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On the Convergence of Three Step Iterative Process for Three Asymptotically Nonexpansive Multi-maps in Banach Spaces

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Abstract
In this paper, we introduced a new three-step iterative scheme with errors for finding common fixed points of three asymptotically nonexpansive multi-maps in Banach spaces and prove a strong convergence theorem of the purposed algorithm under some control conditions. Our results improved and extended many known results existing in the literature.

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1 Introduction

Let D be a nonempty convex subset of a Banach space E. The set D is called *proximinal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x - y\| = d(x, D)$, where $d(x, D) = \inf\{\|x - z\| : z \in D\}$. Let $CB(D), K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for $A, B \in CB(D)$.

A single-valued map $T : D \to D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi- valued map $T : D \to CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called fixed point of $T : D \to D$ (respectively,$T : D \to CB(D)$) if $p = Tp$ (respectively, $p \in Tp$). The set of fixed points of T is denoted by $F(T)$.

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The mapping $T : D \to CB(D)$ is called
(i) *asymptotically nonexpansive* if there exists a sequence $r_n \geq 1, \lim_{n\to\infty} r_n = 1$ and $H(T^n x, T^n y) \leq r_n \|x - y\|$ for all $x, y \in D$ and $n \in \mathbb{N}$;
(ii) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that $H(T^n x, T^n y) \leq L\|x - y\|$ for all $x, y \in D$ and $n \in \mathbb{N}$;

(iii) *hemi compact* if, for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \to CB(D)$ is hemicompact.

In 1953, Mann [8] introduced the following iteration scheme, starting from $x_0 \in D$, to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n)x_n \quad \text{for all} \quad n \in \mathbb{N} \tag{1.1}$$

where $\{\alpha_n\}$ be a sequence in [0,1] satisfies certain conditions. However, we note that Mann's iteration process (1.1) has only weak convergence, in general; for instance, see([1],[4],[11]).
The Ishikawa [5] iteration scheme, starting from $x_0 \in D$, is the sequence $\{x_n\}$ defined by

$$\begin{aligned} y_n &= \beta_n T x_n + (1 - \beta_n)x_n \ \text{ for all } \ n \in \mathbb{N} \\ x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n)x_n \ \text{ for all } \ n \in \mathbb{N} \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in [0,1] satisfies certain conditions.
Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been studied by various authors (see; e.g. [5], [11], [14], [18]) using the Mann iteration or the Ishikawa iteration scheme. For details on the subject, we refer the reader to Berinde [2].

Sastry and Babu [12] defined the Mann and Ishikawa iteration schemes for multi-valued mappings.
Let $T : D \to P(D)$ be a multi-valued map and fix $p \in F(T)$.

(A) The sequence of Mann iterates is defined by $x_0 \in D$,

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n)x_n \quad \text{for all} \ \ n \in \mathbb{N}$$

where $\{\alpha_n\}$ be a sequence in [0,1] and $y_n \in Tx_n$ such that $\|y_n - p\| = d(p, Tx_n)$.

(B) The sequence of Ishikawa iterates is defined by $x_0 \in D$,

$$y_n = \beta_n z_n + (1 - \beta_n)x_n \quad \text{for all} \quad n \in \mathbb{N}$$

where $\{\beta_n\} \in [0, 1]$ and $z_n \in Tx_n$ such that $\|z_n - p\| = d(p, Tx_n)$, and

$$x_{n+1} = \alpha_n z_n^{'} + (1 - \alpha_n)x_n \quad \text{for all} \quad n \in \mathbb{N}$$

where $\{\alpha_n\} \in [0, 1]$ and $z_n^{'} \in Ty_n$ such that $\|z_n^{'} - p\| = d(p, Ty_n)$.

Sastry and Babu [12] proved that the Mann and Ishikawa iteration schemes for a multi-valued map T with a fixed point p converges to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. More precisely, they proved the following result for nonexpansive multi-valued map with compact domain.

Theorem 1. *([12], Theorem 5):Let E be a Hilbert space, D be a nonempty compact convex subset of E, and $T : D \to P(D)$ be a multi-valued map with a fixed point $p \in F(T)$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n\beta_n = \infty$. Then the Ishikawa iterates $\{x_n\}$ defined by (B) converges to a fixed point of T.*

Panyanak [10] extend the above result to uniformly convex Banach spaces but the domain of T remains compact.

Theorem 2. *([10], Theorem 3.1):Let E be a uniformly convex Banach space, D be a nonempty compact convex subset of E, and $T : D \to P(D)$ be a multi-valued map with a fixed point $p \in F(T)$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n\beta_n = \infty$. Then the Ishikawa iterates $\{x_n\}$ defined by (B) converges to a fixed point of T.*

Later, Song and Wang [17] noted that there was a gap in the proof of Theorem 1 (see [12], Theorem 5)and Theorem 2 (see [10], Theorem 3.1). Because the iteration x_n depends on a fixed $p \in F(T)$ as well as T. If $q \in F(T)$ and $q \neq p$, then the iteration x_n defined by q is different from the one defined by p. Therefore, one cannot derive the monotonicity of sequence $\{\|x_n - q\|\}$ from the monotonicity of $\{\|x_n - p\|\}$. So the conclusion of Theorem 1 and Theorem 2 are ambiguous. They further solved/revised the gap and also gave the affirmative answer the question using the following Ishikawa iteration scheme.

(C): Let D be a nonempty convex subset of E, $\alpha_n, \beta_n \in [0, 1]$ and $\gamma_n \in (0, \infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in D$. Then

$$\begin{aligned} y_n &= \beta_n z_n + (1 - \beta_n)x_n \quad \text{for all} \quad n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z_n^{'} + (1 - \alpha_n)x_n \quad \text{for all} \quad n \in \mathbb{N} \end{aligned}$$

where $\|z_n - z_n^{'}\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z_n^{'}\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ for $z_n \in Tx_n$ and $z_n^{'} \in Ty_n$.
Song and Wang [17] proved the following results. In the result, the domain of T is still compact, which is a strong condition.

Theorem 3. *([17], Theorem 1): Let E be a uniformly convex Banach space, D a nonempty compact convex subset of E and $T : D \to CB(D)$ a nonexpansive multi-valued map with $F(T) \neq \phi$ satisfying $TP = \{p\}$ for any $p \in F(T)$. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n\beta_n = \infty$. Then the Ishikawa iterates $\{x_n\}$ defined by (C) converges to a fixed point of T.*

Recall that a multi-valued map $T : D \to CB(D)$ is said to satisfy Condition (I) [14] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in D$.

Theorem 4. *([17], Theorem 2): Let E be a uniformly convex Banach space, D a nonempty compact convex subset of E and $T : D \to CB(D)$ a nonexpansive multi-valued map with $F(T) \neq \phi$ satisfying $Tp = \{p\}$ for any $p \in F(T)$. Assume that T satisfies Condition (I) and $0 \leq \alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then the Ishikawa iterates $\{x_n\}$ defined by (C) converges to a fixed point of T.*

In 2009, Shahzad and Zegeye [15] extended and improved the results of Panyanak [10], Sastry and Babu [12] and Song and Wang [17] to quasi-nonexpansive multi-valued maps. They also relaxed compactness of the domain of T and constructed an iteration scheme which removes the restriction of T namely $Tp = \{p\}$ for any $p \in F(T)$. The results provided an affirmative answer to Panyanak [10] question in a more general setting. They introduced a new iteration as follows:

Let D be a nonempty convex subset of a Banach space E and $\alpha_n, \alpha_n^{'} \in [0,1]$. The sequence of Ishikawa iterates is defined by $x_0 \in D$,

$$\begin{aligned} y_n &= \alpha_n^{'} z_n^{'} + (1-\alpha_n^{'})x_n \quad \text{for all} \quad n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z_n + (1-\alpha_n)x_n \quad \text{for all} \quad n \in \mathbb{N} \end{aligned}$$

where T is a quasi-nonexpansive multi-valued map, $z_n^{'} \in Tx_n$ and $z_n \in Ty_n$.

Since 2003, the iterative schemes with error for a single-valued map in Banach spaces have been studied by many authors, see ([3], [6], [7], [9]). Motivated and inspired by Shahzad and Zegeye [15], we purpose a new three-step iterative scheme for three multi-valued asymptotically nonexpansive maps in Banach spaces and prove strong convergence theorems of the purposed iteration.

2 Preliminaries

In this paper, we use the following iteration scheme.
Let D be a nonempty convex subset of a Banach space E. $\alpha_n, \beta_n, \gamma_n, \alpha_n^{'}, \beta_n^{'}, \gamma_n^{'}, \alpha_n^{''}, \beta_n^{''}, \gamma_n^{''}$ $\in [0.1]$ and $\{u_n\}, \{v_n\}, \{c_n\}$ are bounded sequences in D. Let T_1, T_2, T_3 be three asymptotically nonexpansive multi-valued maps from D into $CB(D)$. Let $\{x_n\}$ be the sequence defined by $x_0 \in D$,

$$\begin{aligned} z_n &= \alpha_n^{''} w_n + \beta_n^{''} x_n + \gamma_n^{''} u_n \quad \text{for all} \quad n \in \mathbb{N} \\ y_n &= \alpha_n^{'} w_n^{'} + \beta_n^{'} x_n + \gamma_n^{'} v_n \quad \text{for all} \quad n \in \mathbb{N} \\ x_{n+1} &= \alpha_n z_n^{'} + \beta_n x_n + \gamma_n c_n \quad \text{for all} \quad n \in \mathbb{N} \end{aligned} \tag{2.1}$$

where $z_n^{'} \in T_1^n y_n, w_n^{'} \in T_2^n z_n$ and $w_n \in T_3^n x_n$

To prove our main results, we shall make use the following definition and lemmas in the sequel.

Definition 5. The mappings $T_1, T_2, T_3 : D \to CB(D)$ with $F := \bigcap_{i=1}^{3} F(T_i) \neq \phi$ are said to satisfy *Condition (II)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that

$$\max\{d(x, T_1x), d(x, T_2x), d(x, T_3x)\} \geq f(d(x, F(T)))$$

for all $x \in D$.

Note that when $T_2 = T_3 = I$, the identity map or $T_1 = T_2 = T_3$ Condition (II) reduces to Condition (I) of Senter and Dotson [14]. Our Condition (II) also contains Condition (A') of Khan and Fakhar-ud-din [3].

Lemma 6. *[18]: Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be the sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty}\beta_n<\infty$ *and* $\sum_{n=1}^{\infty}r_n<\infty$, *then* $\lim_{n\to\infty}\alpha_n$ *exists. In particular,* $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges to zero, then* $\lim_{n\to\infty}\alpha_n=0$

Lemma 7. *[13]: Suppose that E is a uniformly convex Banach space and $0<p\le t_n\le q<1$ for all positive integer n. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty}\|x_n\|\le r$, $\limsup_{n\to\infty}\|y_n\|\le r$ and $\lim_{n\to\infty}\|t_nx_n+(1-t_ny_n\|=r$ hold for some $r\ge 0$. Then $\lim_{n\to\infty}\|x_n-y_n\|=0$.*

3 Main Results

Before proving our main result we shall prove the following crucial lemmas.

Lemma 8. *Let E be a uniformly convex Banach space, and let D be a nonempty closed and convex subset of E. Let T_1,T_2,T_3 be three asymptotically nonexpansive multi-maps from D into $CB(D)$ with the sequence $\{r_{i_n}\}\subset[1,\infty)$ satisfying $\sum_{n=1}^{\infty}r_{i_n}<\infty$ for all $i=1,2,3$ and $F:=\bigcap_{i=1}^{3}F(T_i)\neq\phi$ and $T_ip=\{p\},(i=1,2,3)$. Let $\{x_n\}$ be the sequence defined by (2.1), where $\alpha_n,\beta_n,\gamma_n,\alpha_n^{'},\beta_n^{'},\gamma_n^{'},\alpha_n^{''},\beta_n^{''},\gamma_n^{''}$ be real sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=\alpha_n^{'}+\beta_n^{'}+\gamma_n^{'}=\alpha_n^{''}+\beta_n^{''}+\gamma_n^{''}=1$ and $\{u_n\},\{v_n\},\{c_n\}$ are bounded sequences in D with $\sum_{n=1}^{\infty}\gamma_n<\infty,\sum_{n=1}^{\infty}\gamma_n^{'}<\infty,\sum_{n=1}^{\infty}\gamma_n^{''}<\infty$. Then*

(i) $\lim_{n\to\infty}\|x_n-p\|$ *exists for all $p\in F$;*

(ii) *there exists a constant $M>0$ such that*

$$\|x_{n+m}-p\|\le M\|x_n-p\|+M\sum_{k=n}^{n+m-1}b_k$$

for all $n,m\ge 1$ and $p\in F$, where $M=e^{3\sum_{k=n}^{n+m-1}r_k}$.

Proof. (i) Let $p\in F$ be the common fixed point of $\{T_i\},(i=1,2,3)$. Since $\{u_n\},\{v_n\},\{c_n\}$ are bounded sequences in D , we can put

$$M\ge\max\{\sup_{n\ge1}\|u_n-p\|,\sup_{n\ge1}\|v_n-p\|,\sup_{n\ge1}\|c_n-p\|\}$$

Then M is a finite number for each $n\in N$. For each $n\ge1$, let $r_n=\max\{r_{i_n}:i=1,2,3\}$. Thus we have $r_n\ge0$, $\lim_{n\to\infty}r_{i_n}=0$ and

$$\begin{aligned}\|x_{n+1}-p\|&=\|\alpha_nz_n^{'}+\beta_nx_n+\gamma_nc_n-p\|\\&\le\alpha_n\|z_n^{'}-p\|+\beta_n\|x_n-p\|+\gamma_n\|c_n-p\|\\&\le\alpha_nd(z_n^{'},T_1^np)+\beta_n\|x_n-p\|+\gamma_n\|c_n-p\|\\&\le\alpha_nH(T_1^ny_n,T_1^np)+\beta_n\|x_n-p\|+\gamma_n\|c_n-p\|\\&\le\alpha_nr_n\|y_n-p\|+\beta_n\|x_n-p\|+\gamma_n\|c_n-p\|\end{aligned}\tag{3.1}$$

Similarly, we have

$$\|y_n-p\|\le\alpha_n^{'}r_n\|z_n-p\|+\beta_n^{'}\|x_n-p\|+\gamma_n^{'}\|v_n-p\|\tag{3.2}$$

and

$$\|z_n-p\|\le\alpha_n^{''}r_n\|x_n-p\|+\beta_n^{''}\|x_n-p\|+\gamma_n^{''}\|u_n-p\|\tag{3.3}$$

Substituting (3.3) in (3.2), we get

$$\begin{aligned}\|y_n - p\| &\leq \alpha_n^{'}\alpha_n^{''}r_n^2\|x_n - p\| + \alpha_n^{'}\beta_n^{''}r_n\|x_n - p\| \\ &\quad + \alpha_n^{'}\gamma_n^{''}r_n\|u_n - p\| + \beta_n^{'}\|x_n - p\| + \gamma_n^{'}\|v_n - p\| \\ &= (1 - \beta_n^{'} - \gamma_n^{'})\alpha_n^{''}r_n^2\|x_n - p\| + \beta_n^{'}\|x_n - p\| \\ &\quad + (1 - \beta_n^{'} - \gamma_n^{'})\beta_n^{''}r_n^2\|x_n - p\| + m_n \\ &\leq (1 - \beta_n^{'})\alpha_n^{"}r_n^2\|x_n - p\| + \beta_n^{'}r_n^2\|x_n - p\| \\ &\quad + (1 - \beta_n^{'})\beta_n^{''}r_n^2\|x_n - p\| + m_n \\ &\leq (1 - \beta_n^{'})r_n^2\|x_n - p\| + \beta_n^{'}r_n^2\|x_n - p\| + m_n \\ &= r_n^2\|x_n - p\| + m_n\end{aligned} \tag{3.4}$$

where $m_n = \alpha_n^{'}\gamma_n^{''}r_n\|u_n - p\| + \gamma_n^{'}\|v_n - p\|$

Note that $\sum_{n=1}^{\infty} m_n < \infty$ as $\sum_{n=1}^{\infty}\gamma_n^{'} < \infty$ and $\sum_{n=1}^{\infty}\gamma_n^{''} < \infty$.

Substituting (3.4) in (3.1), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n r_n^3\|x_n - p\| + \alpha_n r_n m_n \\ &\quad + \beta_n\|x_n - p\| + \gamma_n\|c_n - p\| \\ &\leq (\alpha_n + \beta_n)r_n^3\|x_n - p\| + b_n \\ &= r_n^3\|x_n - p\| + b_n\end{aligned} \tag{3.5}$$

where $b_n = \alpha_n r_n m_n + \gamma_n\|c_n - p\|$.

Since $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} m_n < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, which implies that $\sum_{n=1}^{\infty} b_n < \infty$.

It follows that from Lemma (6) that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.This proved that the first part of the lemma.

(ii) Since $1 + x \leq e^x$ for all $\varepsilon > 0$. Then from (3.5)

$$\begin{aligned}\|x_{n+m} - p\| &\leq r_{n+m-1}^3\|x_{n+m-1} - p\| + b_{n+m-1} \\ &\leq e^{3r_{n+m-1}}\|x_{n+m-1} - p\| + b_{n+m-1} \\ &\leq e^{3r_{n+m-1}}\left[e^{3r_{n+m-2}}\|x_{n+m-2} - p\| + b_{n+m-2}\right] + b_{n+m-1} \\ &\leq e^{3(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2} - p\| + e^{3r_{n+m-1}}b_{n+m-2} + b_{n+m-1} \\ &\leq e^{3(r_{n+m-1}+r_{n+m-2})}\|x_{n+m-2} - p\| + e^{3r_{n+m-1}}[b_{n+m-2} + b_{n+m-1}] \\ &\vdots \\ &\leq e^{3\sum_{k=n}^{n+m-1} r_k}\|x_n - p\| + e^{3\sum_{k=n}^{n+m-1} r_k}\sum_{k=n}^{n+m-1} b_k \\ &\leq M\|x_n - p\| + M\sum_{k=n}^{n+m-1} b_k\end{aligned}$$

where $M = e^{3\sum_{k=n}^{n+m-1} r_k}$.

This completes the proof of the lemma. □

Lemma 9. *Let E be a uniformly convex Banach space, and let D be a nonempty closed and convex subset of E. Let T_1, T_2, T_3 be three asymptotically nonexpansive multi-maps from D into $CB(D)$ with the sequence $\{r_{i_n}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} r_{i_n} < \infty$ for all $i = 1, 2, 3$ and $F := \bigcap_{i=1}^{3} F(T_i) \neq \phi$ and $T_i p = \{p\}, (i = 1, 2, 3)$. Let $\{x_n\}$ be the sequence defined by (2.1) with the following restrictions:*

(i) $0 < \alpha \leq \alpha_n, \alpha_n^{'}, \ \alpha_n^{''} \leq 1 - \alpha$ for some $\alpha \in (0, 1)$ and for all $n \geq n_0, \exists \ n_0 \in \mathbb{N}$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n^{'} < \infty$, and $\sum_{n=1}^{\infty} \gamma_n^{''} < \infty$.

Then $\lim_{n\to\infty} \|z_n^{'} - x_n\| = \lim_{n\to\infty} \|w_n^{'} - x_n\| = \lim_{n\to\infty} \|w_n - x_n\| = 0$ for all $n \in \mathbb{N}$.

Proof. Let $p \in F$. It follows from Lemma (8) that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $n \in \mathbb{N}$. Let $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c \geq 0$. For each $n \geq 1$, let $r_n = \max\{r_{i_n} : i = 1, 2, 3\}$. Taking the *limsup* of (3.4), we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = c \tag{3.6}$$

So

$$\begin{aligned}\limsup_{n\to\infty} \|z_n^{'} - p\| &\leq \limsup_{n\to\infty} d(z_n^{'}, T_1^n p) \\ &\leq \limsup_{n\to\infty} H(T_1^n y_n, T_1^n p) \\ &\leq r_n \|y_n - p\| \\ &\leq c\end{aligned} \tag{3.7}$$

Next, we consider

$$\limsup_{n\to\infty} \|z_n^{'} - p + \gamma_n(c_n - x_n)\| \leq \limsup_{n\to\infty} \|z_n^{'} - p\| + \gamma_n \|c_n - x_n\| \tag{3.8}$$

It follows from (3.7) that

$$\limsup_{n\to\infty} \|z_n^{'} - p + \gamma_n(c_n - x_n)\| \leq c \tag{3.9}$$

By the Triangle inequality

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(c_n - x_n)\| \leq c \tag{3.10}$$

Moreover, we note that

$$\begin{aligned}c &= \lim_{n\to\infty} \|x_{n+1} - p\| \\ &= \lim_{n\to\infty} \|\alpha_n z_n^{'} + \beta_n x_n + \gamma_n(c_n - (1 - \alpha_n)p - \alpha_n p)\| \\ &= \lim_{n\to\infty} \|\alpha_n z_n^{'} - \alpha_n p + \alpha_n \gamma_n c_n - \alpha_n \gamma_n x_n + (1 - \alpha_n) x_n \\ &\quad - (1 - \alpha_n)p - \gamma_n x_n + \gamma_n c_n - \alpha_n \gamma_n c_n + \alpha_n \gamma_n x_n\| \\ &= \lim_{n\to\infty} \|\alpha_n (z_n^{'} - p + \gamma_n(c_n - x_n) \\ &\quad + (1 - \alpha_n)(x_n - p + \gamma_n(c_n - x_n)\|\end{aligned} \tag{3.11}$$

It follows from (3.9), (3.10) and Lemma (7) that

$$\lim_{n\to\infty} \|z_n^{'} - x_n\| = 0$$

Next, we prove that $\lim_{n\to\infty} \|w_n^{'} - x_n\| = 0$

For each $n \geq 1$,

$$\begin{aligned} \|x_n - p\| &\leq \|z_n^{'} - x_n\| + \|z_n^{'} - p\| \\ &\leq \|z_n^{'} - x_n\| + d(z_n^{'}, T_1^n p) \\ &\leq \|z_n^{'} - x_n\| + H(T_1^n y_n, T_1^n p) \\ &\leq \|z_n^{'} - x_n\| + r_n\|y_n - p\| \end{aligned} \tag{3.12}$$

Since $\lim_{n\to\infty} \|z_n^{'} - x_n\| = 0 = \lim_{n\to\infty} r_n$, it follows from (3.6) and (3.12) that

$$c = \lim_{n\to\infty} \|x_n - p\| \liminf_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq c \tag{3.13}$$

Hence $\lim_{n\to\infty} \|y_n - p\| = c$

Observe that

$$\|z_n - p\| \leq r_n\|x_n - p\| + \gamma_n^{''}\|v_n - p\|$$

By the boundedness of $\{v_k\}$ and $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} \gamma_n^{''}$, we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq c$$

and so

$$\limsup_{n\to\infty} \|w_n^{'} - p\| \leq \limsup_{n\to\infty} r_n\|z_n - p\| \leq c$$

Next, we consider

$$\|w_n^{'} - p + \gamma_n^{'}(u_n - x_n)\| \leq \|w_n^{'} - p\| + \gamma_n^{'}\|u_n - x_n\| \tag{3.14}$$

Taking *limsup* on both sides, we get

$$\limsup_{n\to\infty} \|w_n^{'} - p + \gamma_n^{'}(u_n - x_n)\| \leq c$$

By the Triangle inequality, we see that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n^{'}(u_n - x_n)\| \leq c$$

Since $\lim_{n\to\infty} \|y_n - p\| = c$, we obtain

$$\begin{aligned} c &= \lim_{n\to\infty} \|x_n - p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{'} w_n^{'} + \beta_n^{'} x_n + \gamma_n^{'} u_n - p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{'}(w_n^{'} - p + \gamma_n^{'}(u_n - x_n)) + (1 - \alpha_n^{'}(x_n - p + \gamma_n^{'}(u_n - x_n))\| \end{aligned} \tag{3.15}$$

By Lemma (7), we obtain $\lim_{n\to\infty} \|w_n^{'} - x_n\| = 0$
Similarly, by using the same argument as in the proof above, we have $\lim_{n\to\infty} \|w_n - x_n\| = 0$, for all $n \in \mathbb{N}$. This completes the proof. □

Theorem 10. *Let E be a uniformly convex Banach space, and let D be a nonempty closed and convex subset of E. Let T_1, T_2, T_3 be three asymptotically nonexpansive multi-maps from D into $CB(D)$ with the sequence $\{r_{i_n}\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} r_{i_n} < \infty$ for all $i = 1, 2, 3$ and $F := \bigcap_{i=1}^{3} F(T_i) \neq \phi$ and $T_i p = \{p\}, (i = 1, 2, 3)$. Let $\{x_n\}$ be the sequence defined by (2.1) with the following restrictions:*

(i) $0 < \alpha \leq \alpha_n, \alpha_n^{'}, \alpha_n^{''} \leq 1 - \alpha$ *for some* $\alpha \in (0, 1)$ *and for all* $n \geq n_0, \exists\ n_0 \in \mathbb{N}$*;*

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n^{'} < \infty$, *and* $\sum_{n=1}^{\infty} \gamma_n^{"} < \infty$.

If $\{T_i\}, (i = 1, 2, 3)$ satisfying Condition (II), then the sequence $\{x_n\}$ converges strongly to a common fixed point of F.

Proof. From Lemma (9), we have

$$\lim_{n\to\infty} \|z_n^{'} - x_n\| = \lim_{n\to\infty} \|w_n^{'} - x_n\| = \lim_{n\to\infty} \|w_n - x_n\| = 0 \tag{3.16}$$

Also,

$$d(x_n, T_3^n x_n) \leq \|w_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty$$

Since $\{x_n\}, \{u_n\}$ are bounded, so is $\{u_n - w_n\}$. Now, let $K = \sup_{n\in\mathbb{N}} \|u_n - w_n\|$. By assumption and (3.16), we get

$$\begin{aligned} \|z_n - w_n\| &\leq \|\alpha_n^{''} w_n + \beta_n^{''} x_n + \gamma_n^{''} u_n - w_n\| \\ &\leq \beta_n^{''} \|x_n - w_n\| + \gamma_n^{''} \|u_n - w_n\| \\ &\leq \beta_n^{''} \|x_n - w_n\| + \gamma_n^{''} K \\ &\to 0 \qquad \text{as} \qquad n \to \infty \end{aligned} \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\|z_n - x_n\| \leq \|z_n - w_n\| + \|w_n - x_n\| \to 0 \qquad \text{as} \qquad n \to \infty \tag{3.18}$$

Again from (3.16) and (3.18), we have

$$\begin{aligned} d(x_n, T_2^n x_n) &\leq d(x_n, T_2^n z_n) + H(T_2^n z_n, T_2^n x_n) \\ &\leq \|x_n - w_n^{'}\| + r_n \|z_n - x_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Now, since $\{y_n\}, \{v_n\}$ are bounded, so is $\{v_n - w_n^{'}\}$. Now, let $K = \sup_{n\in\mathbb{N}} \|v_n - w_n^{'}\|$. By assumption and (3.16), we get

$$\begin{aligned} \|y_n - w_n^{'}\| &\leq \|\alpha_n^{'} w_n^{'} + \beta_n^{'} x_n + \gamma_n^{'} v_n - w_n^{'}\| \\ &\leq \beta_n^{'} \|x_n - w_n^{'}\| + \gamma_n^{'} \|v_n - w_n^{'}\| \\ &\leq \beta_n^{'} \|x_n - w_n^{'}\| + \gamma_n^{'} K \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{3.19}$$

It follows from (3.16) and (3.19) that

$$\|y_n - x_n\| \leq \|y_n - w_n^{'}\| + \|w_n^{'} - x_n\| \to 0 \quad \text{as} \quad n \to \infty \tag{3.20}$$

Therefore from (3.20) and (3.16), we get

$$\begin{aligned} d(x_n, T_1^n x_n) &\le d(x_n, T_1^n y_n) + H(T_1^n y_n, T_1^n x_n) \\ &\le \|x_n - z_n^{'}\| + r_n\|y_n - x_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Now since T_1, T_2, T_3 satisfy Condition (II), we have $d(x_n, F) \to 0$. Thus there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $p_j \subset F$ such that

$$\|x_{n_j} - p_j\| \le 2^{-j} \tag{3.21}$$

Set $M = e^{3\sum_{k=n}^{n+m-1} r_k}$ and write $n_{j+1} = n_j + l$ for some $l \ge 1$. Then we have by (3.5)

$$\begin{aligned} \|x_{n_{j+1}} - p_j\| &= \|x_{n_j+l} - p_j\| \\ &\le M\|x_{n_j} - p_j\| + M \sum_{k=n_j}^{n_j+l-1} b_k \\ &< \frac{M}{2^j} + M \sum_{k=n_j}^{n_j+l-1} b_k \end{aligned}$$

Next we shall show that $\{p_j\}$ is Cauchy sequence in D.
Note that

$$\begin{aligned} \|p_{j+1} - p_j\| &\le \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\ &< \frac{1}{2^{j+1}} + \frac{M}{2^j} + M \sum_{k=n_j}^{n_j+l-1} b_k \\ &< \frac{2M+1}{2^{j+1}} + M \sum_{k=n_j}^{n_j+l-1} b_k \end{aligned}$$

This implies that $\{p_j\}$ is Cauchy sequence in D. Assume that $p_j \to p$ as $j \to \infty$.

Since $d(p_j, T_i^n p) \le H(T_i^n p, T_i^n p_j) \le r_n\|p - p_j\|$ for all $i = 1, 2, 3$ and $p_j \to p$ as $j \to \infty$.
It follows that $d(p_j, T_i^n p) = 0$ for all $i = 1, 2, 3$ and thus $p \in F$. It implies by (3.21) that $\{x_{n_j}\}$ converges strongly to p. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p. This completes the proof of the theorem. □

The main result of this paper holds under the assumption that $Tp = \{p\}$ for all $p \in F$. This condition was introduced by Shahzad and Zegeye [15]. The following examples give an example of nonexpansive multi-map T which satisfies the property that $Tp = \{p\}$ for all $p \in F := \bigcap_{i=1}^{3} F(T_i)$ and Tx is not a singletone for all $x \notin F$.

Example 11. Consider $D = [0, 1] \times [0, 1]$ with the usual norm. Define $T : D \to CB(D)$ by

$$T(x, y) = \begin{cases} \{(x, 0)\}, & \text{if } x \ne 0,\ y = 0; \\ \{(0, y)\}, & \text{if } x = 0,\ y \ne 0; \\ \{(x, 0), (0, y)\}, & \text{if } x, y \ne 0; \\ \{(0, 0)\}, & \text{if } x, y = 0. \end{cases}$$

Example 12. Consider $D = [0, 1]$ with the usual norm. Define $T : D \to CB(D)$ by

$$Tx = \left[\frac{x+1}{2}, 1\right]$$

Example 13. Consider $D = [0, 1] \times [0, 1]$ with the usual norm. Define $T : D \to CB(D)$ by

$$T(x, y) = \{x\} \times \left[\frac{y+1}{2}, 1\right]$$

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