bigwedge_1$*-closed* if $A = \emptyset$ or $A = \uparrow_1 x$ for some $x \in X$. Similarly, a $\leqslant_2$ up-set is $\bigwedge_2$*-closed* if $B = \emptyset$ or $B = \uparrow_2 y$ for some $y \in X$. If A is a $\leqslant_1$ up-set, then we say that A is $\bigwedge$*-closed* if A is $\bigwedge_1$-closed and $r(A)$ is $\bigwedge_2$-closed.

Definition 20 ([2, Definition 2.5])**.** Let $\mathcal{X} = (X, \leqslant_1, \leqslant_2, \tau)$ be a complete doubly-ordered space. A subset $C \subseteq X$ is called *strongly double-closed* if

(i) C and $r(C)$ are closed in the topology τ;

(ii) C is $\bigwedge$-closed.

We denote by $\mathcal{X}^*$ the set of strongly double-closed ℓ-stable subsets of X. These sets will play the same role as the doubly-closed ℓ-stable sets in Urquhart's representation.

Before we are able to define the class of doubly-ordered spaces that will be dual to the class of bounded lattices, we need two more definitions. Below we write $[x]_1$ for the equivalence class of x with respect to the quasi-order $\leqslant_1$ (and similarly for $\leqslant_2$).

Definition 21 ([2, Definitions 2.6 and 2.7])**.** A doubly-ordered set $\mathcal{X}$ is *doubly-order separated* if

(i) $x \nleqslant_1 y \quad \Longrightarrow \quad (\exists C \in \mathcal{X}^*)(x \in C, y \notin C)$;

(ii) $x \nleqslant_2 y \quad \Longrightarrow \quad (\exists C \in \mathcal{X}^*)(x \in r(C), y \notin r(C))$.

Further, $\mathcal{X}$ is *coupled* if whenever $x \leqslant_1 z$ and $y \leqslant_2 z$, then $[x]_1 \cap [y]_2 \neq \emptyset$.

We are now ready to give the definition of the doubly-ordered spaces that will turn out to be the objects of the dual category.

Definition 22 ([2, Definition 2.8])**.** An *enhanced L-space* (EL-space for short) is a complete doubly-ordered space $\mathcal{X} = (X, \leqslant_1, \leqslant_2, \tau)$ where

(i) $\mathcal{X}$ is doubly order-separated, coupled and compact;

(ii) $\mathcal{X}^* \cup \{ r(U) \mid U \in \mathcal{X}^* \}$ is a subbasis for the closed sets of τ;

(iii) if $C, D \in \mathcal{X}^*$, then $C \cap D \in \mathcal{X}^*$ and $\ell(r(C) \cap r(D)) = \ell r(C \cup D) \in \mathcal{X}^*$.

Given an EL-space $\mathcal{X}$, its dual lattice will be the set $\mathcal{X}^*$, ordered by inclusion and with meets and joins give as in item (iii) above. That is $\mathbb{L}(\mathcal{X}) = (\mathcal{X}^*, \cap, \vee, \emptyset, X)$.

Similar to the Urquhart representation of a bounded lattice $\mathbf{L}$, for $a \in L$ we have $u(a) = \{ \langle F, I \rangle \in Y_{\mathbf{L}} \mid x \in F \}$. As we have hinted above, for a bounded lattice $\mathbf{L}$, its dual space will be

$$\mathbb{E}(\mathbf{L}) = (Y_{\mathbf{L}}, \leqslant_1, \leqslant_2, \tau)$$

with τ generated by the subbase

$$\{ Y_{\mathbf{L}} \backslash u(a) \mid a \in L \} \cup \{ Y_{\mathbf{L}} \backslash r(u(a)) \mid a \in L \}.$$

Lemma 23 ([2, Lemma 3.12 and Theorem 3.18])**.** *Let $\mathbf{L}$ be a bounded lattice. Then $\mathbb{E}(\mathbf{L})$ is an EL-space and $u : a \mapsto u(a)$ is an isomorphism from $\mathbf{L}$ onto $\mathbb{L}(\mathbb{E}(\mathbf{L}))$.*

The definition of a morphism in the category of enhanced L-spaces appears below. Unsurprisingly, a morphism is required to be continuous and to preserve the two quasi-orders. It is the third condition that is crucial for the duality theorem to work. In particular, it is needed to show that the inverse of a morphism $f : \mathcal{X} \to \mathcal{Y}$ takes an element of $\mathcal{Y}^*$ to an element of $\mathcal{X}^*$.

Definition 24 ([2, Definition 2.9])**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be EL-spaces. A map $f : \mathcal{X} \to \mathcal{Y}$ is an *EL-space morphism* if

(i) f preserves $\leqslant_1$ and $\leqslant_2$;

(ii) f is continuous;

(iii) f^{-1} commutes with ℓ and r on subbasis elements.

The class of EL-spaces with EL-space morphisms, defines a category, denoted $\boldsymbol{\mathcal{E}}$. If a bijective EL-space morphism exists between $\mathcal{X}, \mathcal{Y} \in \boldsymbol{\mathcal{E}}$, then we say that $\mathcal{X}$ and $\mathcal{Y}$ are order-homeomorphic. If $\mathcal{X}$ is an EL-space, then $\mathcal{X}$ is order homeomorphic to $\mathbb{E}(\mathbb{L}(\mathcal{X}))$.

The theorem below shows that the dual of a bounded lattice homomorphism is an EL-space morphism. Besides the more technical conditions that need to be satisfied, the key fact is that if $h : \mathbf{L} \to \mathbf{K}$ is a bounded lattice homomorphism, and $\langle F, I \rangle$ is a disjoint filter-ideal pair of $\mathbf{K}$, then $\langle h^{-1}(F), h^{-1}(I) \rangle$ is a disjoint filter-ideal pair of $\mathbf{L}$.

Theorem 25 ([2, Theorem 3.22])**.** *Let $h : \mathbf{L} \to \mathbf{K}$ be a bounded lattice homomorphism. Then $\mathbb{E}(h) : \mathbb{E}(\mathbf{K}) \to \mathbb{E}(\mathbf{L})$, defined for $\langle F, I \rangle \in Y_{\mathbf{L}}$ by $\mathbb{E}(h)(\langle F, I \rangle) = h^{-1}(\langle F, I \rangle) = \langle h^{-1}(F), h^{-1}(I) \rangle$, is an EL-space morphism.*

The final step on the path to a duality theorem is a result [2, Theorem 3.23] which shows that if $f : \mathcal{X} \to \mathcal{Y}$ is an EL-space morphism, then by defining $\mathbb{L}(f) = f^{-1}$ one gets a bounded lattice homomorphism. Checking that these functors interact correctly with the identity and composition yields the final theorem.

Theorem 26 ([2, Theorem 3.24])**.** *The categories $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{E}}$ are dually equivalent to one another via the functors $\mathbb{E}$ and $\mathbb{L}$.*

5 Representation and duality via topologised formal contexts

The main source for this section are two papers by Hartung [30, 31]. The first of these papers concerns the basic representation while the second extends the representation to a duality by describing the representation of morphisms in the dual category. There are two main features of Hartung's construction. The first is the fact that, like Urquhart [49], Hartung uses filters and ideals that are maximal in being disjoint from one another. The second is that the morphisms proposed in the second paper are not functions. They are instead pairs of so-called multivalued functions between the dual spaces.

5.1 Formal Concept Analysis

The study of Formal Concept Analysis (FCA) was initiated by Wille [50] and has been studied by many authors since. The book by Ganter and Wille [18] provides comprehensive coverage of the topic. Here we recall the basic facts which are needed in order to understand the representation of Hartung.

Definition 27 ([18, Definition 18])**.** A *formal context* is a triple $\mathcal{K} = (G, M, I)$ consisting of two sets, G and M, and a binary relation $I \subseteq G \times M$. The elements of G are called the *objects* and the elements of M are called the *attributes*. The expression gIm, or $(g, m) \in I$ says that "object g has attribute m".

Given a formal context $\mathcal{K} = (G, M, I)$ we have the following operations on subsets of G and M. If $A \subseteq G$, then

$$A' = \{\, m \in M \mid (\forall g \in A)(gIm) \,\}.$$

For $B \subseteq M$,

$$B' = \{\, g \in G \mid (\forall m \in B)(gIm) \,\}.$$

The first of these sets, A', can be considered as the set of attributes held by all objects in A, while the second set, B', is thought of as the set of objects that have all of the attributes in B. It is usual to write g' rather than $\{g\}'$ (and similarly m' rather than $\{m\}'$).

Definition 28 ([18, Definition 20])**.** If $\mathcal{K} = (G, M, I)$ is a formal context, then a *formal concept* of $\mathcal{K}$ is a pair (A, B) where $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. The set A is known as the *extent*, and B the *intent* of the concept (A, B). We denote by $\mathfrak{B}(G, M, I)$ the set of all concepts of the context (G, M, I).

Notice that if (A, B) is a formal concept then $A = A''$ and $B = B''$. It is not difficult to show that $' : (\wp(G), \subseteq) \to (\wp(M), \subseteq)$ is order-reversing, as is $' : (\wp(M), \subseteq) \to (\wp(G), \subseteq)$, and that these two maps form a Galois connection. Therefore the maps $A \mapsto A''$ and $B \mapsto B''$ are closure operators on $\wp(G)$ and $\wp(M)$ respectively.

We require two main results from FCA. The first is that for any formal context, the formal concepts form a complete lattice. The theorem below is often called the "Basic Theorem on Concept Lattices".

Theorem 29 ([18, Theorem 3])**.** *The set of concepts $\mathfrak{B}(G, M, I)$, ordered by $(A, B) \leqslant (C, D)$ iff $A \subseteq C$ iff $D \subseteq B$, is a complete lattice in which the greatest lower bound and least upper bound of a set of concepts $\{(A_t, B_t) \mid t \in T\}$ are given by:*

$$\bigwedge_{t\in T}(A_t, B_t) = \left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)''\right) \qquad \bigvee_{t\in T}(A_t, B_t) = \left(\left(\bigcup_{t\in T} A_t\right)'', \bigcap_{t\in T} B_t\right).$$

The set of concepts with the lattice structure given by the above theorem is denoted by $\underline{\mathfrak{B}}(G, M, I)$. In the definition below, we note that the term *purified* is sometimes used instead of *clarified*.

Definition 30 ([18, Definitions 23 and 24])**.** A context $\mathcal{K} = (G, M, I)$ is *clarified* if for $g, h \in G$, whenever $g' = h'$ then $g = h$ and for $m, n \in M$, if $m' = n'$ then $m = n$. A clarified context is *reduced* if for each $g \in G$, the concept (g'', g') is completely join-irreducible and for each $m \in M$, the concept (m', m'') is completely meet-irreducible.

Before presenting the next proposition, we remark that an isomorphism between formal contexts (G_1, M_1, I_1) and (G_2, M_2, I_2) is a pair of bijective maps $\alpha : G_1 \to G_2$ and $\beta : M_1 \to M_2$ such that gI_1m iff $(\alpha g)I_2(\beta m)$ for all $g \in G_1$ and $m \in M_1$.

Proposition 31 ([18, Proposition 12])**.** *Let* $\mathbf{L}$ *be a finite lattice. There is, up to isomorphism, a unique reduced context* $\mathbb{K}(\mathbf{L})$ *such that* $\mathbf{L} \cong \underline{\mathfrak{B}}(\mathbb{K}(\mathbf{L}))$*. It is given by:*

$$\mathbb{K}(\mathbf{L}) = (\mathsf{J}(\mathbf{L}), \mathsf{M}(\mathbf{L}), \leqslant).$$

The context in the above proposition is referred to as the *standard context* of the lattice $\mathbf{L}$. It is the above proposition that was generalised by Hartung to the case of not necessarily finite lattices. We provide the details in the next subsection.

5.2 Representation via topological contexts

The way in which the standard context of a finite lattice will be generalised is by making use of a set of filters and a set of ideals. It could be useful for the reader to revisit the connection between MDFIPs of a finite lattice and join- and meet-irreducible elements given by Proposition 8. The starting point for Hartung's representation is to consider as separate sets those filters which are part of an MDFIP (we denote this set by $\mathsf{Filt}_{\mathsf{M}}(\mathbf{L})$), and the ideals that are part of some MDFIP (denoted by $\mathsf{Idl}_{\mathsf{M}}(\mathbf{L})$).

The key to the generalisation of the finite case is not only the use of different sets, but also the addition of topological structure to each of the sets. We note that Hartung defines the topological spaces in terms of closed sets.

Definition 32 ([30, Definition 2.1.1])**.** A structure $\mathcal{K}^\tau = ((G, \rho), (M, \sigma), I)$ is a *topological context* if

(i) (G, ρ) and (M, σ) are topological spaces and $I \subseteq G \times M$;

(ii) if $A \in \rho$ then $A'' \in \rho$ and if $B \in \sigma$ then $B'' \in \sigma$;

(iii) $\{\, A \in \rho \mid A = A'' \text{ and } A' \in \sigma \,\}$ is a subbase of closed sets for ρ and the collection $\{\, B \in \sigma \mid B = B'' \text{ and } B' \in \rho \,\}$ is a subbase of closed sets for σ.

Given a topological context $\mathcal{K}^\tau = ((G, \rho), (M, \sigma), I)$, the set

$$\mathfrak{B}^\tau(\mathcal{K}^\tau) = \{\, (A, B) \in \mathfrak{B}(\mathcal{K}^\tau) \mid A \in \rho, B \in \sigma \,\}$$

is called the set of *closed concepts* of $\mathcal{K}^\tau$. This set, with the order inherited from the concept lattice, forms a lattice which is denoted $\underline{\mathfrak{B}}^\tau(\mathcal{K}^\tau)$.

The next definition gives us the structure that will be the dual space of a bounded lattice $\mathbf{L}$.

Definition 33 ([30, Definition 2.1.6])**.** Let $\mathbf{L}$ be a bounded lattice. Define

$$\mathbb{K}^\tau(\mathbf{L}) := ((\mathsf{Filt}_{\mathsf{M}}(\mathbf{L}), \rho), (\mathsf{Idl}_{\mathsf{M}}(\mathbf{L}), \sigma), \Delta)\,.$$

The relation Δ and the topologies ρ and σ are defined as follows:

(i) $F \Delta I$ if and only if $F \cap I \neq \emptyset$;

(ii) $\mathscr{B}_\rho = \{ F_a \mid a \in L \}$ is a subbasis for ρ, where $F_a = \{ F \in \mathsf{Filt_M}(\mathbf{L}) \mid a \in F \}$;

(iii) $\mathscr{B}_\sigma = \{ I_a \mid a \in L \}$ is a subbasis for σ, where $I_a = \{ I \in \mathsf{Idl_M}(\mathbf{L}) \mid a \in I \}$.

The structure defined above meets the exact requirements of Definition 32.

Proposition 34 ([30, Proposition 2.1.7])**.** *Let* $\mathbf{L}$ *be a bounded lattice. Then the context* $\mathbb{K}^\tau(\mathbf{L})$ *is a topological context.*

The representation of a bounded lattice $\mathbf{L}$ is then given by the closed concepts of the topological context $\mathbb{K}^\tau(\mathbf{L})$. The sets F_a and I_a were used above in Definition 33.

Theorem 35 ([30, Theorem 2.1.8])**.** *Let* $\mathbf{L}$ *be a bounded lattice. Then the map* ι*, defined by* $\iota(a) = (F_a, I_a)$*, is a lattice isomorphism from* $\mathbf{L}$ *to* $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(\mathbf{L}))$.

We note that in the definition below, $(\rho \times \sigma)|_{I^\complement}$ is the product topology restricted to the set $(G \times M)\backslash I$. Further, in the original paper, Hartung uses the term *quasicompact* in item (iii) to emphasize the fact that the space is not necessarily T_2.

Definition 36 ([30, Definition 2.2.3])**.** A topological context $\mathscr{K}^\tau$ is called a *standard topological context* if

(i) $\mathscr{K}^\tau$ is reduced;

(ii) if gIm then there exists a closed concept $(A, B) \in \underline{\mathfrak{B}}^\tau(\mathscr{K}^\tau)$ with $g \in A$ and $m \in B$;

(iii) $\left(I^\complement, (\rho \times \sigma)|_{I^\complement}\right)$ is compact.

Before stating the final representation theorem, we recall that an isomorphism between formal contexts was defined at the end of Section 5.1. In order to be an isomorphism between *topological contexts*, the maps α and β must of course be homeomorphisms as well.

Theorem 37 ([30, Theorem 2.2.5])**.** *Let* $\mathbf{L}$ *be a bounded lattice. Then* $\mathbb{K}^\tau(\mathbf{L})$ *is a standard topological context and* $\mathbf{L}$ *is isomorphic to* $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(\mathbf{L}))$*. Further, given any standard topological context* $\mathscr{K}^\tau$*, the ordered set* $\underline{\mathfrak{B}}^\tau(\mathscr{K}^\tau)$ *is a bounded lattice and* $\mathscr{K}^\tau \cong \mathbb{K}^\tau(\underline{\mathfrak{B}}^\tau(\mathscr{K}^\tau))$.

We observe that later in the paper (see [30, Duality Theorem 3.2.3]), Hartung demonstrates that there is a dual equivalence between the category of bounded lattices with *surjective* homomorphisms and the category of standard topological contexts with standard embeddings (see [30, Definition 3.1.4]). The standard embeddings are pairs of *functions* satisfying certain conditions. This approach was modified in the subsequent paper [31]. We expand on this modification below.

5.3 Morphisms for topological contexts

The work of Allwein and Hartonas (see Section 4) solved the problem of dual representation of lattice homomorphisms by enlarging the dual space by taking all disjoint filter-ideal pairs as elements of the dual space. Hartung [31] took a different approach, instead opting to drop the requirement that the duals of lattice homomorphisms be functions.

A multivalued function $F : X \to Y$ is a binary relation $F \subseteq X \times Y$ such that $\pi_1(F) = X$ (where π_1 is the projection in the first coordinate). For $x \in X$ we have $F(x) = \{\, y \in Y \mid (x, y) \in F \,\}$. For $A \subseteq X$ and $B \subseteq Y$, Hartung further defines:

$$\begin{aligned} F(A) &= \{\, y \in Y \mid (\exists\, a \in A)((a, y) \in F) \,\} \\ F^{-1}(B) &= \{\, x \in X \mid (\exists\, b \in B)((x, b) \in F) \,\} \\ F^{[-1]}(B) &= \{\, x \in X \mid F(x) \subseteq B \,\}. \end{aligned}$$

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be bounded lattices with $\mathbb{K}^\tau(\mathbf{L}_1)$ and $\mathbb{K}^\tau(\mathbf{L}_2)$ their standard topological contexts. If $f : \mathbf{L}_1 \to \mathbf{L}_2$ is a bounded lattice homomorphism, then two multivalued functions will be defined. The two multivalued functions are as follows: $R_f \subseteq \mathsf{Filt}_\mathsf{M}(\mathbf{L}_2) \times \mathsf{Filt}_\mathsf{M}(\mathbf{L}_1)$ and $S_f \subseteq \mathsf{Idl}_\mathsf{M}(\mathbf{L}_2) \times \mathsf{Idl}_\mathsf{M}(\mathbf{L}_1)$. They are defined by:

$$\begin{aligned} (F_2, F_1) \in R_f &\iff F_1 \supseteq f^{-1}(F_2), \\ (I_2, I_1) \in S_f &\iff I_1 \supseteq f^{-1}(I_2). \end{aligned}$$

Definition 38 ([31, Definition 1]). Let $\mathcal{K}_1^\tau$ and $\mathcal{K}_2^\tau$ be standard topological contexts. A pair of multivalued functions $(R, S) : \mathcal{K}_1^\tau \to \mathcal{K}_2^\tau$ is a *multivalued standard morphism* if $R \subseteq G_1 \times G_2$ and $S \subseteq M_1 \times M_2$ are multivalued functions satisfying the following:

(i) $(R^{[-1]}(A), S^{[-1]}(B)) \in \underline{\mathfrak{B}}^\tau(\mathcal{K}_1^\tau)$ for all $(A, B) \in \underline{\mathfrak{B}}^\tau(\mathcal{K}_2^\tau)$

(ii) $R(g) = R(g'') = \overline{R(g)}$ for every $g \in G_1$ and $S(m) = S(m'') = \overline{S(m)}$ for every $m \in M_1$.

The map $\mathbb{K}^\tau$ can now be extended to a functor from $\mathcal{L}$ to the category $\mathcal{C}$ of standard topological contexts with multivalued standard morphisms. For a lattice homomorphism $f : \mathbf{L}_1 \to \mathbf{L}_2$, we set $\mathbb{K}^\tau(f) = (R_f, S_f)$.

Proposition 39 ([31, Proposition 3]). *Let (R, S) be a multivalued standard morphism between two standard topological contexts $\mathcal{K}_1^\tau$ and $\mathcal{K}_2^\tau$. Then the map*

$$f_{RS} : \underline{\mathfrak{B}}^\tau(\mathcal{K}_2^\tau) \to \underline{\mathfrak{B}}^\tau(\mathcal{K}_1^\tau), \qquad (A, B) \mapsto (R^{[-1]}(A), S^{[-1]}(B))$$

is a bounded lattice homomorphism.

Having defined these morphisms, one challenge still remains: to correctly define the composition of two multivalued standard morphisms such that the result is again a multivalued standard morphism. Hartung [31] gives an example where the usual relational composition fails. The key insight is that the composition of two multivalued standard morphisms must be defined as follows:

Definition 40 ([31, Definition 2]). Let $(R_1, S_1) : \mathcal{K}_1^\tau \to \mathcal{K}_2^\tau$ and $(R_2, S_2) : \mathcal{K}_2^\tau \to \mathcal{K}_3^\tau$ be multivalued standard morphisms between topological contexts. Define $\Box$ as follows:

$$(R_2, S_2) \Box (R_1, S_1) := (R_2 \Box R_1, S_2 \Box S_1)$$

where

$$(R_2 \Box R_1)(g_1) := ((R_2 \circ R_1)(g_1))'' \quad \text{and} \quad (S_2 \Box S_1)(m_1) := ((S_2 \circ S_1)(m_1))''.$$

The need to define the composition of a pair of multivalued standard morphisms is to ensure that the map $(R, S) \mapsto f_{RS}$ is (1) compatible with the expected composition of bounded lattice homomorphisms, and also that (2) the composition preserves the identity multivalued standard morphism on each standard topological context.

To be precise, we mean that if $f_1 : \mathbf{L}_1 \to \mathbf{L}_2$ and $f_2 : \mathbf{L}_2 \to \mathbf{L}_3$, then we need that the multivalued standard morphism from $\mathbb{K}^\tau(\mathbf{L}_3)$ to $\mathbb{K}^\tau(\mathbf{L}_1)$ will be given by $(R_{f_2 \circ f_1}, S_{f_2 \circ f_1}) = (R_{f_1}, S_{f_1}) \Box (R_{f_2}, S_{f_2})$.

Theorem 41 ([31])**.** *Using the functors $\mathbb{K}^\tau$ and $\mathbb{B}^\tau$, the category of bounded lattices and bounded lattice homomorphisms is dually equivalent to the category of standard topological contexts with multivalued standard morphisms.*

6 Representation via maximal partial maps

The representation of a bounded lattice via maximal partial maps due to Ploščica [41] is essentially a reframing of Urquhart's representation in the spirit of the theory of natural dualities (cf. [6]). We will not attempt to give a summary of the theory of natural dualities here. Instead we refer the reader to the book by Clark and Davey [6] and a recent survey article by Haviar [32] which appeared in this journal.

We will however highlight one important point: the (quasi)variety of distributive lattices is generated by the two-element lattice, denoted here by $\underline{\mathbf{2}}$. That is $\mathcal{DL} = \mathsf{ISP}(\underline{\mathbf{2}})$. A consequence of this is that if $\mathbf{D} \in \mathcal{DL}$ and $x, y \in D$ with $x \neq y$, then there exists a (distributive) lattice homomorphism $h : \mathbf{D} \to \underline{\mathbf{2}}$ such that $h(x) \neq h(y)$. We further note that for such a homomorphism, $h^{-1}(\{1\})$ is a prime filter and $h^{-1}(\{0\})$ is a prime ideal of $\mathbf{D}$.

The central idea in Ploščica's representation of bounded lattices [41] is the replacement of total maps by maximal partial maps. As before, let $\mathcal{L}$ be the category of all bounded lattices and bounded lattice homomorphisms. A partial map $f : \mathbf{L} \to \mathbf{K}$ between bounded lattices $\mathbf{L}$ and $\mathbf{K}$ is called a *partial homomorphism* if its domain is a bounded sublattice of $\mathbf{L}$ and the restriction $f{\restriction}_{\mathrm{dom}(f)}$ is an $\mathcal{L}$-homomorphism. A partial homomorphism is said to be *maximal* if there is no partial homomorphism properly extending it; such a map is referred to as an MPH, for short. Using Zorn's Lemma, every partial homomorphism can be extended to an MPH. For bounded lattices $\mathbf{L}$ and $\mathbf{K}$, we denote by $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \mathbf{K})$ the set of all MPHs from $\mathbf{L}$ to $\mathbf{K}$.

Our notation in this section is more closely aligned with that used in [9, 10] than in the original article by Ploščica. Let

$$\underline{\mathbf{2}} := \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle \quad \text{and} \quad \underset{\sim}{\mathbf{2}} := \langle \{0, 1\}; \leq \rangle$$

denote, respectively, the two-element bounded lattice and the two-element ordered set with $0 < 1$. The topological structure $\underset{\sim}{\mathbf{2}}_{\mathscr{T}}$ is obtained by adding the discrete topology $\mathscr{T}$ to $\underset{\sim}{\mathbf{2}}$.

Ploščica [41] defines, for any bounded lattice $\mathbf{L}$, the dual space of $\mathbf{L}$ in the following way. We equip the set $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ with the binary relation E defined by the rule

$$(f, g) \in E \quad \text{iff } f(x) \leqslant g(x) \text{ for every } x \in \mathrm{dom}(f) \cap \mathrm{dom}(g).$$

Observe that $(f, g) \in E$ if and only if $f^{-1}(1) \cap g^{-1}(0) = \emptyset$. Ploščica refers to the structure $(\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ as a graph, but we prefer to use the term *digraph*. In Figure 2 we present a number of examples of finite (non-distributive) lattices and their dual digraphs. The labelling of the vertices is explained in the caption. Also, to keep the display simpler,

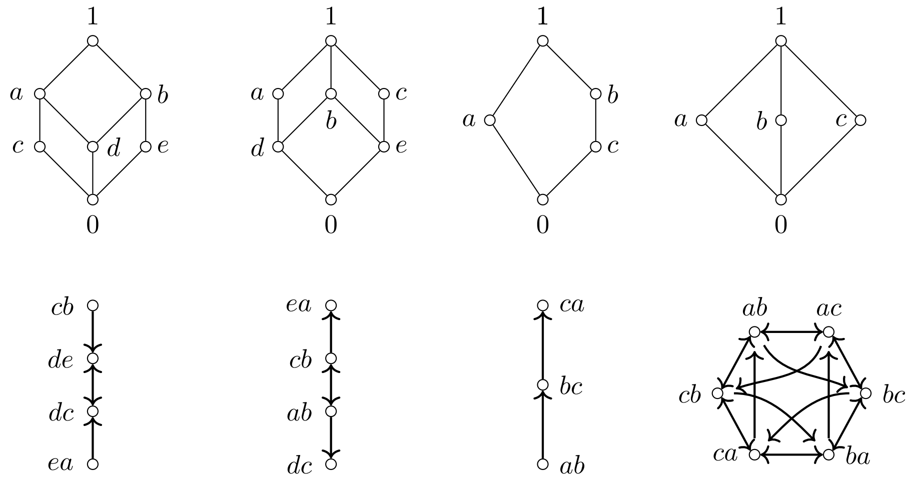

Figure 2. Some finite lattices and their dual digraphs. A vertex is labelled xy if $f^{-1}(1) = {\uparrow}x$ and $f^{-1}(0) = {\downarrow}y$.

we have not included the loop on each vertex. Notice that the directed edge set is not a transitive relation.

Before proceeding, we make the observation that there is a one-to-one correspondence between the set of MPHs from $\mathbf{L}$ to $\underline{\mathbf{2}}$ and the set of maximal disjoint filter-ideal pairs of $\mathbf{L}$. Given $f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, it is not difficult to show that the pair $\langle f^{-1}(1), f^{-1}(0)\rangle$ is an MDFIP of $\mathbf{L}$. Going in the opposite direction, given an MDFIP $\langle F, I\rangle$, the partial map $f : \mathbf{L} \to \underline{\mathbf{2}}$ defined by $f^{-1}(1) = F$ and $f^{-1}(0) = I$ will be an MPH. Further details regarding the connection between the representations of Urquhart and Ploščica can be found in [41, Section 2].

Returning to Ploščica's representation, the set $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ is further equipped with the topology $\mathscr{T}$ which has a subbasis of closed sets consisting of all the sets of the form $V_a = \{\, f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0 \,\}$ and $W_a = \{\, f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1 \,\}$, where $a \in L$. The dual of the lattice $\mathbf{L}$ is then the digraph with topology, $\mathbb{D}(\mathbf{L}) := (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E, \mathscr{T})$. The topology of $\mathbb{D}(\mathbf{L})$ is T_1 and compact. If the lattice $\mathbf{L}$ is distributive, then $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) = \mathcal{L}(\mathbf{L}, \underline{\mathbf{2}})$, the relation E coincides with the pointwise partial order of maps and $\mathbb{D}(\mathbf{L})$ is the usual Priestley dual space of $\mathbf{L}$.

We make the family $\mathcal{G}_{\mathscr{T}}$ of digraphs with topology into a category in the following way. A map $\varphi\colon (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ between digraphs with topology is called a *$\mathcal{G}_{\mathscr{T}}$-morphism* if it preserves the binary relation and is continuous as a map from (X_1, τ_1) to (X_2, τ_2). A partial map $\varphi\colon (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ is called a *partial $\mathcal{G}_{\mathscr{T}}$-morphism* if its domain is a τ_1-closed subset of X_1 and the restriction of φ to its domain is a morphism. A partial $\mathcal{G}_{\mathscr{T}}$-morphism is called maximal, or an MPM, for short, if there is no partial $\mathcal{G}_{\mathscr{T}}$-morphism properly extending it. For a digraph with topology, $\mathbf{X}_{\mathscr{T}} = (X, E, \mathscr{T})$, we denote by $\mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbf{X}_{\mathscr{T}}, \underset{\sim}{\mathbf{2}}_{\mathscr{T}})$ the set of MPMs from $\mathbf{X}_{\mathscr{T}}$ to $\underset{\sim}{\mathbf{2}}_{\mathscr{T}}$.

Just as the dual space was constructed via maximal partial homomorphisms, the lattice will be represented by the maximal partial morphisms described above. Before that is possible, we first need to explore some properties of MPMs.

Lemma 42 ([41, Lemma 1.3])**.** *Let $\mathbf{X}_{\mathscr{T}} = (X, E, \mathscr{T})$ be a digraph with topology and φ a partial E-preserving continuous map from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}_{\mathscr{T}}$. Then $\varphi \in \mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbf{X}_{\mathscr{T}}, \underset{\sim}{\mathbf{2}}_{\mathscr{T}})$ if and* *only if*

(i) $\varphi^{-1}(0) = \{\, x \in X \mid \textit{there is no } y \in \varphi^{-1}(1) \textit{ with } (y,x) \in E \,\}$ *and*

(ii) $\varphi^{-1}(1) = \{\, x \in X \mid \textit{there is no } y \in \varphi^{-1}(0) \textit{ with } (x,y) \in E \,\}$.

Using the above lemma, it is shown that the elements of $\mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbf{X}_{\mathscr{T}}, \underset{\sim}{\mathbf{2}}_{\mathscr{T}})$ are partially ordered as follows:

$$\varphi \leqslant \psi \quad \Longleftrightarrow \quad \varphi^{-1}(1) \subseteq \psi^{-1}(1) \quad \Longleftrightarrow \quad \psi^{-1}(0) \subseteq \varphi^{-1}(0).$$

In fact, this ordering is a lattice ordering with meets and joins of MPMs defined as follows:

$$(\varphi \wedge \psi)(x) = \begin{cases} 1 & \text{if } x \in \varphi^{-1}(1) \cap \psi^{-1}(1) \\ 0 & \text{if for all } z \in \varphi^{-1}(1) \cap \psi^{-1}(1), (z,x) \notin E, \end{cases}$$

$$(\varphi \vee \psi)(x) = \begin{cases} 1 & \text{if for all } z \in \varphi^{-1}(0) \cap \psi^{-1}(0), (x,z) \notin E \\ 0 & \text{if } x \in \varphi^{-1}(0) \cap \psi^{-1}(0) \end{cases}$$

Ploščica's representation of bounded lattices can be summarised as follows.

Proposition 43 ([41, Lemmas 1.2, 1.5 and Theorem 1.7]). *Let* $\mathbf{L}$ *be a bounded lattice and let* $\mathbb{D}(\mathbf{L}) = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{2}), E, \mathscr{T})$ *be the dual of* $\mathbf{L}$. *For* $a \in L$, *let the evaluation map* $e_a \colon \mathbb{D}(\mathbf{L}) \to \underset{\sim}{\mathbf{2}}_{\mathscr{T}}$ *be defined by*

$$e_a(f) = \begin{cases} f(a) & a \in \mathrm{dom}(f), \\ - & \textit{undefined otherwise}. \end{cases}$$

Then the following hold:

(i) *The map* e_a *is an element of* $\mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbb{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathscr{T}})$ *for each* $a \in L$.

(ii) *Every* $\varphi \in \mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbb{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathscr{T}})$ *is of the form* e_a *for some* $a \in L$.

(iii) *The map* $e_{\mathbf{L}} : \mathbf{L} \to \mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbb{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathscr{T}})$ *given by evaluation,* $a \mapsto e_a$ $(a \in L)$, *is an isomorphism of* $\mathbf{L}$ *onto the lattice* $\mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbb{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathscr{T}})$, *ordered by* $\varphi \leqslant \psi$ *if and only if* $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$.

Ploščica does not attempt to characterize the digraphs with topology that arise as the duals of bounded lattices. However, there is a remark [41, p. 87] that a characterization could be obtained by translating the characterization of Urquhart's L-spaces [49].

We remark that recent work by Holliday [34] uses a single set with one binary relation and topology to represent lattices. This is very similar in style to that of Ploščica. Although the focus of Holliday's paper is on lattices with an additional unary operation, an explanation of the representation is given in [34, Section 3.3].

6.1 Dual digraphs of lattices

As mentioned above, the properties of the digraphs with topology that are the dual spaces of bounded lattices were not fully characterised. However, the properties of the untopologised digraphs dual to bounded lattices were described by Craig, Gouveia and Haviar [7]. There they were called *TiRS graphs*, but, in line with our earlier comment, here we prefer to call them *TiRS digraphs*. Before presenting the definition below, we define the following subsets of the vertex set: $xE = \{\, y \in V \mid (x,y) \in E \,\}$ and $Ex = \{\, y \in V \mid (y,x) \in E \,\}$.

Definition 44 ([7, Definition 2.2])**.** A TiRS digraph $G = (V, E)$ is a set V and a reflexive relation $E \subseteq V \times V$ such that:

(S) If $x, y \in V$ and $x \neq y$ then $xE \neq yE$ or $Ex \neq Ey$.

(R) For all $x, y \in V$, if $xE \subset yE$ then $(x, y) \notin E$, and if $Ey \subset Ex$ then $(x, y) \notin E$.

(Ti) For all $x, y \in V$, if xEy then there exists $z \in V$ such that $zE \subseteq xE$ and $Ez \subseteq Ey$.

Recalling the dual digraph of Ploščica's representation as the set of MPHs with the edge set E, we obtain the following result.

Proposition 45 ([7, Proposition 2.3])**.** *For any bounded lattice* $\mathbf{L}$*, its dual digraph* $(\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *is a TiRS digraph.*

Hence, when restricted to the case of finite lattices, the proposition above yields a dual representation. We recall that when $\mathbf{L}$ is distributive, E will simply be a partial order. The representation described below generalises the one for distributive lattices due to Birkhoff [3].

Theorem 46 ([7, Corollary 3.2])**.** *There exists a dual representation of arbitrary finite lattices via finite TiRS graphs generalising the Birkhoff dual representation of finite distributive lattices via finite posets.*

7 Duality via L-frames

An alternative setting for a duality for bounded lattices was pursued by Hartonas and Dunn [26]. Their approach can be viewed in some sense as a hybrid between that of Allwein–Hartonas (Section 4) and that of Hartung (Section 5). Although the basic set-up of the duality here is different to that in Section 4, the reader will notice many similarities in the properties of the dual spaces.

Hartonas and Dunn define a *frame* to be a triple $\mathcal{F} = (X, \perp, Y)$ where X and Y are sets and $\perp \subseteq X \times Y$. As usual, this binary relation gives rise to a Galois connection from $\wp(X)$ to $\wp(Y)$. For $U \subseteq X$ and $V \subseteq Y$ we let:

$$\lambda(U) = \{\, y \in Y \mid (\forall u \in U)(u \perp y) \,\} \quad \text{and} \quad \rho(V) = \{\, x \in X \mid (\forall v \in V)(x \perp v) \,\}.$$

Moreover, $\rho\lambda$ is a closure operator on X while $\lambda\rho$ is a closure operator on Y. Denote by $\mathcal{C}_X$ and $\mathcal{C}_Y$ the closed sets of these closure operators. That is,

$$\mathcal{C}_X = \{\, A \subseteq X \mid \rho\lambda(A) = A \,\} \quad \text{and} \quad \mathcal{C}_Y = \{\, B \subseteq Y \mid \lambda\rho(B) = B \,\}.$$

The sets in $\mathcal{C}_X$ are called the *stable subsets* of X and these will play an important role in the recovery of a lattice from its dual space.

The idea behind the duality is to think of a lattice $\mathbf{L}$ as consisting of a meet semilattice $(L, \wedge)$ and a join semilattice $(L, \vee)$. The dual space of the lattice will consist of dual spaces of each of these components. Hence, in addition to considering the filters of $\mathbf{L}$ topologised as above, the dual space will also have the "filters" of $(L, \vee)$.

Let $X_a = \{\, F \in \mathsf{Filt}(\mathbf{L}) \mid a \in F \,\}$. The set $\mathsf{Filt}(\mathbf{L})$ can be equipped with a topology that has as its subbase the collection:

$$\mathcal{B}_{\mathsf{Filt}(\mathbf{L})} = \{\, X_a \mid a \in L \,\} \cup \{\, X_b^{\complement} \mid b \in L \,\}.$$

Then consider the set $\mathsf{Idl}(\mathbf{L})$ and $Y_a = \{\, I \in \mathsf{Idl}(\mathbf{L}) \mid a \in I \,\}$. Equip $\mathsf{Idl}(\mathbf{L})$ with a topology that has as its subbase the collection:

$$\mathcal{B}_{\mathsf{Idl}(\mathbf{L})} = \{\, Y_a \mid a \in L \,\} \cup \{\, Y_b^{\complement} \mid b \in L \,\}.$$

Given a lattice $\mathbf{L}$, we will denote by $\mathcal{X}_{\mathbf{L}}$ and $\mathcal{Y}_{\mathbf{L}}$ the topological spaces defined above.

Before introducing the characterization of the dual spaces, we remark that the authors use the term "Stone space" to refer to a topological space that is compact and totally separated (i.e. distinct points can be separated by disjoint clopen sets whose union is the whole set).

Definition 47 ([26, Definition 2.1])**.** An *FSpace* $\mathcal{X} = (X, \leqslant, \tau)$ is a partially ordered Stone space such that

(i) X is a complete meet semilattice with respect to the partial order;

(ii) τ has a subbasis of clopen sets $\mathcal{S} = \{\, X_i \mid i \in I \,\} \cup \{\, X \backslash X_j \mid j \in I \,\}$ such that $X_i = {\uparrow} x_i$ where $x_i \in X$;

(iii) the subset $\{\, x_i \mid i \in I \,\}$ is join-dense in X (every $x \in X$ is equal to $\bigvee\{\, x_i \mid x_i \leqslant x \,\}$);

(iv) the collection $X^* = \{\, X_i \mid i \in I \,\}$ is closed under finite intersections.

An *FSpace morphism* $h : \mathcal{X} \to \mathcal{Y}$ is then a continuous function such that h^{-1} preserves (arbitrary) meets and $h^{-1}(Y^*) \subseteq X^*$. The category of FSpaces is denoted by $\mathcal{FSpace}$. From the definition, it is clear that if $h : \mathcal{X} \to \mathcal{Y}$ is an FSpace morphism, then $h^{-1} : Y^* \to X^*$ is a meet-semilattice homomorphism.

We now give the definition of the structures that will act as the dual spaces. In the definition below, structures satisfying only (i)–(iii) were referred to as $\perp$-frames in [26]. The X^* in item (iii) below is that which comes from (iv) in Definition 47.

Definition 48 ([26, Definition 2.3])**.** An *L-frame* is a triple $(\mathcal{X}, \perp, \mathcal{Y})$ where

(i) $\mathcal{X}$ and $\mathcal{Y}$ are FSpaces;

(ii) $\perp \subseteq X \times Y$;

(iii) the Galois connection (λ, ρ) between $\wp(X)$ and $\wp(Y)$ restricts to a Galois connection between X^* and Y^*;

(iv) λ is a dual isomorphism of the intersection semilattices X^* and Y^*.

With the definition of the candidate dual structures in place, we can state the first representation theorem. The theorem below uses the topological spaces $\mathcal{X}_{\mathbf{L}}$ and $\mathcal{Y}_{\mathbf{L}}$ defined above.

Theorem 49 ([26, Theorem 2.4])**.** *Let $\mathbf{L}$ be a lattice and consider the dual structure $(\mathcal{X}_{\mathbf{L}}, \perp, \mathcal{Y}_{\mathbf{L}})$ where for $F \in \mathsf{Filt}(\mathbf{L}), I \in \mathsf{Idl}(\mathbf{L})$, we have $F \perp I$ iff $F \cap I \neq \emptyset$. Then*

(i) *$(\mathcal{X}_{\mathbf{L}}, \perp, \mathcal{Y}_{\mathbf{L}})$ is an L-frame;*

(ii) *$a \mapsto X_a$ is a lattice isomorphism between $\mathbf{L}$ and X^* where $a \wedge b \mapsto X_a \cap X_b$ and $a \vee b \mapsto \rho\lambda(X_a \cup X_b)$;*

(iii) *X^* can be characterised as the stable (i.e. $A = \rho\lambda(A)$) clopen subsets of $\mathcal{X}$.*

Morphisms between FSpaces were defined earlier. That definition is now used in order to define morphisms between L-frames. If $(\mathcal{X}_1, \perp_1, \mathcal{Y}_1)$ and $(\mathcal{X}_2, \perp_2, \mathcal{Y}_2)$ are L-frames, and f and h are FSpace morphisms $f : \mathcal{X}_1 \to \mathcal{X}_2$, $h : \mathcal{Y}_1 \to \mathcal{Y}_2$, then the pair (f, h) is an *L-frame morphism* if for all $A \in X_2^*$ and for all $B \in Y_2^*$:

$$\lambda_1 f^{-1}(A) = h^{-1}\lambda_2(A) \quad \text{and} \quad f^{-1}\rho_2(B) = \rho_1 h^{-1}(B).$$

The category of L-frames and L-frame morphisms is denoted by $\boldsymbol{\mathcal{LF}rame}$. The next result shows that for an L-frame morphism (f, h), both inverse functions will give lattice homomorphisms. We note however that only one of them will be used in the final duality.

Lemma 50 ([26, Lemma 2.10])**.** *Let $(f, h) : (\mathcal{X}_1, \perp_1, \mathcal{Y}_1) \to (\mathcal{X}_2, \perp_2, \mathcal{Y}_2)$ be an L-frame morphism. Then $f^{-1} : X_2^* \to X_1^*$ and $h^{-1} : Y_2^* \to Y_1^*$ are lattice homomorphisms.*

In [26, Proposition 2.12] it is verified that every L-frame is the dual structure of some bounded lattice. The main duality theorem is presented below.

Theorem 51 ([26, Theorem 2.15])**.** *Let $\mathbb{F}$ be the functor which maps a lattice $\mathbf{L}$ to its dual L-frame $\mathbb{F}(\mathbf{L}) = (\mathcal{X}_\mathbf{L}, \perp, \mathcal{Y}_\mathbf{L})$. Let $\mathbb{G}$ be the functor which maps an L-frame $\mathcal{F} = (\mathcal{X}, \perp, \mathcal{Y})$ to its dual lattice $\mathbb{G}(\mathcal{F})$ of stable clopen subsets of $\mathcal{X}$.*

Further, for $h : \mathbf{L} \to \mathbf{K}$ a homomorphism, $\mathbb{F}(h) = (h^{-1}, h^{-1})$ is an L-frame morphism from $\mathbb{F}(\mathbf{K})$ to $\mathbb{F}(\mathbf{L})$. Lastly, if $(f, g) : (\mathcal{X}_1, \perp_1, \mathcal{Y}_1) \to (\mathcal{X}_2, \perp_2, \mathcal{Y}_2)$ is an L-frame morphism, then $\mathbb{G}((f, g)) = f^{-1}$ is a lattice homomorphism from X_2^* to X_1^*.

These functors give a dual equivalence between the categories $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{LF}rame}$.

8 Duality via closure spaces

Around the same time as the work of Hartonas and Dunn [26] (Section 7), another duality was proposed by Hartonas [27]. The focus of [27] was a duality that could be used for lattices with additional operations and hence be applied to establish a duality between the algebraic and Kripke semantics for the associated logics.

The duality in this section is very similar to the duality from Section 7 but, as stated in the introduction, the approach used here uses a single set rather than two sets in the dual structure. The set of filters of the lattice $\mathbf{L}$ is equipped with a closure operator as well as topology. The construction is also closely related to the duality for ortholattices developed by Goldblatt [25]. There the set is equipped with an irreflexive and symmetric binary relation. This relation leads to a closure operator which is then used for the representation. Hartonas chooses rather to equip the set with a closure operator directly.

We recommend the explanations given in the first few pages of [27] which emphasize how the construction is essentially a gluing together of the meet-semilattice and join-semilattice personas of a lattice $\mathbf{L}$. A final note is that [27] does not require the lattices to have a bottom element (see [27, Section 3.1]).

8.1 Closure spaces

Consider a set X with a binary relation $\prec$. This gives rise to a Galois connection on $\wp(X)$ with

$$\lambda(U) = \{\, x \in X \mid (\forall u \in U)(u \prec x) \,\} \quad \text{and} \quad \rho(V) = \{\, x \in X \mid (\forall v \in V)(x \prec v) \,\}$$

for $U, V \subseteq X$. (This is really just a special case of the Galois connection from the start of Section 7.) Then $\Gamma = \lambda\rho$ is a closure operator on X and hence the stable sets, $A = \Gamma(A)$, form a complete lattice.

Definition 52 ([27, Definition 3.1])**.** A *closure space* $\mathcal{X} = (X, \Gamma, \tau)$ is a topological space (X, τ) together with a closure operator Γ on X.

The dual space of a lattice $\mathbf{L}$ is given by $\mathbb{S}(\mathbf{L}) = (\mathsf{Filt}(\mathbf{L}), \tau, \Gamma)$ where $\Gamma = \lambda\rho$ is the closure operator induced by the binary relation $\subseteq$ on $\mathsf{Filt}(\mathbf{L})$. As before, for $a \in L$ we set $X_a = \{ F \in \mathsf{Filt}(\mathbf{L}) \mid a \in F \}$ and let

$$\mathcal{S} = \{ X_a \mid a \in L \} \cup \{ X_b^{\complement} \mid b \in L \}$$

be a subbasis for the topology τ on $\mathbb{S}(\mathbf{L})$.

The proof of the next proposition follows from Goldblatt [25].

Proposition 53 ([27, Proposition 3.4])**.** *The filter space* $(\mathsf{Filt}(\mathbf{L}), \tau)$*, with* τ *induced by the subbasis* $\mathcal{S}$*, is a Stone space (compact and totally separated).*

If $\mathcal{X}$ is a closure space then $\mathbb{K}(\mathcal{X})$ will denote the lattice of stable compact-open subsets of X. A representation theorem can then be stated as follows.

Theorem 54 ([27, Corollary 3.6])**.** *Every lattice* $\mathbf{L}$ *is isomorphic to the lattice* $\mathbb{K}(\mathbb{S}(\mathbf{L}))$ *of stable compact-open subsets of its dual closure space* $\mathbb{S}(\mathbf{L})$*.*

One of the features of this duality of Hartonas is the simple definition of the morphisms between dual structures. We note however that the required properties of the dual structures are by comparison quite complicated. Below we consider the characterisation of the dual closure spaces.

Definition 55 ([27, Definition 4.2])**.** Let $\mathbf{C}$ be a complete lattice and $\mathbf{K}$ a sublattice of $\mathbf{C}$. Then $\mathbf{K}$ is *compact-dense* in $\mathbf{C}$ if it satisfies

(i) Compactness: for any $a \in K$ and $\{ a_j \mid j \in J \} \subseteq C$, if $\bigwedge\{ a_j \mid j \in J \} \leqslant a$ then there exists a finite set $F \subseteq J$ such that $\bigwedge\{ a_j \mid j \in F \} \leqslant a$.

(ii) Density: $\mathbf{K}$ is meet-dense in $\mathbf{C}$, i.e. $a \in C$ implies $a = \bigwedge\{ b \in K \mid a \leqslant b \}$.

Definition 55 above is an entirely order-theoretic definition of how a sublattice is embedded in a complete lattice. It will be used to describe how the compact-open stable sets of a closure space sit inside the complete lattice of stable subsets of the closure space.

Definition 56 ([27, Definition 4.3])**.** Let $\mathcal{X} = (X, \Gamma, \tau)$ be a closure space. Denote by $\mathbb{K}(\mathcal{X})$ the collection of stable compact open subsets. The closure space (X, τ, Γ) is called a *lattice space* (*ℓ-space*) if it satisfies the following:

(i) (X, τ) is compact and totally separated;

(ii) τ is generated by elements of $\mathbb{K}(\mathcal{X})$ and their complements;

(iii) if $A = \Gamma(A)$ then there exists $x \in X$ such that $A = \Gamma(\{x\})$;

(iv) if $A, B \in \mathbb{K}(\mathcal{X})$ then $\Gamma(A \cup B) \in \mathbb{K}(\mathcal{X})$;

(v) the family $\mathbb{K}(\mathcal{X})$ is compact-dense in the complete lattice of stable sets (see Definition 55).

8.2 Stable continuous functions

Having defined above the class of structures that will serve as the dual spaces, we now examine the morphisms, which will be needed in order to obtain a duality theorem.

Definition 57. A function $f : (X, \Gamma, \tau) \to (Y, \Delta, \sigma)$ will be called a *stable continuous map* if it is a continuous function from (X, τ) to (Y, σ) such that f^{-1} takes stable compact open subsets of (Y, Δ, σ) to stable compact opens of (X, Γ, τ).

The category of ℓ-spaces with stable continuous functions will be denoted by $\boldsymbol{\mathcal{LTop}}$. Further, [27, Lemma 4.5] shows that every closure space in $\boldsymbol{\mathcal{LTop}}$ arises as the dual space of some lattice, and also every stable continuous function between two ℓ-spaces arises as the dual of some lattice homomorphism.

For a lattice homomorphism $h : \mathbf{L} \to \mathbf{K}$, define $\mathbb{S}(h) = h^{-1}$. For a stable continuous function $f : \mathcal{X} \to \mathcal{Y}$ between two ℓ-spaces, define $\mathbb{K}(f) = f^{-1} : \mathbb{K}(\mathcal{Y}) \to \mathbb{K}(\mathcal{X})$. With this in place we state the main duality result.

Theorem 58 ([27, Theorem 4.4])**.** *The functors $\mathbb{S} : \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{LTop}}$ and $\mathbb{K} : \boldsymbol{\mathcal{LTop}} \to \boldsymbol{\mathcal{L}}$ form a dual equivalence of categories.*

9 Duality via BL spaces

The duality of Moshier and Jipsen [37] takes a different approach to those covered so far. The major difference is that the dual spaces are purely topological, rather than topological spaces with additional structure (like quasi-orders, binary relations etc.).

The duality in [37] is intended as a duality in the spirit of Stone's duality for distributive lattices [48]. In that 1937 paper, Stone showed that the category of distributive lattices with lattice homomorphisms is dually equivalent to the category of spectral spaces and spectral maps (see Section 2.1). In a similar vein, the dual spaces of Moshier and Jipsen, which they call *BL spaces*, can be described purely in terms of the topology. However, the dual spaces are not a direct generalisation of Stone's duality for distributive lattices as the underlying set of the dual space is the set of *all* filters of the lattice. This set does not coincide with the set of prime filters in the case that the lattice is distributive.

Before defining and discussing BL spaces, we will recall some topological notions and set up some notation. Let $\mathcal{X} = (X, \tau)$ denote a topological space. A subset of X is said to be saturated if it is the intersection of open sets. Following [37], we denote by $\mathsf{O}(\mathcal{X})$ the set of all open sets and by $\mathsf{K}(\mathcal{X})$ the set of all compact saturated sets. We denote by $\mathsf{F}(\mathcal{X})$ the set of all filters of $\mathcal{X}$. Here we emphasize that these are filters of the partially ordered set $(X, \sqsubseteq)$, where $\sqsubseteq$ is the specialization order defined in Section 2.1. Further, since this is only a partially ordered set, these filters are not lattice filters but non-empty up-sets that are down-directed, i.e. whenever $x, y \in F$, there exists $z \in F$ such that $z \sqsubseteq x, y$. Following [37] we combine these notations in the following way: $\mathsf{K}(\mathcal{X}) \cap \mathsf{O}(\mathcal{X}) := \mathsf{KO}(\mathcal{X})$.

Recall that in a partially ordered set $(X, \leqslant)$, a subset $D \subseteq X$ is *directed* if for any $x, y \in D$, there exists an upper bound of x and y in D.

Definition 59 (Cf. [24])**.** Let $(X, \leqslant)$ be a partially ordered set. The *Scott topology*, τ_S, on X is defined as follows:

$$U \in \tau_S \quad\iff\quad U \text{ is an up-set and if } \bigvee D \in U \text{ for } D \text{ directed, then } D \cap U \neq \emptyset.$$

The name used in the definition below was chosen to recognise the contribution made by Hofmann, Mislove and Stralka [33] in their study of duality for semilattices. The spaces in Definition 60 are used for the description of a topological duality for meet semilattices. We will not describe that duality, but the definition provides an important stepping stone to the definition of the dual spaces of bounded lattices.

Definition 60 (cf. [37, Theorem 2.5])**.** An *HMS space* is a topological space $\mathcal{X} = (X, \tau)$ satisfying the following equivalent conditions:

(i) $\mathcal{X}$ is spectral and $\mathsf{OF}(\mathcal{X})$ forms a basis that is closed under finite intersections;

(ii) $\mathcal{X}$ is spectral, $\mathsf{OF}(\mathcal{X})$ forms a basis, $(X, \sqsubseteq)$ is a meet semilattice, and $(X, \sqsubseteq)$ has a least element;

(iii) $\mathcal{X}$ is sober and $\mathsf{KOF}(\mathcal{X})$ forms a basis that is closed under finite intersection.

A subset of a topological space $\mathcal{X}$ is said to be *F-saturated* if it is the intersection of open filters. We will denote by $\mathsf{FSat}(\mathcal{X})$ the collection of F-saturated subsets of $\mathcal{X}$. It is not difficult to show that this is a complete lattice when ordered by set inclusion. Further a closure operator is defined as follows:

$$\mathsf{fsat}(A) := \bigcap\{\, F \in \mathsf{OF}(X) \mid A \subseteq F \,\}.$$

Meets in $\mathsf{FSat}(\mathcal{X})$ are defined by intersections and, as usual, for $\mathcal{A} \subseteq \mathsf{FSat}(\mathcal{X})$ we have $\bigvee \mathcal{A} = \mathsf{fsat}(\bigcup \mathcal{A})$.

Theorem 61 ([37, Theorem 3.2])**.** *Let $\mathcal{X}$ be an HMS space. Then the following are equivalent:*

(i) $\mathsf{OF}(\mathcal{X})$ *forms a sublattice of* $\mathsf{FSat}(\mathcal{X})$*;*

(ii) $\mathsf{KOF}(\mathcal{X})$ *forms a sublattice of* $\mathsf{FSat}(\mathcal{X})$*;*

(iii) $\mathsf{fsat}(U)$ *is open whenever U is open.*

The HMS spaces satisfying the equivalent conditions of Theorem 61 are known as *BL spaces*.

Let $\mathbf{L}$ be a bounded lattice. Denote by $\mathbb{B}(\mathbf{L})$ the topological space whose underlying set is $\mathsf{Filt}(\mathbf{L})$ and whose topology is the Scott topology. The Scott topology has an easy-to-describe basis in this instance. It is given by sets of the form

$$\varphi_a := \{\, F \in \mathsf{Filt}(\mathbf{L}) \mid a \in F \,\}$$

for $a \in L$. It can be shown [37, Lemma 3.6] that $\mathbb{B}(\mathbf{L})$ will always be a BL space.

Theorem 62 ([37, Theorem 3.7])**.** *Let $\mathbf{L}$ be a bounded lattice. Then $\mathbf{L}$ is isomorphic to* $\mathsf{KOF}(\mathbb{B}(\mathbf{L}))$*. If $\mathcal{X}$ is a BL space, then $\mathcal{X}$ is homeomorphic to* $\mathbb{B}(\mathsf{KOF}(\mathcal{X}))$*.*

The maps used in the proof of the above theorem are $a \mapsto \varphi_a$ (a lattice homomorphism) and $x \mapsto \theta_x := \{\, F \in \mathsf{KOF}(X) \mid x \in F \,\}$ (a homeomorphism).

9.1 Morphisms for BL spaces

The description of the functions that will be dual to bounded lattice homomorphisms is done in two steps. The lemma below gives a list of equivalent conditions that will ensure that the dual of such a function is meet-preserving between the lattices $\mathsf{KOF}(\mathcal{Y})$ and $\mathsf{KOF}(\mathcal{X})$.

Lemma 63 ([37, Lemma 5.1])**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a function between HMS spaces. Then the following are equivalent:*

(i) *f^{-1} restricted to $\mathsf{KOF}(\mathcal{Y})$ co-restricts to $\mathsf{KOF}(\mathcal{X})$.*

(ii) *f is spectral and f^{-1} restricted to $\mathsf{OF}(\mathcal{Y})$ co-restricts to $\mathsf{OF}(\mathcal{X})$.*

(iii) *f is spectral and $\mathsf{fsat}(f^{-1}(B)) \subseteq f^{-1}(\mathsf{fsat}(B))$ for all $B \subseteq Y$.*

(iv) *f is spectral and $\mathsf{fsat}(f^{-1}(U)) \subseteq f^{-1}(\mathsf{fsat}(U))$ for all open sets $U \subseteq Y$.*

A function is said to be *F-continuous* if it satisfies the equivalent conditions of Lemma 63. It is shown in [37, Theorem 5.2] that the category of lattices and meet-preserving maps is dually equivalent to the category of BL spaces and F-continuous functions. However, we are most interested in their final duality result, for which we will require one further definition.

Lemma 64 ([37, Lemma 5.3])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be BL spaces and $f\colon \mathcal{X} \to \mathcal{Y}$ be an f-continuous map. Then the following are equivalent:*

(i) *f^{-1} preserves finite joins of compact open filters;*

(ii) *f^{-1} preserves finite joins of open filters;*

(iii) *f^{-1} preserves all joins of open filters;*

(iv) *$f^{-1}(\mathsf{fsat}(U)) \subseteq \mathsf{fsat}(f^{-1}(U))$ for any open $U \subseteq Y$.*

Given that we are interested in functions whose duals will preserve joins, the equivalence of (i) and (iv) in the above lemma leads to a definition of such functions. A spectral function $f\colon X \to Y$ is *F-stable* if $f^{-1}(\mathsf{fsat}(U)) = \mathsf{fsat}(f^{-1}(U))$ for any open set $U \subseteq Y$.

Now, we can define a functor $\mathbb{K}$ from the category of BL spaces with F-stable functions to the category of bounded lattices with lattice homomorphisms where $\mathbb{K}(\mathcal{X}) = (\mathsf{KOF}(\mathcal{X}), \cap, \vee, \emptyset, X)$ and for $f : \mathcal{X} \to \mathcal{Y}$ an F-stable function, $\mathbb{K}(f) = f^{-1}$. The map $\mathbb{B}$ defined on a lattice $\mathbf{L}$ before Theorem 62 can similarly be extended to a functor $\mathbb{B}$ where for $h : \mathbf{L} \to \mathbf{K}$, we have $\mathbb{B}(h) = h^{-1} : \mathbb{B}(\mathbf{K}) \to \mathbb{B}(\mathbf{L})$.

These two functors are used to the produce the main duality theorem, stated below.

Theorem 65 ([37, Theorem 5.4])**.** *The category of lattices and lattice homomorphisms is dually equivalent to the category of BL spaces and F-stable functions.*

10 Duality via irreducible filters

The work of Celani and González [5] aims to achieve a similar result to that of Moshier and Jipsen, i.e. to give a duality for lattices where the dual spaces are purely topological. In addition, they aimed to do so in such a way that when the lattice $\mathbf{L}$ is distributive, then the representation will coincide with that given by Stone's duality for distributive lattices [48].

The first major change from the approach of Moshier and Jipsen is the underlying set of the dual space. A filter P of a lattice $\mathbf{L}$ is said to be *irreducible* if whenever $P = F_1 \cap F_2$ (where $F_1, F_2 \in \mathsf{Filt}(\mathbf{L})$) then $P = F_1$ or $P = F_2$. We will denote by $\mathsf{Filt}_{\mathsf{Irr}}(\mathbf{L})$ the irreducible filters of a lattice $\mathbf{L}$.

As noted earlier, in finite lattices every filter is principal. In a finite lattice $\mathbf{L}$, the irreducible filters are exactly those filters that are principal up-sets of join-irreducible elements.

The duality of Celani and González is first developed for semilattices and then restricted to a duality for bounded lattices. We will present only the results related to bounded lattices. Their name for the class of dual spaces is L-spaces. We will use the notation $\mathbb{L}$-spaces in order to distinguish their spaces from those of Urquhart (Section 3) and Hartonas (Section 8).

We note that in [5] they denote by $(X, \mathcal{K})$ the topological space which has the topology generated by the subbase $\mathcal{K}$. That is, the topology $\tau_{\mathcal{K}}$ which has open sets given by arbitrary unions of finite intersections of elements of $\mathcal{K}$. We will rather write $(X, \tau_{\mathcal{K}})$ for that topological space.

Given a topological space $\mathcal{X} = (X, \tau_{\mathcal{K}})$ (i.e. with $\mathcal{K} \subseteq \wp(X)$ a subbase) consider $S(\mathcal{X}) := \{\, U^{\complement} \mid U \in \mathcal{K} \,\}$. From this collection of subsets $S(\mathcal{X})$ it is possible to define a closure operator. For $Y \subseteq X$:

$$\mathrm{cl}_{\mathcal{K}}(Y) = \bigcap \{\, A \in S(\mathcal{X}) \mid Y \subseteq A \,\}.$$

The set of closed subsets of X under this closure operator is denoted by $\mathcal{C}_{\mathcal{K}}(\mathcal{X})$. For $A, B \in S(\mathcal{X})$, define $A \underline{\vee} B := \mathrm{cl}_{\mathcal{K}}(A \cup B)$. Now if $\mathcal{X}$ is such that $\mathcal{K}$ is closed under finite unions, then the structure defined by $\mathbb{S}(\mathcal{X}) := \langle S(\mathcal{X}); \cap, \underline{\vee}, \emptyset, X \rangle$ is a bounded lattice.

Below is a characterisation of the topological spaces that will act as the dual spaces. As one might suspect from their labels, the conditions (S1) to (S4) are those used for the dual spaces of semilattices. The authors of [5] remark that (S3) will follow from (S2) if the topological space is T_1.

Definition 66 ([5, Definitions 3.5 and 3.25])**.** An $\mathbb{L}$-space is a pair $(X, \tau_{\mathcal{K}})$ which satisfies the following:

(S1) $(X, \tau_{\mathcal{K}})$ is a T_0 topological space and $X = \bigcup \mathcal{K}$.

(S2) $\mathcal{K}$ is a subbase of compact open subsets which is closed under finite unions and $\emptyset \in \mathcal{K}$.

(S3) For all $U, V \in \mathcal{K}$, if $x \in U \cap V$ then there exist $W, D \in \mathcal{K}$ such that $x \notin W$, $x \in D$ and $D \subseteq (U \cap V) \cup W$.

(S4) Let $Y \in \mathcal{C}_{\mathcal{K}}(\mathcal{X})$ and $\mathcal{Z} \subseteq S(\mathcal{X})$ be a Y-family such that $Y \cap A^{\complement} \neq \emptyset$, for all $A \in \mathcal{Z}$; then $Y \cap \bigcap \{\, A^{\complement} \mid A \in \mathcal{Z} \,\} \neq \emptyset$.

(L1) $X \in \mathcal{K}$.

(L2) If $U, V \in \mathcal{K}$, then $\bigcup \{\, W \in \mathcal{K} \mid W \subseteq U \cap V \,\} \in \mathcal{K}$.

Define a map $\sigma : \mathbf{L} \to \wp(\mathsf{Filt}_{\mathsf{Irr}}(\mathbf{L}))$ by $\sigma(a) = \{\, P \in \mathsf{Filt}_{\mathsf{Irr}}(\mathbf{L}) \mid a \in P \,\}$. Further, let $\mathcal{K}_{\mathbf{L}} = \{\, (\sigma(a))^{\complement} \mid a \in L \,\}$. Now define the dual space of a bounded lattice $\mathbf{L}$ to be $\mathbb{X}(\mathbf{L}) = (\mathsf{Filt}_{\mathsf{Irr}}(\mathbf{L}), \tau_{\mathcal{K}_{\mathbf{L}}})$.

The result below presents the desired correspondence between bounded lattices and $\mathbb{L}$-spaces.

Proposition 67 ([5, Propositions 3.12 and 3.27, Corollary 3.28])**.** *Let* $\mathbf{L}$ *be a bounded lattice. Then* $(\mathsf{Filt}_{\mathsf{Irr}}(\mathbf{L}), \tau_{\mathcal{K}_{\mathbf{L}}})$ *is an* $\mathbb{L}$*-space and* $\sigma : \mathbf{L} \to \mathbb{S}(\mathbb{X}(\mathbf{L}))$ *is a lattice isomorphism.*

Let $\mathcal{X} = (X, \tau_{\mathcal{K}})$ *be an* $\mathbb{L}$*-space. Then* $\mathcal{X}$ *is homeomorphic to* $\mathbb{X}(\mathbb{S}(\mathcal{X}))$*.*

The homeomorphism in Proposition 67 is given by $H_{\mathcal{X}}(x) = \{A \in \mathbb{S}(\mathcal{X}) \mid x \in A\}$.

10.1 Morphisms for $\mathbb{L}$-spaces

As we have seen with many of the lattice dualities from the previous sections, it is often not possible to define the duals of homomorphisms in such a way that they will also be functions. Indeed, this is the case again with the duality via irreducible filters. The dual of a homomorphism between two bounded lattices will turn out to be a relation between the dual spaces of the two lattices.

Let $h : \mathbf{L}_1 \to \mathbf{L}_2$ be a bounded lattice homomorphism. A relation $R_h \subseteq \mathsf{Filt}_{\mathsf{Irr}}(\mathbf{L}_2) \times \mathsf{Filt}_{\mathsf{Irr}}(\mathbf{L}_1)$ is defined as follows:

$$(Q, P) \in R_h \quad \iff \quad h^{-1}[Q] \subseteq P.$$

The above definition will be used to construct the dual of a lattice homomorphism. If $R \subseteq X_1 \times X_2$, define the map $\Box_R : \wp(X_2) \to \wp(X_1)$ as follows:

$$\Box_R(B) := \{\, x \in X_1 \mid R(x) \subseteq B \,\}$$

Below is a list of properties that a relation between $\mathbb{L}$-spaces should satisfy.

Definition 68 ([5, Definitions 3.14 and 3.29])**.** Let $\mathcal{X}_1 = (X_1, \tau_{\mathcal{K}_1})$ and $\mathcal{X}_2 = (X_2, \tau_{\mathcal{K}_2})$ be $\mathbb{L}$-spaces. A relation $R \subseteq X_1 \times X_2$ is called an *L-relation* if it satisfies:

(R1) If $B \in S(\mathcal{X}_2)$ then $\Box_R(B) \in S(\mathcal{X}_1)$.

(R2) If $x \in X_1$ then $R(x) \in \mathcal{C}_{\mathcal{K}_2}(\mathcal{X}_2)$.

(R3) $R(x) \neq \emptyset$ for all $x \in X_1$.

(R4) $\Box_R(\mathrm{cl}_{\mathcal{K}_2}(B_1 \cup B_2)) \subseteq \mathrm{cl}_{\mathcal{K}_1}(\Box_R(B_1) \cup \Box_R(B_2))$ for all $B_1, B_2 \in S(\mathcal{X}_2)$.

We note that the following result is originally proved first for semilattice homomorphisms and so-called *meet-relations* in [5, Proposition 3.19 and 3.20], before refining the proof to be applied to lattices and L-relations.

Proposition 69 ([5, Propositions 3.30 and 3.31])**.** *If $\mathcal{X}_1 = (X_1, \tau_{\mathcal{K}_1})$ and $\mathcal{X}_2 = (X_2, \tau_{\mathcal{K}_2})$ are $\mathbb{L}$-spaces with $R \subseteq X_1 \times X_2$ an L-relation, then $\mathbb{S}(R) = \Box_R : \mathbb{S}(\mathcal{X}_2) \to \mathbb{S}(\mathcal{X}_1)$ is a bounded lattice homomorphism.*

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are bounded lattices with $h : \mathbf{L}_1 \to \mathbf{L}_2$ a bounded lattice homomorphism, then $\mathbb{X}(h) = R_h \subseteq \mathbb{X}(\mathbf{L}_2) \times \mathbb{X}(\mathbf{L}_1)$ is an L-relation.

Equipped with the duals of morphisms from above, the functors $\mathbb{S}$ and $\mathbb{X}$ can now be used to provide the main result.

Theorem 70 ([5, Theorem 3.32])**.** *The category of bounded lattices with bounded lattice homomorphisms is dually equivalent to the category of $\mathbb{L}$-spaces with L-relations.*

11 Duality for lattices with admissible homomorphisms

The approach of Gehrke and van Gool is quite different to the rest of the dualities that we have surveyed so far. The duality theorem is proved in two steps. First, the category of lattices with admissible homomorphisms (see Definition 71) is shown to be equivalent to a full subcategory of $\boldsymbol{\mathscr{daDL}}$. The category $\boldsymbol{\mathscr{daDL}}$ is the category of *doubly dense adjoint pairs between distributive lattices*; see Definition 76.

Then, the category $\boldsymbol{\mathscr{daDL}}$ is shown to be equivalent to a category of topological polarities. (A topological polarity has the same basic structure as the topological contexts from Section 5. That is, it consists of two topological spaces and a relation between the underlying sets of the two spaces.) The category of topological polarities to which

$\mathcal{IaDL}$ is equivalent is the totally separated compact polarities, denoted $\mathcal{TSCP}$ (see Definition 78) with morphisms given by pairs of continuous functions. The final step in the process is to identify a full subcategory of $\mathcal{TSCP}$ such that the category of bounded lattices and admissible homomorphisms is equivalent to this subcategory.

One of the facts that the authors highlight is the fact that the topological spaces in their topological polarities are 'nicer' than those in Hartung's topological contexts. An example is given [22, Example 4.1] of a lattice whose dual topological context (in the sense of Definition 33) has spaces that are not sober. The start of [22, Section 4] provides an informative comparison between their objects and those of Hartung.

The first step in the process of constructing the duality is to define two distributive envelopes for any lattice $\mathbf{L}$. Gehrke and van Gool remark that their construction is closely related to the work on injective hulls of semilattices by Bruns and Lakser [4]. In particular, it is remarked that Definition 71 is a finitary version of *admissible* that appears in [4].

The formal definition of a distributive envelope will be given below, but for now we describe the general idea. A distributive envelope is a distributive lattice into which a lattice will be embedded. Moreover, the embedding will preserve as much of the existing distributivity of the lattice as possible.

Definition 71 ([22, Definition 3.1])**.** Let $\mathbf{L}$ be a lattice. A finite subset $M \subseteq L$ will be called *join-admissible* if its join distributes over all meets with elements from L. That is, for all $a \in L$

$$a \wedge \bigvee M = \bigvee_{m \in M} (a \wedge m).$$

Further, a function $f : \mathbf{L}_1 \to \mathbf{L}_2$ is said to *preserve admissible joins* if for every finite join-admissible set $M \subseteq L_1$ the equality $f(\bigvee M) = \bigvee\{ f(m) \mid m \in M \}$ holds.

With the concept of an admissible join in hand, we present the definition of the distributive $\wedge$-envelope.

Definition 72 ([22, Definition 3.2])**.** Let $\mathbf{L}$ be a bounded lattice and $\eta^{\wedge}_{\mathbf{L}} : \mathbf{L} \hookrightarrow D^{\wedge}(\mathbf{L})$ an embedding of $\mathbf{L}$ into a distributive lattice $D^{\wedge}(\mathbf{L})$ which preserves meets and admissible joins. The embedding is a *distributive $\wedge$-envelope* of $\mathbf{L}$ if it satisfies the universal property:

- For any function $f : \mathbf{L} \to \mathbf{D}$ into a distributive lattice $\mathbf{D}$ that preserves finite meets and admissible joins, there exists a unique lattice homomorphism $\hat{f} : D^{\wedge}(\mathbf{L}) \to \mathbf{D}$ such that $\hat{f} \circ \eta^{\wedge}_{\mathbf{L}} = f$.

There are at least two possible constructions of the distributive $\wedge$-envelope of a bounded lattice $\mathbf{L}$. These constructions demonstrate that such an envelope always exists, and by definition it is unique up to isomorphism. We examine one of these constructions. The interested reader can find the second construction in Section 3 of [22].

Definition 73 ([22, Definition 3.3])**.** For a bounded lattice $\mathbf{L}$, a subset $A \subseteq L$ is an *a-ideal* if it is a down-set closed under admissible joins.

For $T \subseteq L$, there exists a smallest a-ideal containing T. This is called the a-ideal generated by T, denoted $\langle T \rangle_{ai}$. Any a-ideal that can be generated by a finite set will be called a finitely generated a-ideal. The intersection of two finitely generated a-ideals is a finitely generated a-ideal. Also, for two finitely generated a-ideals $I = \langle S \rangle_{ai}$ and $J = \langle T \rangle_{ai}$ one gets $I \vee J = \langle S \cup T \rangle_{ai}$ and so the set of finitely generated a-ideals forms a distributive lattice when ordered by inclusion. Lastly, define $\eta^{\wedge}_{\mathbf{L}}(a) = \langle \{a\} \rangle_{ai} = {\downarrow} a$.

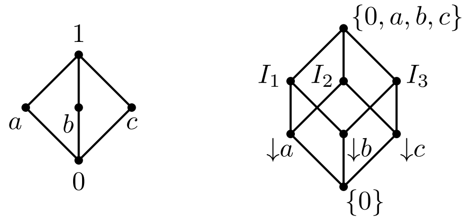

Figure 3. A non-distributive lattice and its distributive $\wedge$-envelope of a-ideals where $I_1 = \{0, a, b\}$, $I_2 = \{0, a, c\}$ and $I_3 = \{0, b, c\}$.

Theorem 74 ([22, Theorem 3.9])**.** *Let* $\mathbf{L}$ *be a bounded lattice. The embedding* $\eta^{\wedge}_{\mathbf{L}}$ *of* $\mathbf{L}$ *into the finitely generated a-ideals of* $\mathbf{L}$ *is a distributive* $\wedge$*-envelope of* $\mathbf{L}$*.*

An example of a lattice and its distributive $\wedge$-envelope is given in Figure 3. It is straightforward to check that none of the sets $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ or $\{a, b, c\}$ are join-admissible.

More generally, we can define, for a bounded lattice $\mathbf{L}$, the structure $\mathbb{A}_1(\mathbf{L}) = (D^{\wedge}(\mathbf{L}), D^{\vee}(\mathbf{L}), u_{\mathbf{L}}, \ell_{\mathbf{L}})$ where $D^{\wedge}(\mathbf{L})$ is the distributive lattice of a-ideals, $D^{\vee}(\mathbf{L})$ is the distributive lattice of a-filters, and

$$u_{\mathbf{L}}(I) = u_{\mathbf{L}}(\langle T \rangle_{ai}) = \langle \{\bigvee T\} \rangle_{af} \text{ and } \ell_{\mathbf{L}}(F) = \ell_{\mathbf{L}}\left(\langle S \rangle_{af}\right) = \langle \bigwedge S \rangle_{ai}.$$

The lattice $\mathbf{L}$ is then isomorphic to the Galois closed sets given by the Galois connection $u_{\mathbf{L}} : D^{\wedge}(\mathbf{L}) \to D^{\vee}(\mathbf{L})$, $\ell_{\mathbf{L}} : D^{\vee}(\mathbf{L}) \to D^{\wedge}(\mathbf{L})$.

Definition 75 ([22, Definition 3.19])**.** Let $\mathbf{L}$ and $\mathbf{K}$ be lattices and $f : \mathbf{L} \to \mathbf{K}$. We call f an *admissible homomorphism* if it is a lattice homomorphism that sends join-admissible subsets of L to join-admissible subsets of K, and meet-admissible subsets of L to meet-admissible subsets of K. The category of lattices with admissible homomorphisms is denoted by $\mathcal{L}_{\mathrm{a}}$.

It is important to note that any surjective homomorphism between lattices is admissible and also that any homomorphism whose codomain is a distributive lattice will also be admissible.

Below is the somewhat lengthy definition of a category whose elements are tuples consisting of lattices and maps between them. Although the definition is long, it is a natural abstraction of the adjoint pairs that have already been constructed. This category will be used as a stepping stone to obtaining the desired topological duality.

Definition 76 ([22, Definition 4.3])**.** Let $\boldsymbol{a\mathcal{DL}}$ denote the category of *adjoint pairs of distributive lattices*. Objects in this category are tuples $(\mathbf{D}, \mathbf{E}, f, g)$ where $\mathbf{D}$ and $\mathbf{E}$ are distributive lattices and $f : D \to E$ and $g : E \to D$ are a pair of maps such that f is lower adjoint to g.

If $(\mathbf{D}_1, \mathbf{E}_1, f_1, g_1)$ and $(\mathbf{D}_2, \mathbf{E}_2, f_2, g_2)$ are objects in $\boldsymbol{a\mathcal{DL}}$, then an $\boldsymbol{a\mathcal{DL}}$ morphism between them will be a pair of homomorphisms $h^{\wedge} : \mathbf{D}_1 \to \mathbf{D}_2$ and $h^{\vee} : \mathbf{E}_1 \to \mathbf{E}_2$ such that $h^{\vee} \circ f_1 = f_2 \circ h^{\wedge}$ and $h^{\wedge} \circ g_1 = g_2 \circ h^{\vee}$.

An object $(\mathbf{D}, \mathbf{E}, f, g)$ of $\boldsymbol{a\mathcal{DL}}$ is *doubly dense* if $g[E]$ is join-dense in $\mathbf{D}$ and $f[D]$ is meet-dense in $\mathbf{E}$. We denote by $\boldsymbol{da\mathcal{DL}}$ the full subcategory of $\boldsymbol{a\mathcal{DL}}$ whose objects are doubly dense adjoint pairs between distributive lattices.

The next proposition achieves the first of the two steps needed to show that $\mathcal{L}_a$ is equivalent to a category of topological polarities. The functor used in the proof maps a lattice $\mathbf{L}$ to $(D^{\wedge}(\mathbf{L}), D^{\vee}(\mathbf{L}), u, l)$ and an admissible homomorphism $h : \mathbf{L}_1 \to \mathbf{L}_2$ to the pair of distributive lattice homomorphisms $(D^{\wedge}(h), D^{\vee}(h))$ (see [22, Corollary 3.13] for an exact description of $D^{\wedge}(h)$ and $D^{\vee}(h)$).

Proposition 77 ([22, Proposition 4.4])**.** *The category $\mathcal{L}_a$ is equivalent to a full subcategory of $\boldsymbol{\mathcal{A}a\mathcal{DL}}$.*

The second step in the duality requires a dual equivalence between $\boldsymbol{\mathcal{A}a\mathcal{DL}}$ and a category $\mathcal{TSCP}$ of topological polarities (see Definition 78). In the final step, a full subcategory of $\mathcal{TSCP}$ will be described, and it is this category that will be dually equivalent to $\mathcal{L}_a$.

For X, Y sets and $R \subseteq X \times Y$, the following operations can be defined. For $x \in X$ and $S \subseteq X$ we have $R[x] = \{ y \in Y \mid xRy \}$ and $R[S] = \{ y \in Y \mid (\exists x \in S)(xRy) \}$. We further let $\Diamond S = R[S]$ and $\Box T = \{ x \in X \mid (\forall y \in Y)(xRy \Rightarrow y \in T) \}$.

Notice that for $x, x' \in X$ we can define a quasi-order induced by R: $x \preceq x'$ if and only if $R[x] \subseteq R[x']$. Similarly, for $y, y' \in Y$ we define $y \preceq y'$ if and only if $R^{-1}[y] \subseteq R^{-1}[y']$.

As remarked above, the objects described below have the same basic structure as the topological contexts of Hartung [30] (Section 5).

Definition 78 ([22, Definition 4.6])**.** A topological polarity $\mathcal{T} = (X, Y, R)$ consists of two sets, X and Y, each with topology and a relation $R \subseteq X \times Y$. It will be called *compact* if both of the topological spaces are compact. It is *totally separated* if it satisfies:

(i) the quasi-orders induced by R on the sets X and Y are partial orders;

(ii) for each clopen down-set U of X, the image $\Diamond U$ is clopen in Y and for each clopen down-set V of Y, the image $\Box V$ is clopen in X;

(iii) for each $x \in X$, $y \in Y$ if $(x, y) \notin R$ then there are clopen sets $U \subseteq X$ and $V \subseteq Y$ with $\Diamond U = V$ and $\Box V = U$ such that $x \in U$ and $y \notin V$.

The initialism TSCP is used as an abbreviation for totally separated compact topological polarity.

Morphisms between two TSCPs (X_1, Y_1, R_1) and (X_2, Y_2, R_2) will be pairs (s_X, s_Y) of continuous functions $s_X : X_1 \to X_2$ and $s_Y : Y_1 \to Y_2$ that must interact in the right way with the relations R_1 and R_2. See [22, Definition 4.12] for full details of the required interaction. These morphisms are used in the definition of the category of TSCPs, denoted $\mathcal{TSCP}$.

Now we describe how the second duality is set up. Let $\mathcal{A} = (\mathbf{D}, \mathbf{E}, f, g) \in \boldsymbol{\mathcal{A}a\mathcal{DL}}$. Define $\mathbb{T}(\mathcal{A})$, its dual topological polarity, by letting $\mathcal{X}$ and $\mathcal{Y}$ be the topological reducts of the dual Priestley spaces of $\mathbf{D}$ and $\mathbf{E}$. Then, define R by:

$$xRy \iff f[x] \subseteq y.$$

For a TSCP $\mathcal{T} = (\mathcal{X}, \mathcal{Y}, R)$, its *dual adjoint pair* is given by $(\mathbf{D}, \mathbf{E}, \Diamond, \Box)$ where $\mathbf{D}$ and $\mathbf{E}$ are the distributive lattices of clopen down-sets of X and Y respectively. It is shown [22, Propositions 4.9 and 4.10] that these two assignments do indeed define maps between the objects of $\boldsymbol{\mathcal{A}a\mathcal{DL}}$ and $\mathcal{TSCP}$.

Theorem 79 ([22, Theorem 4.13])**.** *The category $\boldsymbol{\mathcal{A}a\mathcal{DL}}$ is dually equivalent to the category $\mathcal{TSCP}$.*

Definition 80 ([22, Definition 4.16])**.** Let $(\mathcal{X}, \mathcal{Y}, R)$ be a TSCP, with $U \subseteq X$ a clopen down-set. We call U *R-regular* if for each $x \in X$ with $R[x] \neq R[\{x' \in X \mid x' < x\}]$, we have $R[x] \subseteq R[U]$ implies $x \in U$.

For the order dual condition, we will refer to a down-set $V \subseteq Y$ as being *R-coregular* if for each $y \in Y$ with $R^{-1}[y] \neq R^{-1}[\{y' \in Y \mid y' > y\}]$, we have $R^{-1}[y] \subseteq R^{-1}[U]$ implies $y \in U$.

In order to obtain the final dual equivalence, it is necessary to define a subcategory of $\mathcal{TSCP}$.

Definition 81 ([22, Definition 4.18])**.** A TSCP $(\mathcal{X}, \mathcal{Y}, R)$ is said to be *tight* if all R-regular clopen down-sets in X are R-closed, and all R-coregular clopen down-sets in Y are R-open. The full subcategory of $\mathcal{TSCP}$ whose objects are the tight TSCPs will be denoted by $t\mathcal{TSCP}$.

The proof of the main duality theorem uses both Proposition 77 and Theorem 79.

Theorem 82 ([22, Theorem 4.19])**.** *The category of lattices with admissible homomorphisms, $\mathcal{L}_a$, is dually equivalent to $t\mathcal{TSCP}$, the category of tight totally separated compact polarities.*

12 Canonical extensions

As mentioned in the introduction, there is a strong link between *canonical extensions* of Boolean algebras/distributive lattices/bounded lattices and the associated dualities and representation theories. Here we make this link explicit for the case of bounded lattices by describing the canonical extension in terms of some of the representations described in previous sections.

Canonical extensions were first described in 1951 by Jónsson and Tarski. In a pair of papers [35, 36], they described the completion in the setting of Boolean algebras with operators (additional operations that are join-preserving). If we consider only the Boolean algebra reducts, the canonical extension of a Boolean algebra is given by taking the powerset of the set of ultrafilters of the Boolean algebra. One way of describing this is 'forgetting' the topology of the dual (Stone) space of a Boolean algebra.

For a distributive lattice $\mathbf{D}$, consider the usual Priestley dual space of prime ideals ordered by set inclusion, with its Priestley space topology. Now, ignoring the topology, consider the complete lattice of down-sets of $(\mathsf{Idl}_{\mathsf{P}}(\mathbf{D}), \subseteq)$. This down-set lattice is a canonical extension of $\mathbf{D}$ [21].

Having looked at these two examples, we now give the precise definition of a canonical extension of a lattice. A *completion* of a bounded lattice $\mathbf{L} \in \mathcal{L}$ is a pair $(e, \mathbf{C})$ where $\mathbf{C}$ is a complete lattice and $e \colon \mathbf{L} \hookrightarrow \mathbf{C}$ is an embedding. An element of a completion $(e, \mathbf{C})$ of a bounded lattice $\mathbf{L}$ which is representable as a meet (join) of elements from $e(\mathbf{L})$ is called a *closed element* (*open element*); the sets of closed and open elements of $\mathbf{C}$ are denoted by $\mathcal{C}(\mathbf{C})$ and $\mathcal{O}(\mathbf{C})$, respectively. These names have their origins in the canonical extension of a Boolean algebra. (The closed and open elements of a completion are sometimes also referred to, respectively, as the filter and ideal elements.) A completion $(e, \mathbf{C})$ of $\mathbf{L}$ is said to be *dense* if every element of $\mathbf{C}$ is both a join of meets and a meet of joins of elements from $e(\mathbf{L})$; it is said to be *compact* if, for any sets $A \subseteq \mathcal{C}(\mathbf{C})$ and $B \subseteq \mathcal{O}(\mathbf{C})$ with $\bigwedge A \leqslant \bigvee B$, there exist finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigwedge A' \leqslant \bigvee B'$. (We note that, equivalently, the sets A, B in the definition of compactness above can be taken as arbitrary subsets of L.) A *canonical extension* of $\mathbf{L} \in \mathcal{L}$ has been defined as a dense and compact completion of $\mathbf{L}$ by Gehrke and Harding [20]. They showed that every bounded lattice **L** has a canonical extension and any two canonical extensions of **L** are isomorphic via an isomorphism that fixes the elements of **L**. As a result of this last statement, we can refer to *the* canonical extension of a bounded lattice.

In many cases, the canonical extension of a bounded lattice can be concretely realised by 'forgetting' the topology in a representation theorem, just as was done for Boolean algebras and distributive lattices. To make this statement clearer, we give some examples. In Urquhart's representation (Section 3), a lattice **L** is represented as the doubly-closed ℓ-stable subsets of its dual space. If one ignores the topology and considers only the ℓ-stable subsets of the dual space, this gives the canonical extension of **L** (cf. [20, Remark 2.10], [10, Corollary 2.10]). Similarly, stable sets of Hartonas–Dunn [26] (Section 7) or the ℓ-stable sets of Allwein–Hartonas [2] (Section 4) also yield canonical extensions of **L**.

As suggested by the title of their paper, the Moshier–Jipsen duality [37] (Section 9) allows for a purely topological construction of the canonical extension. Recall that in their set-up, a lattice **L** is isomorphic to $\mathsf{KOF}(\mathscr{X})$ where $\mathscr{X}$ is the dual space of the lattice (the set of lattice filters with the Scott topology). The complete lattice of F-saturated subsets, $\mathsf{FSat}(\mathscr{X})$, is then the canonical extension of **L**.

Another setting in which the canonical extension can be realised by removing the topology is the representation of Ploščica (Section 6). This was shown by Craig, Priestley and Haviar [9]. We first recall from [9] some results concerning general digraphs $\mathbf{X} = (X, E)$. By $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ we denote the set of all maximal partial E-preserving maps from $\mathbf{X}$ to $\underset{\sim}{2}$.

Lemma 83 ([9, Lemma 2.1])**.** *Let $\mathbf{X} = (X, E)$ be a digraph and φ a partial E-preserving map from $\mathbf{X}$ to $\underset{\sim}{2}$. Then $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ if and only if*

(i) $\varphi^{-1}(0) = \{\, x \in X \mid \textit{there is no } y \in \varphi^{-1}(1) \textit{ with } (y, x) \in E \,\}$ *and*

(ii) $\varphi^{-1}(1) = \{\, x \in X \mid \textit{there is no } y \in \varphi^{-1}(0) \textit{ with } (x, y) \in E \,\}$.

Lemma 83 allows us to observe that for a digraph $\mathbf{X} = (X, E)$ and $\varphi, \psi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ we have

$$\varphi^{-1}(1) \subseteq \psi^{-1}(1) \iff \psi^{-1}(0) \subseteq \varphi^{-1}(0).$$

The next result shows that this order on $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ yields a complete lattice.

Theorem 84 ([9, Theorem 2.3])**.** *Let $\mathbf{X} = (X, E)$ be a graph with $x \in X$ and let $\{\, \varphi_i \mid i \in I \,\} \subseteq \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$. Then the set $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ ordered by the rule*

$$\varphi \leqslant \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

is a complete lattice where joins and meets are calculated by

$$\Big(\bigwedge_{i\in I} \varphi_i\Big)(x) = \begin{cases} 1 & \textit{if } \ x \in \bigcap_{i\in I} \varphi_i^{-1}(1), \\ 0 & \textit{if there is no } y \in \bigcap_{i\in I} \varphi_i^{-1}(1) \textit{ with } (y, x) \in E; \end{cases}$$

$$\Big(\bigvee_{i\in I} \varphi_i\Big)(x) = \begin{cases} 1 & \textit{if there is no } y \in \bigcap_{i\in I} \varphi_i^{-1}(0) \textit{ with } (x, y) \in E, \\ 0 & \textit{if } \ x \in \bigcap_{i\in I} \varphi_i^{-1}(0). \end{cases}$$

For $\mathbf{X}_{\mathscr{T}} = (X, E, \mathscr{T})$ a digraph with a T_1-topology and $\mathbf{X} = (X, E)$ its untopologised counterpart, it was shown [9, Proposition 2.5] that $\mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbf{X}_{\mathscr{T}}, \underset{\sim}{2}_{\mathscr{T}}) \subseteq \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$. Moreover, $\mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbf{X}_{\mathscr{T}}, \underset{\sim}{2}_{\mathscr{T}})$ is a sublattice of $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$. For a bounded lattice **L** we had (Proposition 43) that $\mathbf{L} \cong \mathcal{G}^{\mathrm{mp}}_{\mathscr{T}}(\mathbb{D}(\mathbf{L}), \underset{\sim}{2}_{\mathscr{T}})$ and hence **L** can be embedded into $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$

where here $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. It was then shown [9, Theorem 3.11] that this is the canonical extension of $\mathbf{L}$.

The paper by Craig and Haviar [10] details the relationships between many of the different constructions of the canonical extension described in this section.

The next result demonstrates how the representation theory can assist in the understanding of canonical extensions of bounded lattices. This is done via characterisation of their untopologised dual spaces (i.e. their dual digraphs).

Gehrke and Harding showed [20, Lemma 3.4] that in the canonical extension of a bounded lattice $\mathbf{L}$, the completely join-irreducible elements are join dense, and the completely meet-irreducible elements are meet dense. A complete lattice $\mathbf{C}$ for which $J^\infty(\mathbf{C})$ is join-dense and $M^\infty(\mathbf{C})$ is meet-dense was called *perfect* in [15]. The dual digraphs described in Section 6.1 were used by Craig, Gouveia and Haviar to provide a more detailed description of the lattice-theoretic structure of canonical extensions.

Definition 85 ([8, Definition 4.1])**.** We say a perfect lattice $\mathbf{C}$ satisfies the condition (PTi) if for all $x \in J^\infty(\mathbf{C})$ and for all $y \in M^\infty(\mathbf{C})$, if $x \nleqslant y$ then there exist $w \in J^\infty(\mathbf{C})$, $z \in M^\infty(\mathbf{C})$ such that

(i) $w \leqslant x$ and $y \leqslant z$;

(ii) $w \nleqslant z$;

(iii) $(\forall u \in J^\infty(\mathbf{C}))(u < w \Rightarrow u \leqslant z)$;

(iv) $(\forall v \in M^\infty(\mathbf{C}))(z < v \Rightarrow w \leqslant v)$.

The condition above is essentially the translation of the condition (Ti) from RS-frames (see Definition 44) into perfect lattices. The name "(PTi)" was used to indicate that the lattice is both perfect and satisfies the translated (Ti) condition.

Theorem 86 ([8, Theorem 4.7])**.** *Let* $\mathbf{L}$ *be a bounded lattice. Then its canonical extension* $\mathbf{L}^\delta$ *is a PTi lattice.*

We end this section with two remarks. The first is that the reader, now armed with some basic background on canonical extensions, could go and read in the paper of Gehrke and van Gool [22] (Section 11) an alternative method for constructing the distributive $\wedge$-envelope and distributive $\vee$-envelope of a bounded lattice. Their second construction goes via the canonical extension. Lastly, we remark that a useful short survey of representations and canonical extensions of bounded lattices can be found in the recent paper by Hartonas [28, Section 2].

13 Additional operations and representations of finite lattices

In this final section we make some brief remarks about two additional aspects of duality and representation theory for bounded lattices. The first aspect is the representation of lattices with additional operations, and the second is the representation of finite lattices.

One of the motivations for studying duality and dual representations of lattice-based algebras is to aid in the semantic studies of the non-classical logics associated with those algebras. In such cases, it is important for the duality or representation theorems to be able to dually represent additional operations on the lattices.

Here we mention just a few examples of representation or duality theorems being applied to lattices with additional operations. Allwein and Dunn [1] used Urquhart's representation to develop Kripke-style semantics for linear logic. The paper of Hartonas [27], on which Section 8 is based, is largely motivated by applications to logic;

see, in particular, [27, Part III]. Further developments can be found in [29]. The paper of Dzik, Orlowska and van Alten [16] used Urquhart's representation to obtain representations of lattices with an additional unary negation-type operation satisfying various properties. More generally, the book by Orlowska, Radzikowska and Rewitzky [40] covers many applications similar to those in [16].

The work of Moshier and Jipsen covered in Section 9 was followed up with a second paper [38] which looked at lattices with particular additional operations, which were referred to there as *quasi-operators*. This second paper applies the topological duality from [37] to represent quasi-operators as certain functions between products of BL spaces.

There are a number of representation theorems that are designed only for the case of finite lattices. The representation for finite lattices does not require topology and so this greatly simplifies the theory. For readers interested only in finite lattices, some of the representations below might be of interest.

We have already remarked on the one-to-one correspondence between finite lattices and finite TiRS graphs in Section 6.1. This was further refined to a representation for finite join- and meet-semidistributive lattices in [11]. A representation theory for finite semidistributive lattices was also recently developed by Reading, Speyer and Thomas [45]. Nation [39] used the notion of an OD-graph to produce a a representation of finite lattices. This was later extended by Santocanale [46] to a duality. Lastly, Crapo [12] developed a theory for finite lattices with negation.

14 Conclusion

The large number of representation and duality theories published for bounded lattices demonstrates the level of interest in this topic from the mathematical community. The interest stems largely from those interested in universal algebra and/or non-classical logics. What becomes apparent from the various approaches is that no single representation is the best or most useful in all situations. Rather, depending on the interests and possible applications of the researchers, the setting is changed to adapt to their needs. We hope that providing such a survey will help researchers interested in such representations by exposing them to a variety of options, each of which will have strengths and weaknesses in different contexts.

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