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Certain results on a class of Entire functions represented by Dirichlet series having complex frequencies

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Abstract
Consider F to be a class of entire functions represented by Dirichlet series with complex frequencies for which $(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_k|$ is bounded. A study on certain results has been made for this set that is F is proved to be an algebra with continuous quasi-inverse, commutative Banach algebra with identity etc. Moreover, the conditions for the elements of F to possess an inverse, quasi-inverse and the form of spectrum of F are also established.

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1 Introduction

Consider a Dirichlet series of the form

$$f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k, z>}, \qquad z \in \mathbb{C}^n \tag{1.1}$$

where $\{\lambda^k\}$; $\lambda^k = (\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k)$, $k = 1, 2, \ldots$ be a sequence of complex vectors in $\mathbb{C}^n$. Then $< \lambda^k, z >= \lambda_1^k z_1 + \lambda_2^k z_2 + \ldots + \lambda_n^k z_n$. If $a_k's \in \mathbb{C}$ and $\{\lambda^k\}'s$ satisfy the condition $|\lambda^k| \to \infty$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \frac{\log |a_k|}{|\lambda^k|} = -\infty \tag{1.2}$$

$$\limsup_{k\to\infty} \frac{\log\, k}{|\lambda^k|} = D < \infty \tag{1.3}$$

then from [1] the Dirichlet series (1.1) represents an entire function. In this paper let F be the set of series (1.1) for which $(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_k|$ is bounded where $c_1, c_2 \geq 0$ and

c_1, c_2 are simultaneously not zero. Then every element of F represents an entire function. If

$$f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k, z>} \qquad \text{and} \qquad g(z) = \sum_{k=1}^{\infty} b_k\, e^{<\lambda^k, z>}$$

define binary operations that is addition and scalar multiplication in F as

$$f(z) + g(z) = \sum_{k=1}^{\infty} (a_k + b_k)\, e^{<\lambda^k, z>},$$

$$\alpha.f(z) = \sum_{k=1}^{\infty} (\alpha.a_k)\, e^{<\lambda^k, z>},$$

$$f(z).g(z) = \sum_{k=1}^{\infty} \{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k\, b_k\}\, e^{<\lambda^k, z>}.$$

The norm in F is defined as follows

$$\|f\| = \sup_{k \geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_k|. \tag{1.4}$$

If $c_1 = c_2 = 1$ we get the norm as defined in [3] for a class of entire functions represented by Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad s = \sigma + it, \qquad (\sigma, t \in \mathbb{R}) \tag{1.5}$$

whose coefficients belonged to a commutative Banach algebra with identity and $\lambda_n{}'s \in \mathbb{R}$ satisfied the condition $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots < \lambda_n \ldots; \lambda_n \to \infty$ as $n \to \infty$. Further in the same paper authors proved the above class to be a complex FK-space and a Fréchet space. Several other results for a different class of entire Dirichlet series (1.5) may be found in [2].

In the present paper the weighted norm is generalized and various results based on the notions of Banach algebra, Quasi-inverse, Algebra with continuous quasi-inverse, Spectrum of a set have been established.

In the sequel following definitions are required to prove the main results.

Definition 1. A function $g(z) \in F$ is said to be a quasi-inverse of $f(z) \in F$ if $f(z) * g(z) = 0$ where

$$f(z) * g(z) = f(z) + g(z) + f(z).g(z).$$

Definition 2. A topological algebra F is said to be an algebra with continuous quasi-inverse if there exists a neighbourhood of the zero element, every point f of which has a quasi-inverse f' and the mapping $f \to f'$ is continuous.

Definition 3. The set $\sigma(A)$ defined as

$$\sigma(A) = \{k \in K : A - kI \text{ is not invertible}\}$$

is called the spectrum of A.

2 Main Results

In this section main results are proved. For the definitions of terms used refer [4] and [5].

Theorem 4. *An element* $f(z) = \sum_{k=1}^{\infty} a_k \, e^{<\lambda^k, z>} \in F$ *is quasi-invertible if and only if*

$$\inf_{k \geq 1} \{|1 + (k!)^{c_1} \, e^{c_2 k |\lambda^k|} \, a_k|\} > 0. \tag{2.1}$$

The quasi-inverse of $f(z)$ *is the function* $g(z) = \sum_{k=1}^{\infty} b_k \, e^{<\lambda^k, z>}$ *where*

$$b_k = \frac{-a_k}{1 + (k!)^{c_1} \, e^{c_2 k |\lambda^k|} \, a_k}. \tag{2.2}$$

Proof. Let $f(z) \in F$ be quasi-invertible. By Definition 1, there exists $g(z) \in F$ such that $f(z) * g(z) = 0$. This implies

$$a_k + b_k + (k!)^{c_1} \, e^{c_2 k |\lambda^k|} \, a_k \, b_k = 0$$

for all $k \geq 1$. Let (2.1) does not hold that is

$$\inf_{k \geq 1} \{|1 + (k!)^{c_1} \, e^{c_2 k |\lambda^k|} \, a_k|\} = 0. \tag{2.3}$$

There exists a subsequence $\{k_t\}$ of a sequence of indices $\{k\}$ such that $\|f_t\| = 1$ that is

$$(k_t!)^{c_1} \, e^{c_2 k_t |\lambda^{k_t}|} \, |a_{k_t}| = 1 \;\text{ as }\; t \to \infty. \tag{2.4}$$

Thus

$$(k_t!)^{c_1} \, e^{c_2 k_t |\lambda^{k_t}|} \, |b_{k_t}| = \frac{(k_t!)^{c_1} \, e^{c_2 k_t |\lambda^{k_t}|} \, |a_{k_t}|}{|1 + (k_t!)^{c_1} \, e^{c_2 k_t |\lambda^{k_t}|} \, a_{k_t}|}$$

Using (2.3) and (2.4),

$$\|g_t\| \to \infty \;\text{ as }\; t \to \infty$$

which is a contradiction.
Conversely let (2.1) be fulfilled. The function $g(z)$ defined by (2.2) obviously belongs to F. Thus

$$\begin{aligned} f(z) * g(z) &= \sum_{k=1}^{\infty} \{a_k + b_k + (k!)^{c_1} \, e^{c_2 k |\lambda^k|} \, a_k \, b_k\} \, e^{<\lambda^k, z>} \\ &= 0. \end{aligned}$$

Thus $f(z)$ is quasi-invertible which completes the proof of the theorem. □

Theorem 5. *F is an algebra with continuous quasi-inverse.*

Proof. Let $N_\epsilon(0)$ be an ϵ-neighbourhood of 0 where $0 < \epsilon < 1$. Let $p(z) \in N_\epsilon(0)$ where $p(z) = \sum_{k=1}^{\infty} p_k \, e^{<\lambda^k, z>}$. This implies $\|p\| < \epsilon$. Then

$$(k!)^{c_1} \, e^{c_2 k |\lambda^k|} \, |p_k| < \epsilon \;\text{ for all }\; k \geq 1$$

which further implies

$$\inf_{k\geq 1} \{|1 + (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, p_k|\} \geq 1 - \epsilon > 0.$$

Hence by Theorem 4, $p(z)$ possesses a quasi-inverse say $q(z) = \sum_{k=1}^{\infty} q_k\, e^{<\lambda^k, z>}$ where

$$q_k = \frac{-p_k}{1 + (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, p_k}.$$

Now

$$\begin{aligned} \|q\| &= \sup_{k\geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |q_k| \\ &= \sup_{k\geq 1} \frac{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |p_k|}{|1 + (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, p_k|} \\ &< \frac{\epsilon}{1-\epsilon}. \end{aligned}$$

Hence the mapping $p(z) \to q(z)$ is continuous. Thus by Definition 2, F is an algebra with continuous quasi-inverse. Thus the theorem is proved. □

Theorem 6. *F is a commutative Banach algebra with identity.*

Proof. To prove the theorem we first show that F is complete under the norm defined by (1.4). Let $\{f_{m_1}\}$ be a cauchy sequence in F. For given $\epsilon > 0$ we find m such that

$$\|f_{m_1} - f_{m_2}\| < \epsilon \ \text{ where } \ m_1, m_2 \geq m.$$

This implies that

$$\sup_{k\geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_{m_{1k}} - a_{m_{2k}}| < \epsilon \ \text{ where } \ m_1, m_2 \geq m.$$

Clearly $\{a_{m_{1k}}\}$ forms a cauchy sequence in the set of complex numbers for all $k \geq 1$ and thus converges to a_k. Therefore $f_{m_1} \to f$. Also

$$\begin{aligned} \sup_{k\geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_k| \leq{}& \sup_{k\geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_{m_{1k}} - a_k| \\ &+ \sup_{k\geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_{m_{1k}}| \end{aligned}$$

Hence $f(z) \in F$. Thus F is complete. Now if $f(z), g(z) \in F$ then

$$\begin{aligned} \|f.g\| &= \sup_{k\geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |\,(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k\, b_k| \\ &\leq \sup_{k\geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |a_k|\, .\sup_{k\geq 1} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |b_k| \\ &= \|f\|.\|g\| \end{aligned}$$

The identity element in F is

$$e(z) = \sum_{k=1}^{\infty} (k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, e^{<\lambda^k, z>}.$$

Hence the theorem. □

Theorem 7. *The function* $f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k, z>}$ *is invertible in* F *if and only if*

$$\{|\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|\}$$

is a bounded sequence.

Proof. Let $f(z)$ be invertible and $g(z) = \sum_{k=1}^{\infty} b_k\, e^{<\lambda^k, z>}$ be its inverse. Then

$$(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k\, b_k = (k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}$$

Equivalently

$$(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |b_k| = |\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|$$

Clearly since $g(z) \in F$ hence

$$\{|\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|\}$$

is a bounded sequence.
Conversely suppose $\{|\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|\}$ be a bounded sequence. Define $g(z)$ such that

$$g(z) = \sum_{k=1}^{\infty} (k!)^{-2c_1}\, e^{-2c_2 k|\lambda^k|}\, a_k^{-1}\, e^{<\lambda^k, z>}.$$

Obviously $g(z) \in F$. Moreover

$$\begin{aligned} f(z).g(z) &= \sum_{k=1}^{\infty} (k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, \{a_k\, (k!)^{-2c_1}\, e^{-2c_2 k|\lambda^k|}\, a_k^{-1}\}\, e^{<\lambda^k, z>} \\ &= e(z). \end{aligned}$$

Hence the proof of the theorem is completed. □

Theorem 8. *The spectrum* $\sigma(f)$ *where* $f(z) \in F$ *is precisely of the form*

$$\sigma(f) = cl\{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k : k \geq 1\}.$$

Proof. In Theorem 7, $f(z) = \sum_{k=1}^{\infty} a_k\, e^{<\lambda^k, z>} \in F$ is invertible if and only if

$$\{|\,(k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, a_k^{-1}|\}$$

is a bounded sequence. Thus $\{f(z) - \lambda.e(z)\}$ is not invertible if and only if

$$\{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, |\, a_k - \lambda\, (k!)^{-c_1}\, e^{-c_2 k|\lambda^k|}\, |\}^{-1}$$

is not bounded. Therefore by Definition 3, this is possible if and only if there exists a subsequence $\{k_n\}$ of a sequence of indices $\{k\}$ such that

$$|(k_n!)^{c_1}\, e^{c_2 k_n|\lambda^{k_n}|}\, a_{k_n} - \lambda\,|$$

tends to zero as $n \to \infty$. Equivalently

$$\lambda \in cl\{(k!)^{c_1}\, e^{c_2 k|\lambda^k|}\, a_k : k \geq 1\}$$

which proves the theorem. □

The results proved in this section would further be useful in the study of the spaces like FK-space, Fréchet space, Montel space, C^*-algebra etc. and in the study of functions preserving the asymptotic equivalence of functions and sequences that is Pseudo-regularly varying (PRV) functions. Also these results have significant applications in the fields of topology, functional analysis, modern analysis etc.

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