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On stability, boundedness and square integrability conditions for a third-order nonlinear system of differential equations

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Abstract

This paper extends some known results on the stability, boundedness and square integrability of solutions of certain nonlinear vector differential equations of third-order. The Lyapunov's second method is used as basic tool in obtaining the criteria for the stability and boundedness of solutions. Example is included to illustrate the results.

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1 Introduction

The study of the stability and boundedness of ordinary scalar and vector nonlinear differential equations of third order have received tremendous attention. Many works have been done by notable authors, for a comprehensive treatment of this subject see Afuwape [1, 2, 3, 4], Ezeilo [8, 9, 10], Graef [12, 13], Remili [19, 20, 21, 22, 23, 24, 25, 26, 27], Tunç [32, 33, 34, 35, 36, 37, 38, 39], and the references cited therein.

In 1966, 1983,1993 and 2007 respectively, Ezeilo and Tejumola [8], Afuwape [1], Meng [16] and Omeike [17] investigated the ultimately boundedness and existence of periodic solutions of the nonlinear vector differential equation of the form

$$X''' + AX'' + BX' + H(X) = P(t, X, X', X''). \tag{1.1}$$

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Later in 1985, Afuwape [2] demonstrated a result associated with the existence of unique periodic solution of the vector differential equation

$$X''' + AX'' + G(X') + H(X) = P(t, X, X', X'').$$

In 1995, Feng [11] obtained a result associated with the existence of unique periodic solution of the similar type equation

$$X''' + A(t)X'' + B(t)X' + H(X) = P(t, X, X', X''). \tag{1.2}$$

In this paper therefore, using Lyapunov's direct method we obtain criteria for asymptotic stability, boundedness and square integrability of solutions of the equation

$$(H(X(t))X'(t))'' + A(t)X''(t) + B(t)X'(t) + C(t)F(X(t)) = P(t), \tag{1.3}$$

in which $t \in \mathbb{R}^+$ and $X(t), P(t) \in \mathbb{R}^n; A, B,$ and C are continuous $n \times n$ symetric matrices. $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F(0) = 0$, and H is a $n \times n$ symetric differentiable and inversible matrix function. Let $J_F(X)$, $A'(t), B'(t), C'(t)$ and $H'(X)$, denote the jacobian matrices corresponding to $F(X), A(t), B(t), C(t)$ and $H(X)$ respectively, that is, $J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right), A'(t) = \frac{d}{dt}(a_{ij}(t)), B'(t) = \frac{d}{dt}(b_{ij}(t)), C'(t) = \frac{d}{dt}(c_{ij}(t)), H'(X(t)) = \frac{d}{dt}(h_{ij}(X(t)), (i, j = 1, 2, \dots, n)$, where $(x_1, x_2, \dots, x_n), (f_1, f_2, \dots, f_n)$, $(a_{ij}(t)), (b_{ij}(t))$, $(c_{ij}(t))$ and $h_{ij}(X(t))$ are components of X, F, A, B, C and $H(X)$. On the other hand $X(t), Y(t)$ and $Z(t)$ are, respectively, abbreviated as X, Y and Z throughout the paper. Additionally, the symbol $\langle X, Y\rangle$ corresponding to any pair X and Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$.

2 Preliminaries

In this section, we present some lemmas that will be used to establish our main results.

Lemma 1. *[1, 3, 8, 9, 10, 31] Let D be a real symmetric positive definite $n \times n$ matrix. Then for any X in $\mathbb{R}^n$, we have*

$$\delta_d \|XVert^2 \le \langle DX, X\rangle \le \Delta_d \|XVert^2,$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D respectively.

Lemma 2. *[1, 3, 8, 9, 10, 31] Let Q, D be any two real $n \times n$ commuting matrices. Then,*

1) The eigenvalues $\lambda_i(QD)\,(i = 1, 2..., n)$ of the product matrix QD are all real and satisfy

$$\min_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D) \le \lambda_i(QD) \le \max_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D).$$

2) The eigenvalues $\lambda_i(Q + D)\,(i = 1, 2..., n)$ of the sum of matrices Q and D are all real and satisfy.

$$\left\{\min_{1\le j\le n} \lambda_j(Q) + \min_{1\le k\le n} \lambda_k(D)\right\} \le \lambda_i(Q + D) \le \left\{\max_{1\le j\le n} \lambda_j(Q) + \max_{1\le k\le n} \lambda_k(D)\right\}.$$

Lemma 3. *[1, 3, 8, 9, 10, 31, 33] Let $H(X)$ be a continuous vector function with $H(0) = 0$. Then,*

$$1)\quad \frac{d}{dt}\left(\int_0^1 \langle H(\sigma X), X\rangle d\sigma\right) = \left\langle H(X), \frac{dX}{dt}\right\rangle.$$

$$2)\quad \int_0^1 \langle C(t)H(\sigma X), X\rangle d\sigma = \int_0^1\int_0^1 \sigma[\langle C(t)J_H(\sigma\tau X)X, X\rangle] d\sigma d\tau.$$

Lemma 4. *[30] Let $H(X)$ be a continuous vector function with $H(0) = 0$. Then,*

$$\begin{aligned} 1) \quad \langle H(X), H(X)\rangle &= \int_0^1\int_0^1 \sigma\langle J_H(\sigma X)J_H(\sigma\tau X)X, X\rangle d\sigma d\tau. \\ 2) \quad \langle C(t)H(X), X\rangle &= \int_0^1 \langle J_H(\sigma X)C(t)X, X\rangle d\sigma. \end{aligned}$$

3 Stability

We shall state here some assumptions which will be used on the functions that appeared in equation (1.3). Suppose that there are positive constants $\delta_A, \delta_B, \delta_C, \delta_F, \delta_H, \delta_{H^{-1}}, \Delta_A, \Delta_B, \Delta_C, \Delta_F, \Delta_H$, and $\Delta_{H^{-1}}$, such that the matrices $A(t), B(t), C(t), H(X), H^{-1}(X)$ and $J_F(X)$ (Jacobian matrix of $F(X)$) are symmetric and positive definite, and furthermore the eigenvalues $\lambda_i(A(t)), \lambda_i(B(t)), \lambda_i(C(t)), \lambda_i(H(X)), \lambda_i(H^{-1}(X))$ and $\lambda_i(J_F(X))(i = 1, 2, \ldots, n)$ of $A(t), B(t), C(t), H(X), H^{-1}(X)$ and $J_F(X)$, respectively satisfy

$$\begin{aligned} &0 < \delta_A \le \lambda_i(A(t)) \le \Delta_A, \qquad 0 < \delta_H \le \lambda_i(H(X)) \le \Delta_H, \\ &0 < \delta_B \le \lambda_i(B(t)) \le \Delta_B, \qquad 0 < \delta_{H^{-1}} \le \lambda_i\left(H^{-1}(X)\right) \le \Delta_{H^{-1}}, \\ &0 < \delta_C \le \lambda_i(C(t)) \le \Delta_C, \qquad 0 < \delta_F \le \lambda_i(J_F(X)) \le \Delta_F. \end{aligned}$$

Before stating the major theorem, we introduce the following notations

$$\begin{aligned} H_t &= H(X(t)), \\ \theta(t) &= (H_t^{-1})' = -H_t^{-1}H_t'H_t^{-1}. \end{aligned} \tag{3.1}$$

We note that equation (1.3) is equivalent to the following system

$$\begin{cases} X' = H_t^{-1}Y, \\ Y' = Z, \\ Z' = -A(t)H_t^{-1}Z - \left(A(t)\theta(t) + B(t)H_t^{-1}\right)Y - C(t)F(X) + P(t), \end{cases} \tag{3.2}$$

which was obtained by setting

$$X'' = \theta(t)Y + H_t^{-1}Z. \tag{3.3}$$

In this section, we establish some conditions for the asymptotic stability of all solutions of (1.3) in the case $P(t) = 0$. We begin with the following Theorem.

Theorem 5. *In addition to the basic assumptions imposed on the matrices A, B, C, H, H^{-1} and J_F witch commute pairwise, assume that :*

i) $\lambda_i(C') \le 0$.

ii) $\dfrac{(\Delta_C)^2(\Delta_F)^2\Delta_H}{\delta_C\delta_F\delta_B} < d < \delta_A$.

iii) $\dfrac{d}{2}\Delta_{A'} + \dfrac{1}{2}\Delta_{B'}\Delta_H + \Delta_{C'}\Delta_{H^2} < \dfrac{d\delta_B - \Delta_C\Delta_F\Delta_H}{2}$.

iv) $\displaystyle\int_0^{+\infty}\left\|\frac{d}{ds}H_s\right\| ds < +\infty$.

Then any solution of (3.2) is asymptotically stable.

Proof. Let η a positive constant which will be specified later. We define the Lyapunov functional $W = W(t, X, Y, Z)$ as

$$W = V \exp\big(-\frac{1}{\eta}\int_0^t \|\theta(s)\|\, ds\big), \tag{3.4}$$

where

$$\begin{aligned} V &= d\int_0^1 \langle C(t)F(\sigma X), X\rangle d\sigma + \langle C(t)Y, F(X)\rangle + \frac{1}{2}\langle B(t)H_t^{-1}Y, Y\rangle \\ &\quad + d\langle H_t^{-1}Y, Z\rangle + \frac{d}{2}\langle A(t)H_t^{-2}Y, Y\rangle + \frac{1}{2}\langle Z, Z\rangle. \end{aligned} \tag{3.5}$$

It is clear from (3.5) that $V(t, 0, 0, 0) = 0$. By Lemma 1, we have the following inequality

$$\begin{aligned} \langle C(t)Y, F(X)\rangle + \frac{1}{2}\langle B(t)H_t^{-1}Y, Y\rangle &\geq \langle C(t)Y, F(X)\rangle + \frac{\delta_B\delta_{H^{-1}}}{2}\langle Y, Y\rangle \\ &= \frac{\delta_B\delta_{H^{-1}}}{2}\|Y + \frac{1}{\delta_B\delta_{H^{-1}}}C(t)F(X)\|^2 \\ &\quad - \frac{1}{2\delta_B\delta_{H^{-1}}}\langle C^2(t)F(X), F(X)\rangle. \end{aligned}$$

Observe that

$$d\langle H_t^{-1}Y, Z\rangle + \frac{1}{2}\langle Z, Z\rangle = \frac{1}{2}\|Z + dH_t^{-1}Y\|^2 - \frac{d^2}{2}\langle H_t^{-2}Y, Y\rangle.$$

Hence

$$\begin{aligned} V &\geq d\int_0^1 \langle C(t)F(\sigma X), X\rangle d\sigma - \frac{1}{2\delta_B\delta_{H^{-1}}}\langle C^2(t)F(X), F(X)\rangle \\ &\quad + \frac{d}{2}\langle A(t)H_t^{-2}Y, Y\rangle - \frac{d^2}{2}\langle H_t^{-2}Y, Y\rangle + \frac{1}{2}\|Z + dH_t^{-1}Y\|^2. \end{aligned}$$

In view of Lemma 3 and Lemma 4, it follows that

$$\begin{aligned} \langle C^2(t)F(X), F(X)\rangle &\leq (\Delta_C)^2(\Delta_F)^2\|X\|^2\int_0^1\int_0^1 \sigma d\sigma d\tau = \frac{1}{2}(\Delta_C)^2(\Delta_F)^2\|X\|^2, \\ \int_0^1 \langle dC(t)F(\sigma X), X\rangle d\sigma &\geq d\delta_C\delta_F\|X\|^2\int_0^1\int_0^1 \sigma d\sigma d\tau = \frac{1}{2}d\delta_C\delta_F\|X\|^2. \end{aligned}$$

Therefore, since $\delta_{H^{-1}} = \Delta_H^{-1}$ we have

$$\begin{aligned} V &\geq \frac{1}{2}\big(d\delta_C\delta_F - \frac{(\Delta_C)^2(\Delta_F)^2}{\delta_B\Delta_H^{-1}}\big)\|X\|^2 \\ &\quad + \frac{d}{2}\langle\Big(A(t) - dI\Big)H_t^{-2}Y, Y\rangle + \frac{1}{2}\|Z + dH_t^{-1}Y\|^2. \end{aligned}$$

Again, in view of Lemma 1, easily, we obtain that

$$\frac{d}{2}\langle\Big(A(t) - dI\Big)H_t^{-2}Y, Y\rangle \geq \frac{d}{2}(\delta_A - d)\delta_{H^{-2}}\langle Y, Y\rangle.$$

Hence, according to the last estimates, we obtain

$$\begin{aligned} V &\geq \frac{1}{2}\big(d\delta_C\delta_F - \frac{(\Delta_C)^2(\Delta_F)^2\Delta_H}{\delta_B}\big)\,\|X\|^2 + \frac{d}{2}\big(\delta_A - d\big)\delta_{H^{-2}}\|Y\|^2 \\ &\quad + \frac{1}{2}\|Z + dH_t^{-1}Y\|^2, \end{aligned}$$

with the coefficients $\big(d\delta_C\delta_F - \frac{(\Delta_C)^2(\Delta_F)^2\Delta_H}{\delta_B}\big) > 0$ and $\big(\delta_A - d\big) > 0$ in view of condition (ii).

Thus, there exists a constant $k > 0$ small enough such that

$$V \geq k\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right). \tag{3.6}$$

On applying (3.1) and condition (iii) there exists positive constant N such that

$$\begin{aligned} \int_0^t \|\theta(s)\|\, ds &\leq \int_0^t \|H_s^{-1}\|^2 \left\|\frac{d}{ds}H_s\right\| ds \\ &\leq (\Delta_{H^{-1}})^2 \int_0^t \left\|\frac{d}{ds}H_s\right\| ds \leq N, \qquad \text{for all } t \geq 0. \end{aligned} \tag{3.7}$$

On combining this last estimate with (3.4) and (3.6) we get

$$W \geq V\exp(-\frac{N}{\eta}) \geq K_0\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right), \tag{3.8}$$

with $K_0 = k\exp(-\frac{N}{\eta})$.

Now, let $V'_{(3.2)}(t, X, Y, Z) = V'_{(3.2)}$ denote the time derivative of the functional $V(t, X, Y, Z)$ along the trajectories of the system (3.2). An easy computation shows that

$$V'_{(3.2)} = G_1 + G_2 + G_3 + G_4, \tag{3.9}$$

where

$$\begin{aligned} G_1 &= d\int_0^1 \langle C'(t)F(\sigma X), X\rangle d\sigma + \langle C'(t)Y, F(X)\rangle - \langle C'(t)Y, Y\rangle, \\ G_2 &= \left\langle \Big(\frac{d}{2}A'(t) + \frac{1}{2}B'(t)H_t + C'(t)H_t^2 - dB(t) + C(t)J_F H_t\Big)H_t^{-2}Y, Y\right\rangle, \\ G_3 &= \big\langle \big(dI - A(t)\big)H_t^{-1}Z, Z\big\rangle, \\ G_4 &= \frac{d}{2}\big\langle A(t)\theta(t)Y, H_t^{-1}Y\big\rangle + \frac{d}{2}\big\langle A(t)H_t^{-1}Y, \theta(t)Y\big\rangle + \frac{1}{2}\langle B(t)\theta(t)Y, Y\rangle \\ &\quad - d\langle H_t^{-1}Y, A(t)\theta(t)Y\rangle - \langle A(t)\theta(t)Y, Z\rangle + d\langle\theta(t)Y, Z\rangle. \end{aligned}$$

Under the assumption (i) of Theorem 5, we have

$$\begin{aligned} G_1 &\leq d\int_0^1 \langle C'(t)F(\sigma X), X\rangle d\sigma - \left\|C'^{\frac{1}{2}}(t)Y - \frac{1}{2}C'^{\frac{1}{2}}(t)F(X)\right\|^2 + \frac{\Delta_{C'}}{4}\|F(X)\|^2 \\ &\leq d\int_0^1 \langle C'(t)F(\sigma X), X\rangle d\sigma. \end{aligned}$$

By Lemma 3 and Lemma 1 we get

$$\begin{aligned} G_1 &\leq d\int_0^1 \langle C'(t)F(\sigma X), X\rangle d\sigma = \int_0^1\int_0^1 \sigma[\langle dC'(t)J_F(\sigma\tau X)X, X\rangle] d\sigma d\tau \\ &\leq \int_0^1\int_0^1 \sigma[\langle d\Delta_{C'}\Delta_F X, X\rangle] d\sigma d\tau \\ &= d\Delta_{C'}\Delta_F \|XVert^2 \leq 0. \end{aligned} \quad (3.10)$$

If we take into consideration condition (ii) of Theorem 5, we have that

$$\delta_A > d > \frac{(\Delta_C)^2(\Delta_F)^2}{\delta_C\delta_F\delta_B\Delta_H^{-1}} > \frac{\Delta_C\Delta_F\Delta_H}{\delta_B}.$$

Witch implies

$$\begin{aligned} M_1 &= \frac{d\delta_B - \Delta_C\Delta_F\Delta_H}{2} > 0, \\ M_2 &= -(d-\delta_A)\Delta_{H^{-1}} > 0. \end{aligned}$$

Clearly, as a result of the assumption (iii) we have

$$\begin{aligned} G_2 &\leq \Big(\frac{d}{2}\Delta_{A'} + \frac{1}{2}\Delta_{B'}\Delta_H + \Delta_{C'}\Delta_{H^2} - d\delta_B + \Delta_C\Delta_F\Delta_H\Big)\Delta_{H^{-2}}\|Y\|^2 \\ &\leq -M_1\|Y\|^2 \leq 0, \end{aligned}$$

and

$$G_3 \leq (d-\delta_A)\Delta_{H^{-1}}\|Z\|^2 = -M_2\|Z\|^2 \leq 0.$$

By applying the inequality $2VertuvVert \leq \|uVert^2 + \|vVert^2$ we estimate G_4 as follows

$$\begin{aligned} G_4 &= \frac{1}{2}\langle B(t)\theta(t)Y, Y\rangle - \langle A(t)\theta(t)Y, Z\rangle + d\langle\theta(t)Y, Z\rangle \\ &\leq K\|\theta(t)\|V, \end{aligned}$$

where $K = \frac{1}{k}\big(\frac{1}{2}\Delta_B + \frac{1}{2}\Delta_A + \frac{d}{2}\big)$.
Bringing together the estimates just obtained for $G_i (i = 1, 2...4)$ in (3.9) we get

$$V'_{(3.2)} \leq -M_1\|Y\|^2 - M_2\|Z\|^2 + K\|\theta(t)\|V. \quad (3.11)$$

From (3.4), we have

$$W'_{(3.2)} = \left(V'_{(3.2)} - \frac{1}{\eta}\|\theta(t)\|V\right)e^{-\frac{1}{\eta}\int_0^t\|\theta(s)\|ds}.$$

Using (3.11), (3.6) and choosing $\eta = \frac{1}{K}$, we get

$$W'_{(3.2)} \leq \left(-M_1\|Y\|^2 - M_2\|Z\|^2\right)e^{-\frac{1}{\eta}\int_0^t\|\theta(s)\|ds}.$$

Clearly from (3.7) we have $e^{-\frac{1}{\eta}\int_0^t\|\theta(s)\|ds} \geq e^{-\frac{N}{\eta}}$.
Hence

$$W'_{(3.2)} \leq -L\left(\|Y\|^2 + \|Z\|^2\right), \quad (3.12)$$

where $L = e^{-\frac{N}{\eta}} \min\{M_1, M_2\}$. In view of (3.8) and (3.12), it follow from $\big($[6, Theorem 4.1.14]$\big)$ that the solution $\big(X(t), Y(t), Z(t)\big)$ of (3.2) is stable. Now $E = \{(X, Y, Z) : W'_{(3.2)}(X, Y, Z) = 0\} = \{(X, 0, 0) : X \in \mathbb{R}^n\}$ and the largest invariant set contained in E is $F = \{(0, 0, 0)\}$. By LaSalle's invariance principe (see, for example, Haddock [14])

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} Y(t) = \lim_{t\to\infty} Z(t) = 0.$$

□

4 Boundedness

Our main theorem in this section is the following boundedness result which is stated with respect to $P(t) \neq 0$.

Theorem 6. *Let all the conditions of Theorem 5 be satisfied and in addition we assume that there exist positive constants p_1 and P_1 such that:*

$I_1)$ $\|P(t)\| \leq p(t) < p_1, \quad \forall t \geq 0.$

$I_2)$ $\int_0^t p(s)ds < P_1, \quad \forall t \geq 0.$

$I_3)$ $\lim\limits_{t\to\infty} \|H'_t\|$ *exists.*

Then there exists a positive constant P_5 such that any solution $X(t)$ of (1.3) and their derivatives $X'(t)$, and $X''(t)$ satisfy

$$\|X(t)\| \leq P_5, \ \|X'(t)\| \leq P_5, \ \|X''(t)\| \leq P_5. \tag{4.1}$$

Proof. For the case $P(t) \neq 0$, on differentiating (3.5) along the system (3.2) we obtain

$$\begin{aligned} V'_{(3.2)} &\leq -U + K\|\theta(t)\|V + d\langle H_t^{-1}Y, P(t)\rangle + \langle Z, P(t)\rangle \\ &\leq -U + K\|\theta(t)\|V + p(t)\Big(d\|H_t^{-1}\| \ \|Y\| + \|Z\|\Big) \\ &\leq -U + K\|\theta(t)\|V + p(t)K_1\Big(\|Y\| + \|Z\|\Big), \end{aligned}$$

where $K_1 = \max\left\{d\delta_H^{-1}, 1\right\}$ and $U = M_1\|Y\|^2 + M_2\|Z\|^2$.

By using $VertuVert \ \leq \ \|uVert^2 + 1$, it is clear that

$$V'_{(3.2)} \leq -U + K\|\theta(t)\|V + p(t)K_1\Big(\|Y\|^2 + \|Z\|^2 + 2\Big). \tag{4.2}$$

From (3.4) we have

$$W'_{(3.2)} \quad = \quad \Big[V' - \frac{1}{\eta}\|\theta(t)\|V\Big] \exp\big(-\frac{1}{\eta}\int_0^t \|\theta(s)\|\, ds\big). \tag{4.3}$$

Since $K - \dfrac{1}{\eta} = 0$, it follows that

$$W'_{(3.2)} \quad \leq \quad \Big[-U + p(t)K_1\Big(\|Y\|^2 + \|Z\|^2 + 2\Big)\Big] \exp\big(-\frac{1}{\eta}\int_0^t \|\theta(s)\|\, ds\big).$$

In view of (3.12) and the fact that

$$\exp\Big(-\frac{1}{\eta}\int_0^t \|\theta(s)\|\,ds\Big) \leq 1,$$

we have

$$W'_{(3.2)} \leq -L(\|Y\|^2 + \|Z\|^2) + \frac{K_1}{K_0}p(t)\ W + K_2 p(t), \tag{4.4}$$

with $K_2 = 2K_1$. Integrating both sides (4.4) from 0 to t, one can easily obtain

$$W(t) - W(0) \leq K_2\int_0^t p(s)ds + \frac{K_1}{K_0}\int_0^t W(s)p(s)ds.$$

Let

$$P_2 = W(0) + K_2 P_1. \tag{4.5}$$

Thus

$$W(t) \leq P_2 + \frac{K_1}{K_0}\int_0^t W(s)p(s)ds.$$

On applying Gronwall inequality we have

$$W(t) \leq P_2 \exp\Big(\frac{K_1}{K_0}\int_0^t p(s)ds\Big) \leq P_3, \tag{4.6}$$

where $P_3 = P_2 \exp\Big(\frac{K_1}{K_0}P_1\Big)$. Combining (4.6) and (3.8), we have

$$\|X(t)\| \leq P_4,\ \|Y(t)\| \leq P_4, \|Z(t)\| \leq P_4, \tag{4.7}$$

where $P_4 = \sqrt{\frac{P_3}{K_0}}$.
Now, by (3.2) we get

$$\begin{aligned} \|X'(t)\| &= \|H_t^{-1}Y(t)\| \\ &\leq \|H_t^{-1}\|\,\|Y(t)\| \\ &\leq P_4\delta_H^{-1}. \end{aligned}$$

According to condition (I_3) of Theorem 6, there exists positive constant h_1 such that

$$\|H'_t\| < h_1. \tag{4.8}$$

So, by (3.1) we have

$$\|\theta(t)\| \leq \|H_t^{-2}\|\ \|H'_t\| \leq h_1(\delta_H^{-1})^2 = \beta. \tag{4.9}$$

In view of (3.2) and (3.3) we have

$$\begin{aligned} \|X''(t)\| &\leq \|\theta(t)Y(t)\| + \|H_t^{-1}Z(t)\| \\ &\leq \big(h_1(\delta_H^{-1})^2 + \delta_H^{-1}\big)P_4. \end{aligned}$$

Therefore, there exists positive constant P_5 such that

$$\|X(t)\| \leq P_5,\ \|X'(t)\| \leq P_5,\ \|X''(t)\| \leq P_5,\quad \text{for all}\ \ t \geq 0, \tag{4.10}$$

where $P_5 = \max\big\{\big(h_1(\delta_H^{-1})^2 + \delta_H^{-1}\big)P_4,\ P_4\big\}$. This completes the proof of Theorem 6. $\square$

5 Square integrability of solutions

Our next result concerns the square integrability of solutions of equation (1.3).

Theorem 7. *Let all the conditions of Theorem 5 and Theorem 6 satisfied and in addition we assume that*

I$_4$) $2\delta_C\delta_{J_F} - \Delta_A - \Delta_B > 0$.

Then, every solution X of equation (1.3) and their derivatives are elements of $L^2[0,+\infty)$.

Proof. Let $X(t)$ be a solution of (1.3) and define $Q(t) = Q(t, X(t), Y(t), Z(t))$ by

$$Q(t) = W(t) + \lambda \int_0^t \left(\|Y(s)\|^2 + \|Z(s)\|^2\right) ds, \tag{5.1}$$

where $\lambda > 0$ is a constant to be specified later and W(t) is given in (3.4). By differentiating Q(t) and using (4.4) we obtain

$$Q'(t) \leq (\lambda - L)(\|Z(t)\|^2 + \|Y(t)\|^2) + \Big(K_1 W + K_2\Big) p(t).$$

Taking $\lambda - L = 0$ and using (4.6) we get

$$Q'(t) \leq K_3 p(t), \tag{5.2}$$

where $K_3 = K_1 P_3 + K_2$. Integrating (5.2) from 0 to t, $t \geq 0$, and using condition (I_2) of Theorem 6 we obtain

$$Q(t) - Q(0) = \int_0^t Q'(s) ds \leq K_3 P_1.$$

With (4.5) and equality $Q(0) = W(0)$ we get

$$Q(t) \leq K_3 P_1 + P_2 - K_2 P_1.$$

We can conclude by (5.1) that

$$\int_0^t (\|Z(s)\|^2 + \|Y(s)\|^2) ds < \frac{K_3 P_1 + P_2 - K_2 P_1}{\lambda},$$

which imply the existence of positive constants μ_1 and μ_2 such that

$$\int_0^t \|Y(s)\|^2 ds \leq \mu_1 \quad \text{and} \quad \int_0^t \|Z(s)\|^2 ds \leq \mu_2.$$

Observe that from (3.2)

$$\begin{aligned} \int_0^\infty \|X'(s)\|^2 ds &= \int_0^\infty \left\|H_s^{-1} Y(s)\right\|^2 ds \\ &\leq \int_0^\infty \left\|H_s^{-1}\right\|^2 \|Y(s)\|^2 ds \\ &\leq (\Delta_{H^{-1}})^2 \mu_1 = l_1 < \infty. \end{aligned} \tag{5.3}$$

On the other hand, using (4.9) and (3.3) we obtain

$$\begin{aligned}\int_0^t \|X''(s)\|^2 ds &= \int_0^t \|H_s^{-1}\|^2 \|Z(s)\|^2 ds + \int_0^t \|\theta(s)\|^2 \|Y(s)\|^2 ds \\ &\quad +2\int_0^t \langle \theta(s)Y(s), H_s^{-1}Z(s)\rangle ds \\ &\leq \big(\Delta_{H^{-1}}^2 + \beta\Delta_{H^{-1}}\big)\int_0^t \|Z(s)\|^2 ds \\ &\quad +\big(\beta^2 + \beta\Delta_{H^{-1}}\big)\int_0^t \|Y(s)\|^2 ds \\ &\leq M(\mu_1+\mu_2) = l_2 < \infty, \end{aligned} \tag{5.4}$$

where $M = \max\{\Delta_{H^{-1}}^2 + \beta\Delta_{H^{-1}}, \beta^2 + \beta\Delta_{H^{-1}}\}$.
Next, multiply (1.3) by $X(t)$ and integrate by parts from 0 to t all the terms on the LHS of (1.3) obtaining

$$\int_0^t \langle C(s)F(X(s)), X(s)\rangle ds = I(t) + J(t), \tag{5.5}$$

where

$$\begin{aligned} I(t) &= -\langle H_t' X'(t) + H_t X''(t), X(t)\rangle + \langle H_t X'(t), X'(t)\rangle - \int_0^t \langle H_s X'(s), X''(s)\rangle ds + k_1, \\ J(t) &= \int_0^t \langle \big(-A(s)X''(s) - B(s)X'(s) + P(s)\big), X(s)\rangle\rangle ds, \end{aligned}$$

and

$$k_1 = \langle H_0' X'(0), X(0)\rangle + \langle H_0 X''(0), X(0)\rangle - \langle H_0 X'(0), X'(0)\rangle.$$

Using (4.8) and (4.10) we get

$$|-\langle H_t' X'(t) + H_t X''(t), X(t)\rangle + \langle H_t X'(t), X'(t)\rangle| \leq P_5^2\Big(h_1 + 2\Delta_H\Big).$$

It is clear that

$$\begin{aligned}\int_0^t \langle H_s X'(s), X''(s)\rangle ds &\leq \frac{\Delta_H}{2}\int_0^t \Big(\|X'(s)\|^2 + \|X''(s)\|^2\Big) ds. \\ &\leq \frac{\Delta_H}{2}(l_1 + l_2) = l_3. \end{aligned}$$

Hence

$$I(t) \leq l_3 + P_5^2\Big(h_1 + 2\Delta_H\Big) + |\, k_1 \,| = l_4. \tag{5.6}$$

By using assumption (I_2) of Theorem 6, Lemma 1, and inequality $2uv \leq u^2 + v^2$, we get

$$\begin{aligned} J(t) &\leq \frac{\Delta_A}{2}\int_0^t \Big(\|X''^2 + \|X(s)\|^2\Big) ds + \frac{\Delta_B}{2}\int_0^t \Big(\|X'^2 + \|X(s)\|^2\Big) ds \\ &\quad + P_5 \int_0^t \|P(s)\| ds \\ &\leq l_5 + \frac{\Delta_A + \Delta_B}{2}\int_0^t \|X(s)\|^2 ds, \end{aligned} \tag{5.7}$$

where $l_5 = \frac{\Delta_A}{2} l_2 + \frac{\Delta_B}{2} l_1 + P_1 P_5$. By Lemma 4 we have

$$\langle C(t)F(X(t)), X(t)\rangle \geq \delta_C \delta_{J_F} \|X(t)\|^2 .$$

Thus, (5.6), (5.7) and condition (I_4) imply that

$$\int_0^t \|X(s)\|^2 \, ds \leq l_0,$$

where $l_0 = \frac{2(l_4 + l_5)}{2\delta_C \delta_{J_F} - (\Delta_A + \Delta_B)}$. This fact completes the proof of Theorem. □

Example 8. As a special case of the following equation

$$(H(X(t))X'(t))'' + A(t)X''(t) + B(t)X'(t) + C(t)F(X(t)) = P(t) \tag{5.8}$$

where

$$\begin{aligned}
X(t) &= \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad F(X) = \begin{pmatrix} 0.2 \arctan x \\ 0.16y \end{pmatrix}, \\
J_F(X) &= \begin{pmatrix} \frac{0.2}{1+x^2} & 0 \\ 0 & 0.16 \end{pmatrix}, H_t = \begin{pmatrix} h_{11}(x(t)) & 0 \\ 0 & h_{22}(y(t)) \end{pmatrix}, \\
P(t) &= \begin{pmatrix} \frac{1}{1+t^2} \\ \frac{\sin t}{3+\cos^2 t} \end{pmatrix}, \quad A(t) = \begin{pmatrix} \frac{e^{\sin t}}{10} + \frac{1}{4} & 0 \\ 0 & \frac{9\cos t}{100} + \frac{1}{3} \end{pmatrix}, \\
B(t) &= \begin{pmatrix} \frac{e^{-t^2}+1}{2} & 0 \\ 0 & \frac{\sin t}{4} + \frac{1}{2} \end{pmatrix}, \quad C(t) = \begin{pmatrix} e^{-2t} + 5 & 0 \\ 0 & e^{-t} + 5 \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
h_{11}(x(t)) &= \frac{0.02}{6}\Big(\frac{\sin(x(t))}{(1+x^2(t))} + 3\Big), \\
h_{22}(y(t)) &= \frac{0.02}{6}\Big(\frac{\cos(y(t))}{(1+y^2(t))} + 5\Big).
\end{aligned}$$

Clearly, $H(X), A, B$ and $J_F(X)$ are diagonal matrices, hence they are symmetric and commute pairwise. Then, by an easy calculation, we obtain eigenvalues of the matrices $H,\ A,\ B,\ C$ and $J_F(X)$ as follows:

$$\begin{aligned}
\delta_h &= \frac{0.02}{3} \leq \lambda_1(H) = \frac{0.02}{6}\Big(\frac{\sin x}{(1+x^2)} + 3\Big), \\
\lambda_2(H) &= \frac{0.02}{6}\Big(\frac{\cos x}{(1+x^2)} + 5\Big) \leq 0.02 = \Delta_h, \\
\delta_A &= 0.24333 \leq \lambda_1(A(t)) = \frac{9}{100}\cos t + \frac{1}{3}, \quad \lambda_2(A(t)) = \frac{e^{\sin t}}{10} + \frac{1}{4} \leq 0.52183 = \Delta_A, \\
\delta_B &= 0.5 \leq \lambda_1(B(t)) = \frac{\sin t}{4} + \frac{3}{4}, \quad \lambda_2(B(t)) = \frac{e^{-t^2}}{2} + \frac{1}{2} \leq 1 = \Delta_B, \\
\delta_C &= 5 \leq \lambda_1(C(t)) = e^{-2t} + 5, \quad \lambda_2(C(t)) = e^{-3t} + 5 \leq 6 = \Delta_C, \\
\delta_F &= \frac{1.6}{10} = \lambda_1(J_F(X)), \quad \lambda_2(J_F(X)) = \frac{0.2}{1+x^2} \leq \frac{2}{10} = \Delta_F.
\end{aligned}$$

A simple computation gives

$$\begin{aligned}
\lambda_1\left(A'(t)\right) &= -\frac{9}{100}\sin t, \qquad \lambda_2\left(A'(t)\right) = \frac{\cos t}{10}e^{\sin t} \leq \frac{e}{10} = \Delta_{A'},\\
\lambda_1\left(B'(t)\right) &= -te^{-t^2}, \qquad \lambda_2\left(B'(t)\right) = -\frac{\cos t}{4} \leq \frac{1}{4} = \Delta_{B'},\\
\lambda_1\left(C'(t)\right) &= -2e^{-2t}, \qquad \lambda_2\left(C'(t)\right) = -e^{-t} \leq 0 = \Delta_{C'}.
\end{aligned}$$

A trivial verification shows that H is nonsingular matrix and we have

$$\frac{d}{dt}H_t = \begin{pmatrix} \frac{d}{dt}h_{11}(x(t)) & 0 \\ 0 & \frac{d}{dt}h_{22}(y(t)) \end{pmatrix},$$

where

$$\begin{aligned}
\frac{d}{dt}h_{11}(x(t)) &= \frac{0.02}{6}\Big(\frac{\cos(x(t))}{(1+x^2(t))} - \frac{2x(t)\sin(x(t))}{\left(1+x^2(t)\right)^2}\Big)x'(t),\\
\frac{d}{dt}h_{22}(y(t)) &= \frac{0.02}{6}\Big(\frac{-\sin(y(t))}{(1+y^2(t))} - \frac{2y\cos(y(t))}{\left(1+y^2(t)\right)^2}\Big)y'(t).
\end{aligned}$$

Thus

$$\|\frac{d}{dt}H_t\| = \max\left\{\left|\frac{d}{dt}h_{11}(x(t))\right|, \left|\frac{d}{dt}h_{22}(y(t))\right|\right\} = D(t),$$

and

$$\|\theta(t)\| \leq \frac{1}{\delta_h^2}D(t), \quad \text{for all } t \geq 0.$$

A straightforward calculation give

$$\begin{aligned}
\int_0^t \|\theta(s)\|ds &\leq 2.25\times 10^4\int_0^t D(s)ds\\
&= 2.25\times 10^4\int_0^t \max\left\{\left|\frac{d}{ds}h_{11}(x(s))\right|, \left|\frac{d}{ds}h_{22}(y(s))\right|\right\}ds\\
&\leq 2.25\times 10^4\int_0^t \frac{0.02}{6}\left|(\frac{\cos x}{1+x^2} - \frac{2x\sin x}{\left(1+x^2\right)^2})x'(s)\right|ds\\
&\quad + \int_0^t \frac{0.02}{6}\left|(\frac{-\sin y}{1+y^2} - \frac{2y\cos y}{\left(1+y^2\right)^2})y'(s)\right|ds\\
&\leq 300\Bigg(\int_{\omega_1(t)}^{\omega_2(t)}\left|(\frac{\cos u}{1+u^2} - \frac{2u\sin u}{\left(1+u^2\right)^2})du\right|\\
&\quad + \int_{\varphi_1(t)}^{\varphi_2(t)}\left|(\frac{-\sin v}{1+v^2} - \frac{2v\cos v}{\left(1+v^2\right)^2})dv\right|\Bigg)\\
&< 300\Bigg(\int_{-\infty}^{+\infty}\left|\frac{1+u^2+2u}{\left(1+u^2\right)^2}\right|du + \int_{-\infty}^{+\infty}\left|\frac{1+u^2+2u}{\left(1+u^2\right)^2}\right|du\Bigg)\\
&= 150(\pi+2),
\end{aligned}$$

where

$$\omega_1(t) = \min\{x(0), x(t)\}, \qquad \omega_2(t) = \max\{x(0), x(t)\},$$

and

$$\varphi_1(t) = \min\{y(0), y(t)\}, \qquad \varphi_2(t) = \max\{y(0), y(t)\}.$$

Now, it is easy see that

$$\|P(t)\| = \sqrt{P_1^2(t) + P_2^2(t)} \le P_1(t) + P_2(t) = p(t) < \frac{4}{3} = p_1,$$

where $P_1(t) = \frac{1}{1+t^2}$, $P_2(t) = \frac{\sin t}{3+\cos^2 t}$. So, we have for $t \in [0, +\infty)$

$$\int_0^t \|p(s)\| ds = \int_0^t \|P_1(s)\| ds + \int_0^t \|P_2(s)\| ds < \infty.$$

By taking $d = 0.23$, it follows easily that

$$\begin{aligned} &\frac{(\Delta_C)^2(\Delta_F)^2\Delta_H}{\delta_B\delta_C\delta_F} = 0.072 < d < \delta_A = 0.24333. \\ &\frac{d}{2}\Delta_{A'} + \frac{1}{2}\Delta_{B'}\Delta_H + \Delta_{C'}\Delta_{H^2} < \frac{d}{2}\Delta_{A'} + \frac{1}{2}\Delta_{B'}\Delta_H \\ &= 3.376 \times 10^2 < \frac{d\delta_B - \Delta_C\Delta_F\Delta_H}{2} = 0.0455. \end{aligned}$$

We have also

$$\delta_C\delta_F - \frac{\Delta_A + \Delta_B}{2} = 3.9086 \times 10^2 > 0.$$

Thus, all the conditions of Theorem 7 are satisfied.

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