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Polynomials, sign patterns and Descartes' rule

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Abstract

The famous Descartes' rule of signs from 1637 giving an upper bound on the number of positive roots of a real univariate polynomial in terms of the number of sign changes of its coefficients, has been an indispensable source of inspiration for generations of mathematicians. Trying to extend and sharpen this rule, we consider below the set of all real univariate polynomials of a given degree, a given collection of signs of their coefficients, and given numbers of positive and negative roots. In spite of the elementary definition of the main object of our study, it is a non-trivial question for which sign patterns and numbers of positive and negative roots the corresponding set is non-empty. The main result of the present paper is a discovery of a new infinite family of non-realizable combinations of sign patterns and the numbers of positive and negative roots.

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1 Introduction

This paper continues the line of study of Descartes' rule of signs initiated in [4]. The basic set-up under consideration is as follows.

Consider the affine space Pol_d of all real monic univariate polynomials of degree d. Below we concentrate on polynomials from Pol_d with all coefficients non-vanishing. An arbitrary ordered sequence $\overline{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_d)$ of $\pm$-signs is called a *sign pattern*. When working with monic polynomials we will use their *shortened sign patterns* $\widehat{\sigma}$ representing the signs of all coefficients except the leading term which equals 1. For the actual sign pattern $\overline{\sigma}$, we write $\overline{\sigma} = (1, \widehat{\sigma})$ to emphasise that we consider monic polynomials.

Given a shortened sign pattern $\widehat{\sigma}$, we say that its *Descartes pair* $(p_{\widehat{\sigma}}, n_{\widehat{\sigma}})$ is the pair of non-negative integers counting sign changes and sign preservations of $\overline{\sigma} = (1, \widehat{\sigma})$. By Descartes' rule of signs, $p_{\widehat{\sigma}}$ (resp. $n_{\widehat{\sigma}}$) gives the upper bound on the number of positive (resp. negative) roots of any monic polynomial from $Pol_d(\widehat{\sigma})$. (Observe that, for any $\widehat{\sigma}$, $p_{\widehat{\sigma}}+n_{\widehat{\sigma}} = d$.) To any monic polynomial $q(x)$ with the sign pattern $\overline{\sigma} = (1, \widehat{\sigma})$, we associate the pair (pos_q, neg_q) giving the numbers of its positive and negative roots counted with multiplicities. Obviously the pair (pos_q, neg_q) satisfies the standard restrictions

$$pos_q \leq p_{\overline{\sigma}},\ pos_q \equiv p_{\overline{\sigma}} \pmod 2,\ neg_q \leq n_{\overline{\sigma}},\ neg_q \equiv n_{\overline{\sigma}} \pmod 2. \qquad (1.1)$$

We call pairs (pos, neg) satisfying (1.1) *admissible* for $\overline{\sigma}$. Conversely, for a given pair (pos, neg), we call a sign pattern $\overline{\sigma}$ such that (1.1) is satisfied *admitting* the latter pair. It turns out that there exist couples $(\overline{\sigma}, (pos, neg))$, where $\overline{\sigma}$ is a sign pattern and (pos, neg) is a pair admissible for $\overline{\sigma}$, which are not realizable by polynomials. Namely, D. J. Grabiner [5] found the first example of non-realizable combination for polynomials of degree 4. He has shown that the sign pattern $(+, -, -, -, +)$ does not allow to realize the pair $(0, 2)$ and the sign pattern $(+, +, -, +, +)$ does not allow to realize $(2, 0)$. Observe that their Descartes pairs equal $(2, 2)$.

His argument is very simple. (Due to symmetry induced by $x \mapsto -x$ it suffices to consider only the first case.) Observe that a fourth-degree polynomial with only two negative roots for which the sum of roots is positive could be factored as $a(x^2 + bx + c)(x^2 - sx + t)$ with $a, b, c, s, t > 0,\ s^2 < 4t$, and $b^2 \geq 4c$.

The product of these factors equals $a(x^4 + (b - s)x^3 + (t + c - bs)x^2 + (bt - cs)x + ct)$. To get the correct sign pattern, we need $b < s$ and $bt < cs$, which gives $b^2t < s^2c$ and thus $b^2/c < s^2/t$. But we have $b^2/c \geq 4 > s^2/t$.

The following basic question and related conjecture were formulated in [4]. (Apparently for the first time Problem 1 was mentioned in [3].)

Problem 1. *For a given sign pattern $\overline{\sigma}$, which admissible pairs (pos, neg) are realizable by polynomials whose signs of coefficients are given by $\overline{\sigma}$?*

Observe that we have the natural $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on the space of monic polynomials and on the set of all sign patterns respectively. The first generator acts by reverting the signs of all monomials in second, fourth etc. position (which for polynomials means $P(x) \to (-1)^d P(-x)$); the second generator acts by reading the pattern backwards (which for polynomials means $P(x) \to x^d P(1/x)$). If one wants to preserve the set of monic polynomials one has to divide $x^d P(1/x)$ by its leading term. We will refer to the latter action as the *standard* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-action*. Up to some trivialities, the properties we will study below are invariant under this action. The following initial results were partially proven in [3, 1] and in complete generality in [4].

Theorem 1.

(i) *Up to degree $d \leq 3$, for any sign pattern $\overline{\sigma}$, all admissible pairs (pos, neg) are realizable.*

(ii) *For $d = 4$, (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) the only non-realizable combination is $(1, -, -, -, +)$ with the pair $(0, 2)$;*

(iii) *For $d = 5$, (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) the only non-realizable combination is $(1, -, -, -, -, +)$ with the pair $(0, 3)$;*

(iv) *For $d = 6$, (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) the only non-realizable combinations are $(1, -, -, -, -, -, +)$ with $(0, 2)$ and $(0, 4)$; $(1, +, +, +, -, +, +)$ with $(2, 0)$; $(1, +, -, -, -, -, +)$ with $(0, 4)$.*

The next two results can be found in [4] and [8].

Theorem 2. *For $d = 7$, among the 1472 possible combinations of a sign pattern and a pair (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action), there exist exactly 6 which are non-realizable. They are:*

$$(1, +, -, -, -, -, -, +) \quad with \quad (0, 5); \qquad (1, +, -, -, -, -, +, +) \quad with \quad (0, 5);$$

$$(1,+,-,+,-,-,-,-) \quad \textit{with} \quad (3,0); \quad (1,+,+,-,-,-,-,+) \quad \textit{with} \quad (0,5);$$
$$\textit{and}, \quad (1,-,-,-,-,-,-,+) \quad \textit{with}\ (0,3)\ \textit{and}\ (0,5).$$

Theorem 3. *For $d = 8$, among the 3648 possible combinations of a sign pattern and a pair (up to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action), there exist exactly 13 which are non-realizable. They are:*

$$(1,+,-,-,-,-,-,+,+) \quad \textit{with} \quad (0,6); \quad (1,-,-,-,-,-,-,+,+) \quad \textit{with} \quad (0,6);$$
$$(1,+,+,+,-,-,-,-,+) \quad \textit{with} \quad (0,6); \quad (1,+,+,-,-,-,-,-,+) \quad \textit{with} \quad (0,6);$$
$$(1,+,+,+,-,+,+,+,+) \quad \textit{with} \quad (2,0); \quad (1,+,+,+,+,+,-,+,+) \quad \textit{with} \quad (2,0);$$
$$(1,+,+,+,-,+,-,+,+) \quad \textit{with}\ (2,0)\ \textit{and}\ (4,0)\ ; \quad (1,-,-,-,+,-,-,-,+) \quad \textit{with}$$
$$(0,2)\ \textit{and}\ (0,4); \quad (1,-,-,-,-,-,-,-,+) \quad \textit{with}\ (0,2), (0,4),\ \textit{and}\ (0,6).$$

Finally, it was shown in [9] that for $d = 11$, the sign pattern

$$(+,-,-,-,-,-,+,+,+,+,+,-)$$

is not realizable with the admissible pair $(1,8)$. This is the first example found of non-realizability in which both components of the admissible pair are nonzero.

The first goal of the present paper is to present a new infinite series of non-realizable patterns, defined for odd degrees, this is why we call it *the odd series.* (Two other series can be found in [4]; one of them, defined for even degrees, is called *the even series.*) Namely, for a fixed odd degree $d \geq 5$ and $1 \leq k \leq (d-3)/2$, denote by σ_k the sign pattern beginning with two pluses followed by k pairs "$-,+$" and then by $d-2k-1$ minuses. Its Descartes pair equals $(2k+1, d-2k-1)$.

Theorem 4.

(i) *The sign pattern σ_k is not realizable with any of the pairs $(3,0)$, $(5,0)$, $\ldots$, $(2k+1,0)$;*

(ii) *the sign pattern σ_k is realizable with the pair $(1,0)$;*

(iii) *the sign pattern σ_k is realizable with any of the pairs $(2\ell+1, 2r)$, $\ell = 0, 1, \ldots, k$, $r = 1, 2, \ldots, (d-2k-1)/2$.*

Theorem 4 is proved in § 2. Notice that Cases (i), (ii) and (iii) exhaust all possible admissible pairs (pos, neg). It is also worth mentioning that Theorem 4 covers the only non-realizable case for degree 5 (up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) and the third and the last two non-realizable cases for degree 7 mentioned above. Indeed,

1) for $d = 5$, the sign pattern $\sigma^\bullet := (+,-,-,-,-,+)$ is not realizable with the admissible pair $(0,3)$ if and only if the sign pattern $\sigma^\circ := (+,+,-,+,-,-)$ is not realizable with the pair $(3,0)$ (should the couple $(\sigma^\bullet, (0,3))$ be realizable by some polynomial $P(x)$, then $(\sigma^\circ, (3,0))$ should be realizable by the polynomial $-P(-x)$); see part (iii) of Theorem 1;

2) for $d = 7$, the third example of Theorem 2 is exactly the case of σ_1 with the pair $(3,0)$ of Theorem 4;

3) by analogy with 1), the sign pattern $(+,-,-,-,-,-,-,+)$ is not realizable with the admissible pair $(0,3)$ or $(0,5)$ (see the last case in Theorem 2) if and only if the sign pattern $(+,+,-,+,-,+,-,-)$ is not realizable with the pair $(3,0)$ or $(5,0)$ (this is the case of σ_2, see Theorem 4).

The second aim of this paper is to present discriminant loci for families of polynomials of degree $d \leq 4$, see § 3. These loci and the coordinate hyperplanes partition the space of coefficients of monic degree d polynomials into open domains in each of which one and the same couple (sign pattern, admissible pair) is realized. We explain the correspondence between the couples and the domains and for degree 4, we explain the non-realizability of the case mentioned in part (ii) of Theorem 1 by the absence of the corresponding domain.

2 Proofs

Proof of Theorem 4. Part (i): Suppose that a polynomial $P := \sum_{j=0}^{d} a_j x^{d-j}$ has the sign pattern σ_k and realizes the pair $(2s+1, 0)$, $1 \leq s \leq k$. Denote by

$$P_e := \sum_{\nu=0}^{(d-1)/2} a_{2\nu+1} x^{d-2\nu-1} \quad \text{and} \quad P_o := \sum_{\nu=0}^{(d-1)/2} a_{2\nu} x^{d-2\nu}$$

its even and odd parts respectively. In each of the sequences $\{a_{2\nu+1}\}_{\nu=0}^{(d-1)/2}$ and $\{a_{2\nu}\}_{\nu=0}^{(d-1)/2}$ there is exactly one sign change. Therefore by Descartes' rule of signs each of the polynomials P_e and P_o has exactly one real positive root (denoted by x_e and x_o respectively) which is simple.

Remarks 5.

(i) The polynomial P_e (resp. P_o) is positive and increasing on (x_e, ∞) (resp. on (x_o, ∞)) and negative on $[0, x_e)$ (resp. on $(0, x_o)$).

(ii) One has $x_o \neq x_e$, otherwise $P(-x_o) = 0$, i.e. P has a negative root which is a contradiction.

Without loss of generality we assume all positive roots of P to be distinct. Indeed, if P has a positive root h of multiplicity $\kappa > 1$, then P is of the form $P = (x-h)^{\kappa} P^{\sharp}$, where $P^{\sharp}(h) \neq 0$. Then for $\varepsilon > 0$ small enough, the polynomial $(x-h)^{\kappa-1}(x-h-\varepsilon)P^{\sharp}$ realizes the same sign pattern as P, with the same admissible pair $(2s+1, 0)$, but has one simple positive root more than P. Continuing like this one can obtain after $\leq 2s$ steps a polynomial with $2s+1$ simple positive roots, defining the same sign pattern as P and realizing the admissible pair $(2s+1, 0)$

Denote the smallest three of the positive roots of P by $0 < \xi_1 < \xi_2 < \xi_3$. Hence at any point $\zeta \in (\xi_1, \xi_2)$ one has the $P(\zeta) > 0$; clearly P is negative on (ξ_2, ξ_3). One can choose $\zeta \neq x_e$ and $\zeta \neq x_o$. Hence it is impossible to have $P_e(\zeta) \leq 0$ and $P_o(\zeta) \leq 0$ (with at most one equality, see part (ii) of Remarks 5). It is also impossible to have $P_e(\zeta) \geq 0$ and $P_o(\zeta) \geq 0$. Indeed, this would imply that $x_e \leq \zeta$ and $x_o \leq \zeta$. Thus one would get $P_e(x) \geq 0$ and $P_o(x) \geq 0$, i.e. $P(x) > 0$, for $x \in (\xi_2, \xi_3)$ – a contradiction.

The two remaining possibilities are (one can skip the possibilities to have equalities, they were already taken into account):

a) $P_e(\zeta) > 0$, $P_o(\zeta) < 0$;

b) $P_e(\zeta) < 0$, $P_o(\zeta) > 0$.

The first one is impossible because it would imply that

$$P(-\zeta) = P_e(\zeta) - P_o(\zeta) > 0 \ ,$$

and since $P(0) < 0$ and $P(x) \to -\infty$ for $x \to -\infty$, the polynomial P would have at least one negative root in $(-\infty, -\zeta)$ and at least one in $(-\zeta, 0)$ – a contradiction.

So suppose that possibility b) takes place. In this case one must have $x_o < \zeta < x_e$. Without loss of generality one can assume that $\xi_1 = 1$; this can be achieved by a rescaling $x \mapsto \xi_1 x$. Hence $P_o(1) = \beta > 0$ and $P_e(1) = -\beta$. Considering the polynomial P/β instead of P, one can assume that $\beta = 1$. Lemma 6 below immediately implies that there are no real roots of P larger than 1 (one can use the Taylor series of P at 1) which is a contradiction finishing the proof of Part (i).

Lemma 6. *Under the above assumptions, $P^{(m)}(1) > 0$, for any $m = 1, 2, \ldots, d$.*

Proof of Lemma 6. In the proof we use minimization arguments which can be applied to compact sets, so we allow zero values of the coefficients as well. For any $m = 1, 2, \ldots, d$, it is true that if the sum of the coefficients $\delta := a_2 + a_4 + \cdots + a_{d-1}$ is fixed (recall that all these coefficients are negative), then $P_o^{(m)}(1)$ is minimal for $a_2 = \delta$, $a_4 = a_6 = \cdots = a_{d-1} = 0$. Indeed, when taking derivatives and computing their values at $x = 1$, monomials of larger degree in x are multiplied by larger factors (equal to these degrees); we apply $(d-3)/2$ times the fact that if $A \geq 0$, $B \geq 0$ and $\lambda > \mu > 0$, then for $A + B$ fixed, the sum $\lambda A + \mu B$ is maximal when $B = 0$. Therefore in what follows we assume that $a_4 = a_6 = \cdots = a_{d-1} = 0$, and hence $a_2 = 1 - a_0 < 0$.

Similarly, consider $P_e^{(m)}(1)$. Recall that $a_1 > 0$, $a_3 > 0$, …, $a_{2k+1} > 0$, $a_{2k+3} < 0$, $a_{2k+5} < 0$, …, $a_d < 0$. Hence for fixed sums $\delta_* := a_1 + a_3 + \cdots + a_{2k+1}$ and $\delta_{**} := a_{2k+3} + a_{2k+5} + \cdots + a_d$, the value of $P_e^{(m)}(1)$ is minimal if

$$\begin{cases} a_1 = \cdots = a_{2k-1} = 0 \ , \ a_{2k+1} = \delta_* \\ a_{2k+5} = \cdots = a_d = 0 \ , \ a_{2k+3} = \delta_{**}. \end{cases} \tag{2.1}$$

Let us now assume that conditions (2.1) are valid. Thus $P_e = a_{2k+1}x^{d-2k-1} + a_{2k+3}x^{d-2k-3}$ and $a_{2k+1} + a_{2k+3} = -1$. One can further decrease $P_e^{(m)}(1)$ by assuming that $a_{2k+1} = 0$, $a_{2k+3} = -1$. Thus $P(x) = a_0x^d + a_2x^{d-2} - x^{d-2k-3}$ and $a_0 + a_2 = 1$.

But then $P^{(m)}(x) = u_m a_0 x^{d-m} + v_m a_2 x^{d-2-m} - w_m x^{d-2k-3-m}$ and $P^{(m)}(1) = u_m a_0 + v_m a_2 - w_m$ for some numbers $0 \leq w_m \leq v_m < u_m$. Therefore

$$\begin{aligned} P^{(m)}(1) &= w_m(a_0 + a_2 - 1) + (v_m - w_m)(a_0 + a_2) + (u_m - v_m)a_0 \\ &= (v_m - w_m)(a_0 + a_2) + (u_m - v_m)a_0 > 0 \ . \end{aligned}$$

□

Proof of Part (ii): The polynomial $x^d - 1$ has the necessary signs of the leading coefficient and of the constant term. It has a single real simple root at 1. One can construct a polynomial of the form $S := x^d - 1 + \varepsilon \sum_{j=1}^{d-1} c_j x^j$, where $c_j = 1$ (resp. $c_j = -1$) if the sign at the corresponding position of σ_k is + (resp. −). For a small enough $\varepsilon > 0$, the polynomial S has a single simple real root close to 1, and its coefficients have the sign pattern σ.

Finally, our approach to settling Part (iii) is based on the following lemma borrowed from [4].

Lemma 7 (See Lemma 14 in [4])**.** *Suppose that the monic polynomials P_1 and P_2 of degrees d_1 and d_2 with sign patterns $\bar{\sigma}_1 = (1, \hat{\sigma}_1)$ and $\bar{\sigma}_2 = (1, \hat{\sigma}_2)$, respectively, realize the pairs (pos_1, neg_1) and (pos_2, neg_2).*

Then

(i) *if the last position of* $\hat{\sigma}_1$ *is* $+$, *then for any small enough* $\varepsilon > 0$, *the polynomial* $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ *realizes the sign pattern* $(1, \hat{\sigma}_1, \hat{\sigma}_2)$ *and the pair* $(pos_1 + pos_2, neg_1 + neg_2)$.

(ii) *if the last position of* $\hat{\sigma}_1$ *is* $-$, *then for any* $\varepsilon > 0$ *small enough, the polynomial* $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ *realizes the sign pattern* $(1, \hat{\sigma}_1, -\hat{\sigma}_2)$ *and the pair* $(pos_1 + pos_2, neg_1 + neg_2)$. *(Here* $-\hat{\sigma}$ *is the sign pattern obtained from* $\hat{\sigma}$ *by changing each* $+$ *by* $-$ *and vice versa.)*

Remark 8. Example 15 in [4] explains some of the possible applications of Lemma 7. We present and extend this example below. If

$$P_2 = x - 1 \ , \ x + 1 \ , \ x^2 + 2x + 2 \ , \ x^2 + 2x + 0.5 \ , \ x^2 - 2x + 2 \quad \text{or} \quad x^2 - 2x + 0.5 \ ,$$

then $(pos_2, neg_2) = (1, 0)$, $(0, 1)$, $(0, 0)$, $(0, 2)$, $(0, 0)$ and $(2, 0)$ respectively. Denote by τ the last entry of $\hat{\sigma}_1$. When $\tau = +$, then one has respectively $\hat{\sigma}_2 = (-)$, $(+)$, $(+, +)$, $(+, +)$, $(-, +)$ and $(-, +)$ and the sign pattern of $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ equals

$$(1, \hat{\sigma}_1, -) \ , \ (1, \hat{\sigma}_1, +) \ , \ (1, \hat{\sigma}_1, +, +) \ , \ (1, \hat{\sigma}_1, +, +) \ , \ (1, \hat{\sigma}_1, -, +) \quad \text{or} \quad (1, \hat{\sigma}_1, -, +) \ .$$

If $\tau = -$, then $-\hat{\sigma}_2 = (+)$, $(-)$, $(-, -)$, $(-, -)$, $(+, -)$ and $(+, -)$ and the sign pattern of $\varepsilon^{d_2} P_1(x) P_2(x/\varepsilon)$ equals

$$(1, \hat{\sigma}_1, +) \ , \ (1, \hat{\sigma}_1, -) \ , \ (1, \hat{\sigma}_1, -, -) \ , \ (1, \hat{\sigma}_1, -, -) \ , \ (1, \hat{\sigma}_1, +, -) \quad \text{or} \quad (1, \hat{\sigma}_1, +, -) \ .$$

Proof of Part (iii): Recall that the sign pattern σ_k ends with $d - 2k - 1$ minuses. Set $\sigma_k = (+, +, \sigma^*, \sigma^\dagger)$, where the sign patterns σ^* (resp. $\sigma^\dagger$) consist of a minus followed by k pairs $(+, -)$ (resp. of $d - 2k - 2$ minuses).

The sign pattern $(+, +)$ is realizable by the polynomial $x + 1$ (hence with the pair $(0, 1)$). To obtain a polynomial realizing the sign pattern $(+, +, \sigma^*)$ with the pair $(2\ell + 1, 1)$ one applies Lemma 7, first $k - \ell$ times with $P_2 = x^2 - 2x + 2$, and then $2\ell + 1$ times with $P_2 = x - 1$. After this one applies Lemma 7, first $2r - 1$ times with $P_2 = x + 1$, and then $(d - 2k - 1)/2 - r$ times with $P_2 = x^2 + 2x + 2$ to realize the sign pattern σ_k with the pair $(2\ell + 1, 2r)$. □

3 Discriminant loci of cubic and quartic polynomials under a microscope

The goal of this section is mainly pedagogical. For the convenience of our readers, we present below detailed descriptions and illustrations of cases of (non)realizability of sign patterns and admissible pairs for polynomials of degree up to 4.

Define the *standard real discriminant locus* $\mathcal{D}_d \subset Pol_d$ as the subset of all polynomials having a real multiple root. (Detailed information about a natural stratification of $\mathcal{D}_d$ can be found in e.g., [6].) It is a well-known and simple fact that $Pol_d \setminus \mathcal{D}_d$ consists of $\left[\frac{d}{2}\right] + 1$ components distinguished by the number of real simple roots. Moreover, each such component is contractible in Pol_d. Obviously, the number of real roots in a family of monic polynomials changes if and only if this family crosses the discriminant locus $\mathcal{D}_d$.

3.1 Degrees 1 and 2

Clearly, a polynomial $x + u$ has a single real root $-u$ whose sign is opposite to the sign of the constant term. For degrees 2, 3 and 4 we will use the invariance of the zero set of the family of polynomials $x^n + a_1 x^{n-1} + \cdots + a_n$ with respect to the group of quasi-homogeneous dilatations $x \mapsto tx$, $a_j \mapsto t^j a_j$, to set the subdominant coefficient to 1. Namely, for $a_1 \neq 0$, if we set $x \mapsto a_1 x$, then this changes the family of polynomials

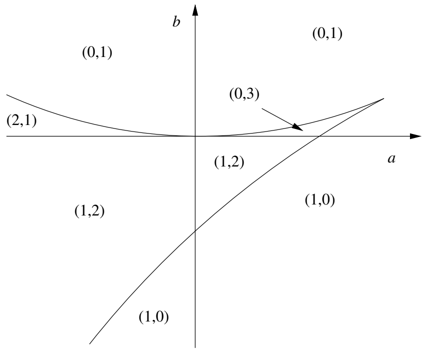

Figure 1. The discriminant locus of the family $x^3 + x^2 + ax + b$.

into $(a_1)^n(x^n + x^{n-1} + \cdots)$ which upon division by $(a_1)^n$ (which preserves the zero set) gives $x^n + x^{n-1} + \cdots$. Thus for $n = 2$, we consider the family $P_2 := x^2 + x + a$. For $a \leq 1/4$, it has two real roots; for $a < 1/4$, these are distinct. For $a \in (0, 1/4)$, they are both negative while for $a < 0$, they are of opposite signs.

3.2 Degree 3

For $n = 3$, we consider the family $P_3 := x^3 + x^2 + ax + b$. Its discriminant locus Σ is defined by the equation $4a^3 - a^2 + 4b - 18ab + 27b^2 = 0$. This is a curve shown in Fig. 1. It has an ordinary cusp for $(a, b) = (1/3, 1/27)$ and an ordinary tangency to the a-axis at the origin. In the eight regions of the complement to its union with the coordinate axes, the polynomial has roots as indicated in Fig. 1. (Here $(0, 1)$ means 0 positive and 1 negative real roots hence there exists a complex conjugate pair as well.) The point of the cusp corresponds to a triple root at $-1/3$, the upper arc corresponds to the case of one double real root to the right and a simple one to the left (and vice versa for the lower arc).

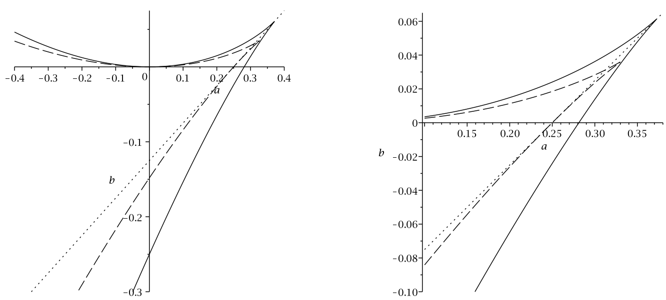

Figure 2. The projection of the discriminant locus of $x^4 + x^3 + ax^2 + bx + c$ to the plane of parameters (a, b). (Picture on the right shows the enlarged portion of the projection near the cusp point.)

3.3 Degree 4

For $n = 4$, we consider the family $P_4 := x^4 + x^3 + ax^2 + bx + c$. In Fig. 2 we show the projection $\tilde{\Phi}$ of its discriminant locus Φ in the (a, b)-plane. (For the other sets their projections in (a, b) are denoted by the same letters with tilde.) By the dashed line we show the set Σ for the family P_3. One has

$$\Phi \cap \{c = 0\} = \Sigma \cup \{b = c = 0\}.$$

By the solid line we represent the projection

$$\tilde{\Lambda} \ : \ 64a^3 - 18a^2 + 54b - 216ab + 216b^2 = 0$$

of the subset $\Lambda \subset \Phi$ for which the polynomial P_4 has a real root of multiplicity at least 3. The ordinary cusp point of $\tilde{\Lambda}$ is the projection of the point $(3/8, 1/16, 1/256)$ which defines the polynomial $x^4 + x^3 + 3x^2/8 + x/16 + 1/256 = (x + 1/4)^4$ to the plane (a, b).

At this point the set Φ has a swallowtail singularity, see e.g. [2]. On the upper arc of Λ the polynomial P_4 has one triple root to the right and a simple one to the left (and vice versa for the lower arc). The upper arc of $\tilde{\Lambda}$ has an ordinary tangency to the a-axis at the origin. Along the curve Λ the intersections of the hypersurface Φ with planes transversal to Λ have cusp points.

The cusp point of Σ belongs to Λ. At this point Λ intersects the (a, b)-plane. The tangent line $\tilde{L} : b = a/2 - 1/8$ to $\tilde{\Lambda}$ at its cusp at $(3/8, 1/16)$ is tangent to the curve Σ at $(1/4, 0)$. ($\tilde{L}$ is shown by the dotted line.) The set L corresponds to polynomials having two double roots. For $a < 3/8$, these roots are real, and for $a > 3/8$, they are complex conjugate. The curve L is tangent to the (a, b)-plane at the point $(1/4, 0, 0)$. It belongs to the half-space $\{c \geq 0\}$.

Now we consider the intersections of Φ with the planes parallel to the (b, c)-plane. For $a < 3/8$, they have two ordinary cusps (which are the points of Λ) and a transversal self-intersection point (which belongs to L). The first three pictures in Fig. 3 show this intersection with the plane $a = -0.1$ in different scales. The curves are tangent to the a-axis. Inside the curvilinear triangle (denoted by H_4) the polynomial has four distinct real roots. In the domain H_2 which surrounds H_4, the polynomial P_4 has two distinct real roots and a complex conjugate pair. In the domain H_0 above the self-intersection point it has two complex conjugate pairs. These domains are defined in the same way for all $a < 3/8$. For $a > 3/8$, the domain H_4 does not exist.

The set $\Phi \cap \{a < 0, b < 0, c > 0\}$ divides the set $\{a < 0, b < 0, c > 0\}$ into four sectors, see the first picture in Fig. 3. The intersection $\{a < 0, b < 0, c > 0\} \cap H_2$ consists of two contractible components. They correspond to the two cases $(0, 2)$ (the right sector, bordering $\{a < 0, b > 0, c > 0\}$) and $(2, 0)$ (the left sector) realizable with the sign pattern $(+, +, -, -, +)$. The other two cases realizable in $\{a < 0, b < 0, c > 0\}$ are $(2, 2)$ (the sector below) and $(0, 0)$ (the sector above).

For $a < 0$, $b > 0$, $c > 0$, and when the polynomial P_4 belongs respectively to H_4, H_2 or H_0, it realizes the cases $(2, 2)$, $(0, 2)$ and $(0, 0)$. The set $\{a < 0, b > 0, c > 0\} \cap H_2$ is contractible, so only one of the cases $(0, 2)$ and $(2, 0)$ (namely, $(0, 2)$) is realizable with the sign pattern $(+, +, -, +, +)$ (see the first picture in Fig. 3).

In $\{a < 0, b < 0, c < 0\}$ one can realize the cases $(1, 3)$ and $(1, 1)$. They correspond to the domains $\{a < 0, b < 0, c < 0\} \cap H_4$ (the curvilinear triangle) and $\{a < 0, b < 0, c < 0\} \cap H_2$ (its complement).

In $\{a < 0, b > 0, c < 0\}$ one can similarly realize the cases $(3, 1)$ (the curvilinear triangle) and $(1, 1)$ (its complement).

On the fourth and fifth pictures in Fig. 3 we present the intersection of Φ with the plane $\{a = 0.15\}$. The figures are quite similar to the first three pictures in Fig. 3, and the realizable pairs are the same with one exception. Namely, for $a > 0$, $b > 0$, $c > 0$ in the domain H_4 it is the pair $(0, 4)$ which is realized. And, clearly, the third component of the sign patterns changes from $-$ to $+$.

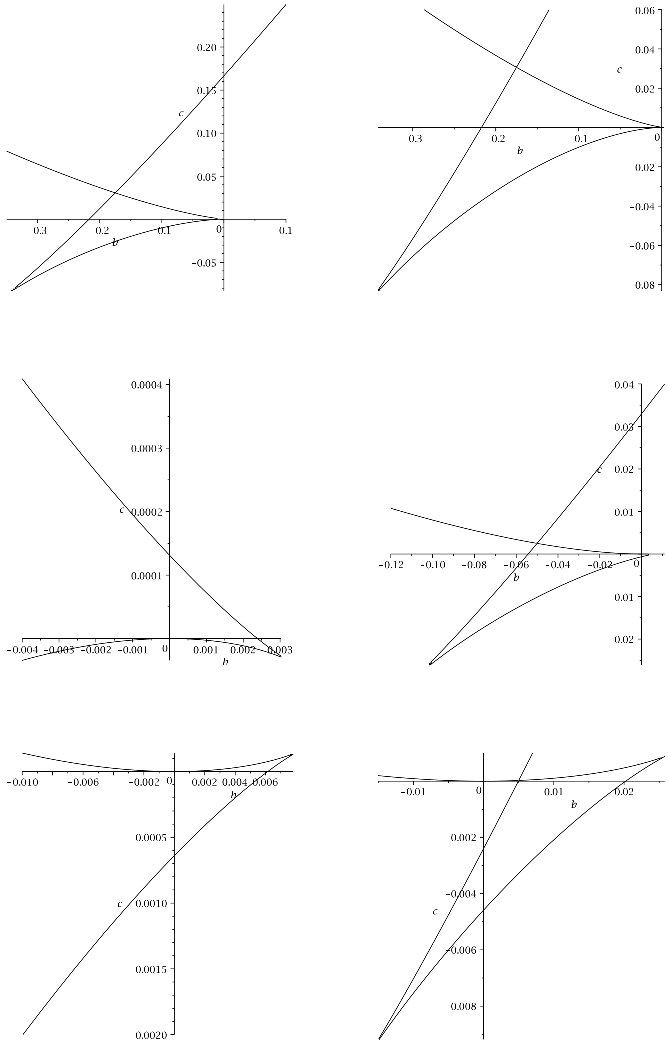

Figure 3. Intersections of the discriminant locus of $x^4 + x^3 + ax^2 + bx + c$ with the planes $a = -0.1$ (the first three pictures); $a = 0.15$ (the fourth and the fifth pictures); and $a = 0.26$ (the last picture).

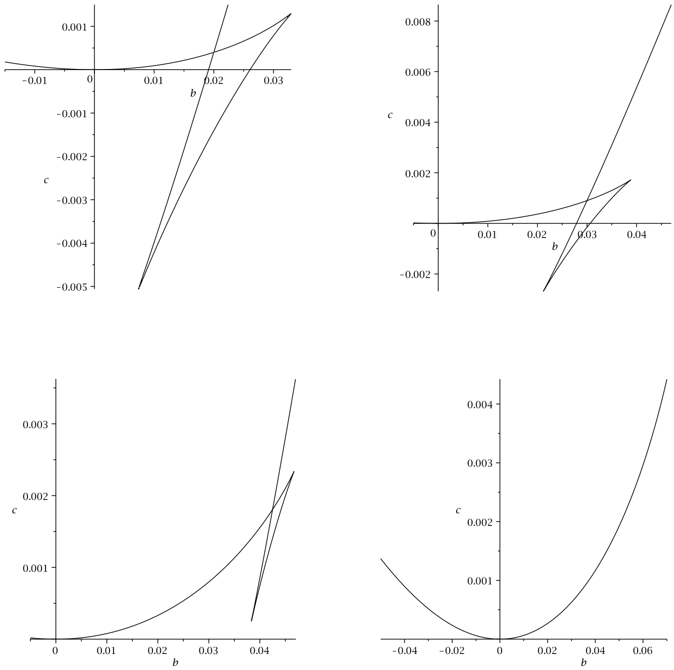

Figure 4. The intersection of the discriminant locus of $x^4 + x^3 + ax^2 + bx + c$ with the planes $a = 0.29; 0.31; 0.335; 0.4$.

The intersections of Φ with the planes $\{a = 0.26\}$, $\{a = 0.29\}$, $\{a = 0.31\}$ and $\{a = 0.335\}$ are shown on the last picture in Fig. 3 and in Fig. 4. For $a_0 > 0.375$, the intersections of Φ with the planes $\{a = a_0\}$ resemble the lower right picture in Fig. 4.

4 Final Remarks

The following important questions closely related to the main topic of the present paper remained unaddressed above.

Problem 2. *Is the set of all polynomials realizing a given pair* (pos, neg) *and having a sign pattern* $\overline{\sigma}$ *path-connected (if non-empty)?*

Given a real polynomial p of degree d with all non-vanishing coefficients, consider the sequence of pairs

$$\{(pos_0(p), neg_0(p)), (pos_1(p), neg_1(p)), (pos_2(p), neg_2(p)), \ldots, (pos_{d-1}(p), neg_{d-1}(p))\},$$

where $(pos_j(p), neg_j(p))$ is the numbers of positive and negative roots of $p^{(j)}$ respectively. Observe that if one is given the above sequence of pairs, then one knows the sign pattern of a polynomial p which is assumed to be monic. Additionally it is easy to construct examples when the converse fails.

Problem 3. *Which sequences of pairs are realizable by real polynomials of degree d with all non-vanishing coefficients?*

Notice that a similar problem for the sequence of pairs of real roots (without division into positive and negative) was considered in [7]. One can easily find examples of non-realizable sequences $\{(pos_j(p), neg_j(p))\}_{j=0}^{d-1}$. E. g. for $d = 4$ this is the sequence $(2, 0)$,

$(2,1)$, $(1,1)$, $(0,1)$. Indeed, the sign pattern must be $(+,+,-,+,+)$ about which we know that it is not realizable with the pair $(2,0)$. However it is not self-evident that all non-realizable sequences are obtained in this way.

Our final question is as follows.

Problem 4. *Is the set of all polynomials realizing a given sequence of pairs as above path-connected (if non-empty)?*

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