Acta Universitatis Matthiae Belii, series Mathematics
Issue 2014, 45–55, ISSN 1338-7111
Online Edition, http://actamath.savbb.sk

Fuzzy Soft Convex Subalgebras on Residuated Lattices

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Abstract
In this paper, the notions of fuzzy soft subalgebra and fuzzy soft convex subalgebra of a residuated lattice are introduced and some related properties are investigated. Then, we define fuzzy soft congruence on a residuated lattice and obtain the relation between fuzzy soft convex subalgebras and fuzzy soft congruence relations on residuated lattices. The concept of soft homomorphism is defined and some related results are obtained.

Received January 2, 2014
Revised November 6, 2014
Accepted in final form November 9, 2014
Published online November 10, 2014
Communicated with Miroslav Haviar.

Keywords residuated lattice, fuzzy soft convex subalgebra, fuzzy soft congruence relation, soft homomorphism.
MSC(2010) 03G25, 03E72, 08A72, 06D99.

1 Introduction

M. Ward and R. P. Dilworth [12] introduced the concept of residuated lattice in the 1930's. Their investigation stemmed from attempts to generalize properties of the lattice of ideals of a ring. Residuated lattices provide an algebraic semantics for logics without contraction also known as resource sensitive logic. For the previous study of these algebras see [1, 9, 10, 11].

Molodtsov [4] introduced the concept of soft sets in 1999 as a new mathematical tool for dealing with uncertainties. He established the fundamental results of the new theory and successfully applied the soft theory into several directions, such as game theory, theory of probability, smoothness of functions, etc. Maji et al. [5] described first practical application of soft sets in decision making problems which is based on the notion of knowledge reduction in rough set theory. Maji et al. [6] defined and studied several basic notions of soft set theory and several operations on the theory of soft sets.

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [13]. The notion of fuzzy soft sets, as a generalization of the standard soft sets, is introduced in [7], and Roy et al. [8] presented an application of fuzzy soft sets in a decision making problem.

Fuzzy equivalence relations were introduced by Zadeh [14] as a generalization of the concept of an equivalence relation. They have been studied as a way to measure the degree of similarity between the objects of a given universe of discourse. It has been shown that they are useful in different contexts such as fuzzy control, approximate reasoning, fuzzy cluster analysis, etc.

In this paper, we deal with the algebraic structure of residuated lattice by applying the notion of fuzzy soft sets and study fuzzy soft congruence relations.

In Section 2, some basic definitions and results are mentioned. In Section 3, we introduce the notions of fuzzy soft (convex) subalgebras of a residuated lattice and fuzzy soft congruence relation and investigate some related properties. Then we study the relation between them. Finally, we define soft homomorphisms of residuated lattices and study them.

2 Preliminaries

We recall some definitions and theorems which will be needed in this paper.

Definition 1 ([1, 12]). A *residuated lattice* is an algebraic structure $(L, \wedge, \vee, \rightarrow, *, e)$ such that

(1) $(L, \wedge, \vee)$ is a lattice,

(2) $(L, *, e)$ is a commutative monoid where e is a unit element,

(3) $x * y \leq z$ iff $x \leq y \rightarrow z$, for all $x, y, z \in L$.

In the rest of this paper, we denote the residuated lattice $(L, \wedge, \vee, *, \rightarrow, e)$ by L. If e is the greatest element of L, then L is called an integral residuated lattice.

A *fuzzy set* of a non-empty set X is a mapping $\mu : X \rightarrow [0, 1]$. For each $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in X : \mu(x) \geq \alpha\}$ is called *α-level subset* of μ.

Definition 2 ([3]). A fuzzy subset S of a residuated lattice L is called a *fuzzy subalgebra* of L, if

(1) $S(e) \geq S(x)$,

(2) $S(y \rightarrow x) = \min\{S(x), S(y)\}$,

(3) $S(x * y) = \min\{S(x), S(y)\}$,

(4) $S(x \wedge y) = \min\{S(x), S(y)\}$,

(5) $S(x \vee y) = \min\{S(x), S(y)\}$,

for all $x, y \in L$.

Definition 3 ([3]). A fuzzy subalgebra S of a residuated lattice L is said to be a *fuzzy convex subalgebra* of L if $a \in S_\alpha$, $b \in S_\beta$ and $a \leq c \leq b$, then there exists γ between α and β such that $c \in S_\gamma$.

Definition 4 ([14]). Let X, Y be to sets and $\mathfrak{F}(X \times Y)$ be the set of all fuzzy subset of $X \times Y$. Then a fuzzy subset $R \in \mathfrak{F}(X \times Y)$ is called a *fuzzy binary relation* from X to Y and $R(x, y)$ is called the *degree of relation* between x and y, where $(x, y) \in X \times Y$. If $X = Y$, then R is called a *fuzzy relation* on X.

Definition 5 ([14]). A *fuzzy equivalence relation* R on a non-empty set X is a fuzzy relation on X satisfying the following conditions:

(R1) $R(x, x) = \sup\{R(y, z) : y, z \in X\}$ (reflexive),

(R2) $R(x, y) = R(y, x)$ (symmetric),

(R3) $R(x,z) = \min\{R(x,y), R(y,z)\}$, for all $x, y, z \in X$ (transitive).

Definition 6 ([3]). A fuzzy equivalence relation θ on a residuated lattice L is called a *fuzzy congruence relation* on L if

(C1) $\theta(y \to x, w \to z) = \min\{\theta(x,z), \theta(y,w)\}$,

(C2) $\theta(x * y, z * w) = \min\{\theta(x,z), \theta(y,w)\}$,

(C3) $\theta(x \wedge y, z \wedge w) = \min\{\theta(x,z), \theta(y,w)\}$,

(C4) $\theta(x \vee y, z \vee w) = \min\{\theta(x,z), \theta(y,w)\}$,

for all $x, y, z, w \in L$.

Definition 7 ([4]). Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subseteq E$. A pair (F, A) is called a *soft set* over U, where $F : A \to P(U)$ is a map. In other words, a soft set over U is a parametrized family of subsets of the universe U.

Definition 8 ([7]). Let U be an initial universe set and E be a set of parameters. Let $\mathfrak{F}(U)$ denote the set of all fuzzy sets in U. Then (F, A) is called a *fuzzy soft set* over U where $A \subseteq E$ and $F : A \to \mathfrak{F}(U)$ is a map.

In general, for every $u \in A$, $F[u]$ is a fuzzy set in U and it is called *fuzzy value set* of parameter u. If for every $u \in A$, $F[u]$ is a fuzzy subset of U, then (F, A) is degenerated to be the fuzzy soft set. Thus, from the above definition, it is clear that fuzzy soft sets are a generalization of soft sets.

Definition 9. Let X and Y be two non-empty subsets of some universal set U and E be a set of parameters. A pair (R, E) is called a *soft relation* where $R : E \to \mathfrak{F}(X \times Y)$ is a map.

3 Fuzzy soft convex subalgebras

In what follows, let E be a set of parameters unless otherwise specified and L be a residuated lattice.

Definition 10. Let (F, A) be a fuzzy soft set over a residuated lattice L where A is a subset of E. If there exists $u \in A$ such that $F[u]$ is a fuzzy subalgebra of L, we say that (F, A) is a *fuzzy soft subalgebra based on a parameter* u over L. If (F, A) is a fuzzy soft subalgebra based on a parameter u over L for all $u \in A$, we say that (F, A) is a *fuzzy soft subalgebra* over L.

Example 11. Suppose that there are five players in the universe, that is

$$U = \{a, b, c, e, d\}.$$

Let $\Cap$, $\Cup$, $\circledast$, $\twoheadrightarrow$ be four soft game machines for two players to play accordingly in such a way, we have the following results:

$a \Cap x = a$ and $d \Cap x = x$ for all $x \in U$,

$$b \Cap x = \begin{cases} a & if \quad x = a \\ b & if \quad x \in \{b, c, e, d\} \end{cases} \quad c \Cap x = \begin{cases} x & if \quad x \in \{a, b\} \\ c & if \quad x \in \{c, e, d\} \end{cases}$$

$$e \Cap x = \begin{cases} x & if \quad x \in \{a,b,c\} \\ e & if \quad x \in \{e,d\} \end{cases}$$

$a \Cup x = x$ and $d \Cup x = d$ for all $x \in U$,

$$b \Cup x = \begin{cases} b & if \quad x \in \{a,b\} \\ x & if \quad x \in \{c,e,d\} \end{cases} \quad c \Cup x = \begin{cases} c & if \quad x \in \{a,b,c\} \\ x & if \quad x \in \{e,d\} \end{cases}$$

$$e \Cup x = \begin{cases} e & if \quad x \in \{a,b,c,e\} \\ d & if \quad x = d \end{cases}$$

$a \circledast x = a$ and $e \circledast x = x$ for all $x \in U$,

$$c \circledast x = \begin{cases} a & if \quad x = a \\ b & if \quad x = b \\ c & if \quad x \in \{c,e\} \\ d & if \quad x = d \end{cases} \quad d \circledast x = \begin{cases} a & if \quad x = a \\ b & if \quad x = b \\ d & otherwise \end{cases}$$

$$b \circledast x = \begin{cases} a & if \quad x = a \\ b & otherwise \end{cases}$$

$a \twoheadrightarrow x = d$ and $e \twoheadrightarrow x = x$ for all $x \in U$,

$$c \twoheadrightarrow x = \begin{cases} a & if \quad x = a \\ b & if \quad x = b \\ e & if \quad x \in \{b,e\} \\ d & if \quad x = d \end{cases} \quad d \twoheadrightarrow x = \begin{cases} a & if \quad x = a \\ d & if \quad x = d \\ b & otherwise \end{cases}$$

$$b \twoheadrightarrow x = \begin{cases} a & if \quad x = a \\ d & otherwise \end{cases}$$

Then $(U, \Cap, \Cup, \circledast, \twoheadrightarrow, e)$ is a residuated lattice. Consider a set of parameters:

$$E = \{Clever, Agile\}.$$

(1) Let (F,E) be a fuzzy soft set over U. Then $F[Clever]$ and $F[Agile]$ are fuzzy sets in U. Define them as follows:

F	a	b	c	e	d
Clever	0.4	0.8	0.5	0.9	0.8
Agile	0.2	0.3	0.8	0.6	0.7

Then (F,E) is a fuzzy soft subalgebra based on a parameter "Clever" over U but it is not a fuzzy soft subalgebra based on a parameter "Agile" over U. Hence (F,E) is not a fuzzy soft subalgebra over U.

(2) Let (G,E) be a fuzzy soft set over U. Then $G[Clever]$ and $G[Agile]$ are fuzzy sets in U. Define them as follows:

G	a	b	c	e	d
Clever	0.5	0.7	0.5	1	0.7
Agile	0.1	0.3	0.2	0.4	0.3

Then $G[Clever]$ and $G[Agile]$ are fuzzy soft subalgebras based on parameters "Clever" and "Agile" over U, respectively. Hence (G,E) is a fuzzy soft subalgebra over U.

Definition 12. Let (F, A) be a fuzzy soft subalgebra over a residuated lattice L where A is a subset of E. If there exists $u \in A$ such that $F[u]$ is a fuzzy convex subalgebra of L, we say that (F, A) is a *fuzzy soft convex subalgebra based on a parameter* u over L. If (F, A) is a fuzzy soft convex subalgebra based on a parameter u over L for all $u \in A$, we say that (F, A) is a *fuzzy soft convex subalgebra* over L.

Example 13. Suppose that there are five players in the universe, that is

$$U = \{a, b, c, e, d\}.$$

Let $\Cap$, $\Cup$, $\divideontimes$, $\twoheadrightarrow$ be four soft game machines for two players to play accordingly in such a way, we have the following results:

$a \Cap x = a$, $d \Cap x = x$, for all $x \in U$,

$$b \Cap x = \begin{cases} a & if \quad x \in \{a, c\} \\ b & \quad otherwise \end{cases} \quad c \Cap x = \begin{cases} a & if \quad x \in \{a, b\} \\ c & \quad otherwise \end{cases}$$

$$e \Cap x = \begin{cases} x & if \quad x \in \{e, d\} \\ b & \quad otherwise \end{cases}$$

$a \Cup x = x$, $d \Cup x = d$ for all $x \in U$,

$$b \Cup x = \begin{cases} b & if \quad x \in \{a, b\} \\ e & if \quad x \in \{c, e\} \\ d & if \quad x = d \end{cases} \quad c \Cup x = \begin{cases} c & if \quad x \in \{a, c\} \\ e & if \quad x \in \{e, b\} \\ d & if \quad x = d \end{cases}$$

$$e \Cup x = \begin{cases} d & if \quad x = d \\ x & \quad otherwise \end{cases}$$

$a \divideontimes x = a$ and $e \divideontimes x = x$ for all $x \in U$,

$$b \divideontimes x = \begin{cases} a & if \quad x \in \{a, c\} \\ b & if \quad otherwise \end{cases} \quad c \divideontimes x = \begin{cases} a & if \quad x \in \{a, b\} \\ c & if \quad otherwise \end{cases}$$

$$d \divideontimes x = \begin{cases} d & if \quad x \in \{e, d\} \\ x & if \quad otherwise \end{cases}$$

$a \twoheadrightarrow x = d$ and $e \twoheadrightarrow x = x$ for all $x \in U$,

$$b \twoheadrightarrow x = \begin{cases} c & if \quad x \in \{a, c\} \\ d & if \quad otherwise \end{cases} \quad c \twoheadrightarrow x = \begin{cases} b & if \quad x \in \{a, b\} \\ d & if \quad otherwise \end{cases}$$

$$d \twoheadrightarrow x = \begin{cases} a & if \quad x = a \\ b & if \quad x = b \\ c & if \quad x = c \\ d & \quad otherwise \end{cases}$$

Then $(U, \Cap, \Cup, \divideontimes, \twoheadrightarrow, e)$ is a residuated lattice. Consider a set of parameters:

$$E = \{Clever, Smart\}.$$

(1) Let (F, E) be a fuzzy soft set over U. Then $F[Clever]$ and $F[Smart]$ are fuzzy sets in U. Define them as follows:

F	a	b	c	e	d
$Clever$	0.5	0.5	0.6	0.7	0.7
$Smart$	0.3	0.5	0.1	0.8	0.7

Then (F, E) is a fuzzy soft convex subalgebra on the parameter "Clever" over U but it is not a fuzzy soft convex subalgebra based on the parameter "Smart" over U. Hence (F, E) is not a fuzzy soft convex subalgebra over U.

(2) Consider fuzzy soft subalgebra (G, E) over U where $G[Clever]$ and $G[Agile]$ are fuzzy sets in U:

G	a	b	c	e	d
$Clever$	0.3	0.3	0.65	0.82	0.7
$Smart$	0.11	0.223	0.11	0.4	0.34

Then $G[Clever]$ and $G[Smart]$ are fuzzy soft convex subalgebras of residuated lattice based on parameters "Clever"and "Smart" over U, respectively. Hence (G, E) is a fuzzy soft convex subalgebra over U.

Remark 14. We notice that each fuzzy soft subalgebra may not be a fuzzy soft convex subalgebra of a residuated lattice. Consider a fuzzy soft subalgebra (G, E) over U in Example 3.2 which is not a fuzzy soft convex subalgebra over U.

Definition 15 ([5])**.** Let (F, A) and (G, B) be two fuzzy soft sets over a common universe U. The *union* of (F, A) and (G, B) is defined to be the fuzzy soft set (H, C) satisfying the following conditions:

(i) $C = A \cup B$,

(ii) for all $u \in C$, $H[u] = \begin{cases} F[u] & if \quad u \in A \backslash B \\ G[u] & if \quad u \in B \backslash A \\ F[u] \cup G[u] & if \quad u \in A \cap B \end{cases}$

where $F[u] \cup G[u]$ is union of fuzzy sets. In this case, we write $(F, A) \sqcup (G, B) = (H, C)$.

Theorem 16. *Let (F, A) and (G, A) be two fuzzy soft convex subalgebras over a residuated lattice L. If A and B are disjoint, then the union $(F, A) \sqcup (G, B)$ is a fuzzy soft convex subalgebra over L.*

Proof. Suppose that $(F, A) \sqcup (G, B) = (H, C)$, where $C = A \cup B$ and for all $u \in C$,

$$H[u] = \begin{cases} F[u] & if \quad u \in A \backslash B \\ G[u] & if \quad u \in B \backslash A \\ F[u] \cup G[u] & if \quad u \in A \cap B \end{cases}$$

By assumption, $A \cap B = \emptyset$. Hence we have either $u \in A \backslash B$ or $u \in B \backslash A$ for all $u \in C$. Consider the following cases:

(1) If $u \in A \backslash B$, then $H[u] = F[u]$ is a fuzzy convex subalgebra over L because (F, A) is a fuzzy soft convex subalgebra over L.

(2) If $u \in B \backslash A$, then $H[u] = G[u]$ is a fuzzy convex subalgebra over L because (G, A) is a fuzzy soft convex subalgebra of over L.

Therefore $(H, C) = (F, A) \sqcup (G, B)$ is a fuzzy soft convex subalgebra over L. □

The following example shows that Theorem 16 is not valid, if A and B are not disjoint.

Example 17. Let $(U, \Cap, \Cup, \circledast, \twoheadrightarrow, e)$ be the residuated lattice in Example 13. Consider two sets of parameters:

$$A = \{Attentive, Brave\},\ B = \{Attentive\}$$

Then A and B are not disjoint.

Let (F, A) be a fuzzy soft set over U. Then $F[Attentive]$, $F[Brave]$ are fuzzy sets in U. Define them as follows:

F	a	b	c	e	d
Smart	0.2	0.4	0.2	0.7	0.5
Attentive	0.1	0.1	0.4	0.6	0.5

Then (F, A) is a fuzzy soft convex subalgebra over U.

Let (G, B) be a fuzzy soft set over U. Then $G[Attentive]$ is a fuzzy set in U. Define it as follows:

G	a	b	c	e	d
Attentive	0.3	0.5	0.3	0.8	0.7

Then (G, B) is a fuzzy soft convex subalgebra over U.

But the union (F, A) and (G, B) is not a fuzzy soft convex subalgebra over U. Suppose that $u = Attentive$. Then

$$\begin{gathered}(F[u] \cup G[u])(b \circledast c) = (F[u] \cup G[u])(a) = max\{F[u](a), G[u](a)\} = 0.3 \\ \min\{(F[u] \cup G[u])(b), (F[u] \cup G[u])(c)\} = 0.4\end{gathered}$$

but $0.3 \not\geq 0.4$.

Definition 18 ([5])**.** Let (F, A) and (G, B) be two fuzzy sets over a common universe U. The *extended intersection* of (F, A) and (G, B) is defined to be the fuzzy soft set (H, C) satisfying the following conditions:

(i) $C = A \cup B$

(ii) for all $u \in C$,

$$H[u] = \begin{cases} F[u] & if \quad u \in A \backslash B \\ G[u] & if \quad u \in B \backslash A \\ F[u] \cap G[u] & if \quad u \in A \cap B \end{cases}$$

where $F[u] \cap G[u]$ is intersectin of fuzzy sets. In this case, we write $(F, A) \sqcap_e (G, B) = (H, C)$.

Theorem 19. *Let (F, A) and (G, B) be two fuzzy soft convex subalgebras over a residuated lattice L. Then the extended intersection of (F, A) and (G, B) is a fuzzy soft convex subalgebra over L.*

Proof. **Proof.** Let $(F, A) \sqcap_e (G, B) = (H, C)$ be the extended intersection of (F, A) and (G, B). We have $C = A \cup B$. Suppose that $u \in C$ be arbitrary.

(i) if $u \in A \backslash B$, then $H[u] = F[u]$ is a fuzzy convex subalgebra over L,

(ii) if $u \in B \backslash A$, then $H[u] = G[u]$ is a fuzzy convex subalgebra over L,

(iii) if $A \cap B \neq \emptyset$, then $H[u] = F[u] \cap G[u]$ is a fuzzy convex subalgebra for all $u \in A \cap B$, because the intersection of two fuzzy convex subalgebras in L is an fuzzy convex subalgebra. Therefore (H, C) is a fuzzy soft convex subalgebra over L.

□

Corollary 20. *Let (F, A) and (G, A) be two fuzzy soft convex subalgebras over a residuated lattice L. Then the extended intersection of (F, A) and (G, A) is a fuzzy soft convex subalgebra over L.*

Definition 21 ([5])**.** Let (F, A) and (G, B) be two fuzzy soft sets over a common universe U such that $A \cap B \neq \emptyset$. The *restricted intersection* of (F, A) and (G, B) is defined to be the fuzzy soft set (H, C) satisfying the following conditions:

(i) $C = A \cap B$,

(ii) $H[u] = F[u] \cap G[u]$ for all $u \in C$.

In this case, we write $(F, A) \sqcap (G, B) = (H, C)$.

Corollary 22. *The restricted intersection of two fuzzy soft convex subalgebras over a residuated lattice L is a fuzzy soft convex subalgebra over L.*

Notation The set of all fuzzy soft convex subalgebras over a residuated lattice L is denoted by $FSC(L)$.

Clearly $FSC(L)$ is a lattice, because if $(F, A), (G, B) \in FSC(L)$, then $(F, A) \bigvee (G, B)$ (i.e. the intersection of all fuzzy soft convex subalgebras containing $(F, A), (G, B)$) is the least upper bound of (F, A) and (G, B). Also, $(F, A) \sqcap_e (G, B) \in FSC(L)$ is the greatest lower bound of (F, A) and (G, B). Since we can replace the set (F, A) and (G, B) by an arbitrary family of fuzzy soft convex subalgebras, so the lattice $(FSC(L), \sqcap, \bigvee)$ is a complete lattice.

Definition 23. Let (R, A) be a fuzzy soft relation on a residuated lattice L where A be a subset of E.

(1) If $R[u]$ is a fuzzy equivalence relation on L for all $u \in A$, we say that (R, A) is a *fuzzy soft equivalence relation* over L.

(2) A fuzzy soft equivalence relation (θ, A) over L is called *fuzzy soft congruence relation*, if $\theta[u]$ is a fuzzy congruence relation on L for all $u \in A$.

Example 24. Consider the residuated lattice which is defined in Example 11. Suppose that $A = \{Smart\}$ and $u = Smart$. Define

$$\theta[u](a, b) = \theta[u](a, d) = \theta[u](a, c) = \theta[u](a, e) = 0.2,$$
$$\theta[u](d, b) = \theta[u](b, c) = \theta[u](e, d) = \theta[u](b, e) = \theta[u](c, d) = 0.45,$$
$$\theta[u](c, e) = 0.67,$$
$$\theta[u](e, e) = \theta[u](a, a) = \theta[u](b, b) = \theta[u](c, c) = \theta[u](d, d) = 0.87.$$

Then (θ, A) is a fuzzy soft congruence relation on L.

Definition 25. Let (F, A) be a fuzzy soft convex subalgebras over L. Fuzzy soft relation (θ_F, A) on L which is defined by

$$\theta_F[u](x, y) = min\{F[u]((y \to x) \wedge e), F[u]((x \to y) \wedge e)\}$$

is called the *fuzzy soft relation induced* by (F, A).

Proposition 26. *Let (F, A) be a fuzzy soft convex subalgebras over a residuated lattice L. Then the fuzzy soft relation (θ_F, A) induced by (F, A) is a fuzzy soft congruence relation on L.*

Proof. By assumption (F, A) is a fuzzy soft congruence relation on L. Thus $F[u]$ is a fuzzy convex subalgebra of L for all $u \in A$. By Theorem 3.19 in [3],

$$min\{F[u]((y \to x) \wedge e), F[u]((x \to y) \wedge e)\}$$

is a fuzzy congruence relation on L for all $u \in A$. We get that $\theta_F[u]$ is a fuzzy congruence relation on L for all $u \in A$. Therefore (θ_F, A) is a fuzzy soft congruence relation on L. □

Definition 27. Let (θ, A) be a fuzzy soft congruence relation on a residuated lattice L. Then the fuzzy subset (F_θ, A) which is defined by

$$F_\theta[u](x) = \theta[u](x, e)$$

is called the *fuzzy soft subset induced* by (θ, A).

Proposition 28. *Let (θ, A) be a fuzzy soft congruence relation on a residuated lattice L. Then the fuzzy soft subset (F_θ, A) induced by (θ, A) is a fuzzy soft convex subalgebra over L.*

Proof. Since (θ, A) is a fuzzy soft congruence relation on L, then $\theta[u]$ is a fuzzy congruence relation on L for all $u \in A$. By Theorem 3.22 in [3], $\theta[u](x, e)$ is a fuzzy convex subalgebra of L for all $u \in A$. Hence (F_θ, A) is a fuzzy soft convex subalgebra over L. □

Theorem 29. *There is a bijection between the set of all fuzzy soft convex subalgebras over a residuated lattice L and the set of all fuzzy soft congruence relations on L.*

Proof. It follows from Proposition 26 and Proposition 28. □

Definition 30. Let (θ, A) be a fuzzy soft congruence relation on a residuated lattice L, $u \in A$ and $x \in L$. Define the fuzzy set $[x]_{\theta[u]}$ by $[x]_{\theta[u]}(y) = \theta[u](x, y)$. The fuzzy set $[x]_{\theta[u]}$ is called a *fuzzy soft congruence class* of x by $\theta[u]$ in L.

Proposition 31. *Let (θ, A) be a soft congruence relation over a residuated lattice L and $u \in A$. Define $L/\theta[u] = \{[x]_{\theta[u]} : x \in L\}$. Then $(L/\theta[u], \wedge, \vee, *, \to, [e]_{\theta[u]})$ is a residuated lattice where*

$$_{\theta[u]} \wedge [y]_{\theta[u]} = [x \wedge y]_{\theta[u]}$$
$$[x]_{\theta[u]} \vee [y]_{\theta[u]} = [x \vee y]_{\theta[u]}$$
$$[x]_{\theta[u]} * [y]_{\theta[u]} = [x * y]_{\theta[u]}$$
$$[x]_{\theta[u]} \to [y]_{\theta[u]} = [x \to y]_{\theta[u]}$$

for all $x, y \in L$.

Definition 32. Let (F, A) and (G, B) be two fuzzy soft subalgebras over residuated lattices L and M respectively and let $f : L \to M$ and $g : A \to B$ be two functions. Then a pair (f, g) is called a *soft homomorphism* , if the following conditions hold:

(1) f is a residuated lattice homomorphism from L to M,

(2) $f(F[u]) = G[g(u)]$ for all $u \in A$.

Then we say that (F, A) is a *soft homomorphic* to (G, B).

Proposition 33. *Let L and M be two residuated lattices, (F, A) be a fuzzy soft subalgebra over L and let $f : L \to M$ be a surjective homomorphism of residuated lattices. Then*

(1) $(f(F), A)$ is a fuzzy soft subalgebra over M.

(2) If L is an integral residuated lattice, then $(f(F), A)$ is a fuzzy soft convex subalgebra over M.

Proof. (1) Since (F, A) is a fuzzy soft subalgebra over L, then $F[u]$ is a fuzzy subalgebra in L for all $u \in L$. Hence $f(F) : A \to \mathfrak{F}(M)$ is a mapping given by

$$f(F)[u](y) = \begin{cases} \sup_{y=f(x)} F[u](x) & if \quad f^{-1}(G) \neq \emptyset \\ 0 & if \quad f^{-1}(G) = \emptyset \end{cases}$$

for all $u \in A$. We will show that $f(F)[u]$ is a fuzzy subalgebra of M for all $u \in A$.

Suppose that $\star \in \{\wedge, \vee, *, \to\}$. Let $x, y \in M$. Since f is onto, there exist $a, b \in L$ such that $f(a) = x$ and $f(y) = b$. Since f is a homomorphism, $x \star y = f(a) \star f(b) = f(a \star b)$. Hence $a \star b \in f^{-1}(x \star y)$. We have

$$\begin{aligned} f(F)[u](x \star y) &= \sup\{f(F)[u](z) : z \in f^{-1}(x \star y)\} \\ &\geq \sup\{f(F)[u](a \star b) : a \in f^{-1}(x), b \in f^{-1}(y)\} \\ &\geq \sup\{\min\{f(F)[u](a), f(F)[u](b)\} : a \in f^{-1}(x) \text{ and } b \in f^{-1}(y)\} \\ &= \min\{\sup\{f(F)[u](a) : a \in f^{-1}(x)\}, \sup\{f(F)[u](b) : b \in f^{-1}(y)\}\} \\ &= \min\{f(F)[u](x), f(F)[u](y)\}. \end{aligned}$$

Hence $f(F)[u]$ is a fuzzy subalgebra of M, for all $u \in A$.

(2) Now, suppose that $a \in f(F)[u]_\alpha$, $b \in f(F)[u]_\beta$ and $a \leq c \leq b$. We will show that $f(F)[u](a) \leq f(F)[u](c)$.

Suppose that $f(F)[u](a) > f(F)[u](c)$. Then there exists $a_0 \in L$ such that $f(a_0) = a$ and $F[u](a_0) > \sup\{F[u](y) : y \in f^{-1}(c)\}$. We have $f(a_0) = a \leq c = f(y)$ for all $y \in f^{-1}(c)$. Let $y \in f^{-1}(c)$ be arbitrary. Since f is a homomorphism, then $c = a \vee c = f(a_0) \vee f(y) = f(a_0 \vee y)$, that is $a_0 \vee y \in f^{-1}(c)$. We get that $F[u](a_0 \vee y) < F[u](a_0)$. Since L is an integral commutative residuated lattice, then $F[u](a_0 \vee y) \geq \max\{F[u](a_0), F[u](y)\} = F[u](a_0)$ which is a contradiction. Hence $f(F)[u](a) \leq f(F)[u](c)$.

Similarly, we can prove that $f(F)[u](c) \leq f(F)[u](b)$. Hence we have $f(F)[u](a) \leq f(F)[u](c) \leq f(F)[u](b)$. Put $\gamma_1 = f(F)[u](c)$ and $\gamma = \min\{\gamma_1, \beta\}$. We have $\alpha \leq \gamma \leq \beta$ and $c \in f(F)[u]_\gamma$. Therefore $f(F)[u]$ is a fuzzy convex subalgebra of M, for all $u \in A$. So $(f(F), A)$ is a fuzzy soft convex subalgebra over M. □

Proposition 34. *Let L and M be two residuated lattices, (G, A) be a fuzzy soft subalgebra over residuated lattice M and let $f : L \to M$ be a homomorphism of residuated lattices. Then*

(1) $(f^{-1}(G), A)$ is a fuzzy soft subalgebra over L.

(2) If M is an integral residuated lattice, then $(f^{-1}(G), A)$ is a fuzzy soft convex subalgebra over L.

Proof. (1) By assumption (G, A) is a fuzzy soft subalgebra over M, then $G[u]$ is a fuzzy subalgebra in M for all $u \in M$. Since f is a homomorphism, then $f^{-1}(G[u])$ is a fuzzy set in L where $f^{-1}(G[u])(x) = G[u](f(x))$. We have $f^{-1}(G) : A \to \mathfrak{F}(L)$. Let $a, b \in L$ and $\star \in \{\wedge, \vee, *, \to\}$. We have

$$f^{-1}(G[u])(a \star b) = G[u](f(a) * f(b)) \geq \min\{G[u](f(a)), G[u](f(b))\}$$
$$= \min\{f^{-1}(G[u])(a), f^{-1}(G[u])(b)\}.$$

Then $(f^{-1}(G), A)$ is a fuzzy soft subalgebra over L.

(2) Now, suppose that $a \in f^{-1}(G[u])_\alpha$, $b \in f^{-1}(G[u])_\beta$ and $a \leq c \leq b$. Since f is a homomorphism, $f(a) \leq f(c) \leq f(b)$. By assumption M is an integral commutative residuated lattice, then we get that $G[u](f(a)) \leq G[u](f(c)) \leq G[u](f(b))$, that is $f^{-1}(G[u])(a) \leq f^{-1}(G[u])(c) \leq f^{-1}(G[u])(b)$. Put $\gamma_1 = f^{-1}(G[u])(c)$ and $\gamma = \min\{\gamma_1, \beta\}$. We have $\alpha \leq \gamma \leq \beta$ and $c \in f^{-1}(G[u])_\gamma$. Therefore $f^{-1}(G[u])$ is a fuzzy convex subalgebra of L, for all $u \in A$. Hence $(f^{-1}(G), A)$ is a fuzzy soft convex subalebra over L. □

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