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Uniform bounds on locations of zeros of partial theta function

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Abstract

We consider the partial theta function $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, where $(q,z) \in \mathbb{C}^2$, $|q| < 1$. We show that for any $0 < \delta_0 < \delta < 1$, there exists $n_0 \in \mathbb{N}$ such that for any q with $\delta_0 \leq |q| \leq \delta$ and for any $n \geq n_0$ the function θ has exactly n zeros with modulus $< |q|^{-n-1/2}$ counted with multiplicity.

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1 Introduction

We consider the bivariate series $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, where $(q,z) \in \mathbb{C}^2$, $|q| < 1$. This series defines a *partial theta function*. The terminology is explained by the fact that the Jacobi theta function is defined by the series $\sum_{j=-\infty}^{\infty} q^{j^2} z^j$ and the following equality holds true: $\theta(q^2, z/q) = \sum_{j=0}^{\infty} q^{j^2} z^j$. The word “partial” is justified by the summation in θ ranging from 0 to ∞ and not from $-\infty$ to ∞. In what follows we consider z as a variable and q as a parameter. For each fixed value of the parameter q the function θ is an entire function in the variable z.

The function θ finds applications in various domains, such as statistical physics and combinatorics (see [17]), Ramanujan type q-series (see [18]), the theory of (mock) modular forms (see [3]), asymptotic analysis (see [2]), and also in problems concerning real polynomials in one variable with all roots real (such polynomials are called *hyperbolic*, see [4], [5], [15], [14], [6], [13] and [7]). Other facts about θ can be found in [1].

The zeros of θ depend on the parameter q. For some values of q (called *spectral*) confluence of zeros occurs, so it would be correct to regard the zeros as multivalued functions of q; about the spectrum of θ see [13], [11] and [12].

We denote by $\mathbb{D}_\rho$ the open disk in the q-space centered at 0 and of radius ρ, by $\mathcal{C}_\rho$ the corresponding circumference, and by $A_{\delta_0,\delta}$ the closed annulus $\{q \in \mathbb{C} \mid \delta_0 \leq |q| \leq \delta\}$.

In the present paper we prove the following theorem:

Theorem 1. *For any couple of numbers (δ_0, δ) such that $0 < \delta_0 < \delta < 1$, there exists $n_0 \in \mathbb{N}$ such that for any $q \in A_{\delta_0,\delta}$ and for any $n \geq n_0$ the function θ has exactly n zeros in $\mathbb{D}_{|q|^{-n-1/2}}$ counted with multiplicity.*

Remark 2. 1. The proof of the theorem is based on a comparison between θ and the function

$$u(q,z) := \prod_{\nu=1}^{\infty}(1+q^{\nu}z) \tag{1.1}$$

We use the equality

$$u = \sum_{j=0}^{\infty} q^{j(j+1)/2}z^j/(q;q)_j \ , \tag{1.2}$$

where $(q;q)_j := (1-q)(1-q^2)\cdots(1-q^j)$ is the q-Pochhammer symbol; it follows directly from Problem I-50 of [16] (see pages 9 and 186 of [16]). The analog of the above theorem for the *deformed exponential function* $\sum_{j=0}^{\infty} q^{j(j+1)/2}z^j/j!$ is proved in a non-published text by A. E. Eremenko using a different method.

2. For q close to 0 the zeros of θ are of the form $-q^{-\ell}(1+o(1))$, $\ell \in \mathbb{N}$, see more details about this in [8], [9] and [10].

2 Proofs

Proof of Theorem 1. It is shown in [8] that for $0 < |q| \leq 0.108$ the zeros of θ can be expanded in convergent Laurent series. Recall that the function u (defined by (1.1)) satisfies equality (1.2), i.e. the zeros of u are the numbers $-q^{-\ell}$, $\ell \in \mathbb{N}$. We show that for $n \in \mathbb{N}$ sufficiently large the functions u and θ have one and the same number of zeros in the open disk $\mathbb{D}_{|q|^{-n-1/2}}$. To this end we show that for the restrictions u^0 and θ^0 of u and θ to the circumference $\mathcal{C}_{|q|^{-n-1/2}}$ one has $|u^0 - \theta^0/(q;q)_n| < |u^0|$ after which we apply the Rouché theorem.

For $0 < |q| \leq 0.108$ one can establish a bijection between the zeros of θ and u, because their ℓth zeros are of the form $-q^{-\ell}(1+o(1))$ and the moduli of the zeros increase with ℓ, see part 2 of Remark 2.

Set $P_k(|q|) := \prod_{\ell=0}^{k}(1-|q|^{\ell+1/2})$, $k \in \mathbb{N}\cup\infty$. For $|u^0|$ one obtains the estimation

$$|u^0| \geq |q|^{-n^2/2}P_{n-1}(|q|)P_{\infty}(|q|) > |q|^{-n^2/2}(P_{\infty}(|q|))^2 \geq |q|^{-n^2/2}(P_{\infty}(\delta))^2 \ . \tag{2.1}$$

Indeed, for $|z| = |q|^{-n-1/2}$ one can set $z := |q|^{-n-1/2}\omega$, $|\omega| = 1$. For $1 \leq \nu \leq n$ (resp. for $\nu > n$), the factor $(1+q^{\nu}z)$ in (1.1) is of the form $(1-|q|^{-\ell-1/2}\omega_{\ell})$, where $\ell = n-\nu$ and $|\omega_{\ell}| = 1$ (resp. of the form $(1-|q|^{\ell+1/2}\omega_{\ell}^*)$, where $\ell = \nu-n-1$ and $|\omega_{\ell}^*| = 1$). Thus

$$u(q,|q|^{-n-1/2}\omega^{-n-1/2}) = \prod_{\ell=0}^{n-1}(1-|q|^{-\ell-1/2}\omega_{\ell})\prod_{\ell=0}^{\infty}(1-|q|^{\ell+1/2}\omega_{\ell}^*) \ .$$

The first of the factors in the right-hand side can be represented in the form $|q|^{-n^2/2}\tilde{\omega}\prod_{\ell=0}^{n-1}(1-|q|^{\ell+1/2}\omega_{\ell}^{**})$ with $|\tilde{\omega}| = |\omega_{\ell}^{**}| = 1$. Therefore

$$u(q,|q|^{-n-1/2}\omega^{-n-1/2}) = |q|^{-n^2/2}\tilde{\omega}\prod_{\ell=0}^{n-1}(1-|q|^{\ell+1/2}\omega_{\ell}^{**})\prod_{\ell=0}^{\infty}(1-|q|^{\ell+1/2}\omega_{\ell}^*) \ .$$

The modulus of the right-hand side is minimal for $\omega_{\ell}^* = \omega_{\ell}^{**} = 1$ in which case one obtains the leftmost inequality in (2.1).

Consider the monomial $\beta_j := \alpha_j z^j$ in the series $u - \theta/(q;q)_n$. Hence for $j = n$ it vanishes and for $j > n$ one has

$$\begin{array}{rcl} \alpha_j & = & q^{j(j+1)/2}(1/(q;q)_j - 1/(q;q)_n) = q^{j(j+1)/2}U_{j,n} \ , \ \text{ where} \\ U_{j,n} & := & (1-\prod_{\ell=n+1}^{j}(1-q^\ell))/(q;q)_j \ , \end{array}$$

so for $|z| = |q|^{-n-1/2}$ one has $|\beta_j| = |q|^{-n^2/2+(j-n)^2/2}|U_{j,n}|$. One can observe that $U_{j,n} = q^{n+1} + O(q^{n+2})$. Set

$$U_{j,n} := \sum_{\nu \geq n+1} u_{j,n;\nu}q^\nu \ \text{ and } \ U := ((\prod_{\ell=1}^{\infty}(1+q^\ell)) - 1)/(q;q)_\infty = \sum_{\nu=1}^{\infty} u_\nu q^\nu \ .$$

The Taylor series of U converges for $|q| < 1$ because the infinite products defining U converge. Clearly $u_{j,n;\nu} \in \mathbb{Z}$, $u_\nu \in \mathbb{N}$ (because all coefficients of the series $1/(q;q)_j$ and $1/(q;q)_\infty$ are positive integers) and $u_{j,n;n+1} = u_1 = 1$.

The following lemma explains in what sense the series U majorizes the series $U_{j,n}$.

Lemma 3. *One has* $|u_{j,n;n+\nu}| \leq u_\nu$, $\nu \in \mathbb{N}$.

Before proving Lemma 3 (the proof is given at the end of the paper) we continue the proof of Theorem 1.

Set $R(|q|) := \sum_{j>n}|q|^{(j-n)^2/2}$. The following inequality results immediately from the lemma:

$$Z_1 := \sum_{j>n}|\beta_j| \leq |q|^{-n^2/2}|q|^n U(|q|)R(|q|) \leq |q|^{-n^2/2}\delta^n U(\delta)R(\delta) \ . \tag{2.2}$$

The first condition which we impose on the choice of n is the following inequality to be fulfilled:

$$\delta^n U(\delta)R(\delta) < (P_\infty(\delta))^2/4 \ . \tag{2.3}$$

For $j < n$ and $|z| = |q|^{-n-1/2}$ one has $|\beta_j| = |q|^{-n^2/2+(j-n)^2/2}|\tilde{U}_{j,n}|$, where

$$\tilde{U}_{j,n} := (\prod_{\ell=j+1}^{n}(1-q^\ell) - 1)/(q;q)_n \ . \tag{2.4}$$

Hence $|\tilde{U}_{j,n}| \leq T(|q|) := (\prod_{\ell=1}^{\infty}(1+|q|^\ell)+1)/(|q|;|q|)_\infty$ and

$$|\beta_j| \leq |q|^{-n^2/2}|q|^{(j-n)^2/2}T(\delta) \tag{2.5}$$

Choose $m \in \mathbb{N}$ such that $T(\delta)\sum_{s=m}^{\infty}\delta^{s^2/2} \leq (P_\infty(\delta))^2/4$. Inequality (2.5) implies that

$$Z_2 := \sum_{j=0}^{n-m}|\beta_j| \leq |q|^{-n^2/2}(P_\infty(\delta))^2/4 \tag{2.6}$$

Notice that for $n < m$ the above sum is empty and the inequality trivially holds true.

The finite sum

$$Z_3 := \sum_{j=n-m+1}^{n-1}|\beta_j| \tag{2.7}$$

is of the form $|q|^{-n^2/2}O(|q|^n)$. Indeed, consider formula (2.4). There exists $M > 0$ depending only on δ_0 and δ such that

$$0 < |1/(q;q)_n| \leq 1/(|q|;|q|)_n < 1/(|q|;|q|)_\infty \leq M \quad \text{for} \quad \delta_0 \leq |q| \leq \delta \ .$$

Thus

$$|\tilde{U}_{j,n}| \leq M\big(\prod_{\ell=j+1}^{n}(1+|q|^\ell)-1\big) \ .$$

The index j can take only the values $n-m+1, \ldots, n-1$. In the last product each monomial $|q|^\ell$ can be represented in the form $|q|^n|q|^{\ell-n}$, where $\ell-n = 2-m, \ldots, 0$. The modulus of each factor $|q|^{\ell-n}$ is not larger than $1/\delta_0^{\max(0,m-2)}$. Therefore

$$|\tilde{U}_{j,n}| \leq M((1+|q|^n/\delta_0^{\max(0,m-2)})^{m-1}-1) = O(|q|^n) \ .$$

The sum Z_3 (see (2.7)) can be made less than $|q|^{-n^2/2}(P_\infty(\delta))^2/4$ by choosing n large enough. Thus inequalities (2.1), (2.2) and (2.6) yield

$$|u^0 - \theta^0/(q;q)_n| \leq Z_1+Z_2+Z_3 \leq (3/4)|q|^{-n^2/2}(P_\infty(\delta))^2 < |q|^{-n^2/2}(P_\infty(\delta))^2 \leq |u^0|$$

which proves the theorem. □

Proof of Lemma 3. We first compare the coefficients of the series

$$\prod_{\ell=p}^{r}(1+q^\ell)-1 = \sum_{\nu\geq p}\gamma_\nu^1 q^\nu \quad \text{and} \quad \prod_{\ell=p}^{r}(1-q^\ell)-1 = \sum_{\nu\geq p}\gamma_\nu^2 q^\nu \quad , \quad p \leq r \ .$$

They are obtained respectively as a sum of the non-negative coefficients of monomials and as a linear combination of the same coefficients some of which are taken with the $+$ and the rest with the $-$ sign. Therefore $\gamma_\nu^1 \geq |\gamma_\nu^2|$, $\nu \geq p$. This means that $|u_{j,n;\nu}| \leq v_{j,n;\nu} \leq v_{\infty,n;\nu}$, where

$$V_{j,n} := \big(\prod_{\ell=n+1}^{j}(1+q^\ell)-1\big)/(q;q)_j = \sum_{\nu\geq n+1} v_{j,n;\nu}q^\nu \quad , \quad V_{\infty,0} = U \quad \text{and} \quad v_{\infty,0;\nu} = u_\nu \ .$$

To prove the lemma it suffices to show that

$$v_{\infty,n;n+\nu} \leq v_{\infty,0;\nu} \ . \tag{2.8}$$

Consider the series $S_r := \prod_{\ell=r+1}^{\infty}(1+q^\ell)-1 = \sum_{\nu\geq r+1} s_{r;\nu}q^\nu$ for $r=0$ and $r=n$. Compare the coefficients $s_{0;\nu}$ and $s_{n;n+\nu}$. The coefficient $s_{0;\nu}$ is equal to the number of ways in which ν can be represented as a sum of distinct natural numbers forming an increasing sequence whereas $s_{n;n+\nu}$ is the number of ways in which $n+\nu$ can be represented as a sum of distinct natural numbers $\geq n+1$ forming an increasing sequence. Clearly $s_{n;n+\nu} \leq s_{0;\nu}$. This implies inequality (2.8) and the lemma, because one has $V_{\infty,r} = S_r/(q;q)_\infty$ and the coefficients of the series $1/(q;q)_\infty$ are all positive. □

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