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Modified Chromatic Schultz Polynomial of Some Cycle Related Graphs

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Abstract

Let $\mathcal{C} = c_1, c_2, \ldots, , c_\ell$ be a proper colouring of a connected graph G with chromatic number ℓ. Then, the chromatic Schultz polynomial $S(G, x)$ of G is defined as $S(G, x) = \sum\limits_{v_i, v_j \in V(G)} (\zeta(v_i) + \zeta(v_j))x^{d(u,v)}$, where $\zeta(v_i) = s$, when the vertex v_i has the colour c_s under $\mathcal{C}$. In this paper, we study the chromatic Schultz polynomials of certain cycle related graph classes.

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MSC(2010) 05C31, 05C15, 05C12.

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [1, 5, 12, 13] and for graph classes, we refer to [2, 4]. Further, for graph colouring, see [3, 6, 9]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

A *vertex colouring* is an assignment $c : V(G) \to \mathcal{C}$ which assigns the vertices of G, to a set of colours (or labels or weights) $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\}$. The vertex colouring c is said to be *proper* if no two adjacent vertices of G have same colours with respect to that colouring. The number of colours required in a minimum proper colouring of G is

*corresponding author

called the *chromatic number* of G and is denoted $\chi(G)$. A colour class of G is the set of all vertices of G which have the same colour. The cardinality of the colour class of a colour c_i is said to be the *strength* of that colour in G and is denoted by $\theta(c_i)$. We can also define a function $\zeta : V(G) \to \{1, 2, 3, \dots, \ell\}$ such that $\zeta(v_i) = s$ if and only if $c(v_i) = c_s, c_s \in \mathcal{C}$.

If we colour the vertices of G in such a way that c_1 is assigned to maximum possible number of vertices, then c_2 is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called a χ^--*colouring* of G (see [7]). In a similar manner, if c_ℓ is assigned to maximum possible number of vertices, then $c_{\ell-1}$ is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called a χ^+-*colouring* of G (see [7, 8].

With respect to a proper colouring $c : V(G) \to \mathcal{C}$, a function $\zeta : V(G) \to \mathbb{N}_0$ is defined in [10] as $\zeta(v) = s$ if $c(v) = c_s \in \mathcal{C}$.

The chromatic version of Schultz polynomial was introduced in [10] as follows:

Definition 1 (Chromatic Schultz Polynomial of Graphs)**.** [10] Let G be a connected graph with chromatic number $\chi(G)$. Then, the *chromatic Schultz polynomial* of G denoted by $S_\chi(G, x)$ is defined as

$$S\chi(G,x) = \sum_{u,v \in V(G)} (\zeta(u) + \zeta(v)) x^{d(u,v)}.$$

A modified version of the chromatic Schultz polynomial was also introduced in [10] as given below:

Definition 2 (Modified Chromatic Schultz Polynomial of Graphs)**.** [10] Let G be a connected graph with chromatic number $\chi(G)$. Then, the chromatic Schultz polynomial of G denoted by $S_\chi(G, x)$ is defined as

$$S\chi(G,x) = \sum_{u,v \in V(G)} (\zeta(u) \cdot \zeta(v)) x^{d(u,v)}.$$

The two versions of chromatic Schultz polynomials of some fundamental graph classes were determined in [10]. Following that article, in this paper, we investigate the chromatic Schultz polynomials of certain related graph classes.

Definition 3. Let G be a connected graph with chromatic number $\chi(G)$. Then, the *modified chromatic Schultz polynomial* of G, denoted by $S^*_\chi(G, x)$, is defined as

$$S^*_\chi(G,x) = \sum_{u,v \in V(G)} (\zeta(u)\zeta(v)) x^{d(u,v)}$$

Definition 4. Let G be a connected graph with chromatic number φ^- and φ^+ be the minimal and maximal parameter colouring of G. Then,

(i) the *modified χ^--chromatic Schultz polynomial* of G, denoted by $S^*_{\chi^-}$, is defined as

$$S^*_{\chi^-}(G,x) = \sum_{u,v \in V(G)} (\zeta_{\varphi^-}(u) \cdot \zeta_{\varphi^-}(v)) x^{d(u,v)};$$

and

(ii) the χ^+-*chromatic Schultz polynomial* of G, denoted by $S^*_{\chi^-}$, is defined as

$$S^*_{\chi^+}(G,x) = \sum_{u,v\in V(G)} (\zeta_{\varphi^+}(u)\cdot\zeta_{\varphi^+}(v))x^{d(u,v)}.$$

The two versions of chromatic Schultz polynomials of some fundamental graph classes namely paths, cycles and complete graphs were determined in [10]. The chromatic Schultz polynomial of certain other graph classes namely wheel graphs, helm graphs, closed helm graphs, sunflower graphs, flower graphs and sunflower graphs were determined in [11]. Following those articles, in this paper, we investigate the modified chromatic Schultz polynomials of certain cycle related graph classes.

2 Discussion and New Results

2.1 The Modified Chromatic Schultz Polynomial of Wheel Graphs

A *wheel graph, denoted by* W_n, is a graph obtained by joining all vertices of a cycle C_{n-1} to an external vertex. That is, $W_n = C_{n-1} + K_1$. The vertices on the cycle of W_n is called its rim vertices and the vertex K_1 is called the central vertex. The wheel graph on 9 vertices is shown in Figure 1.

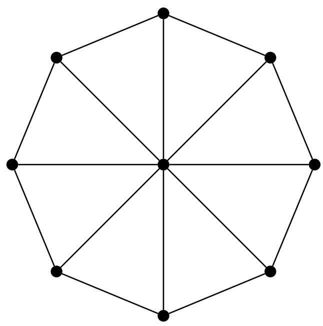

Figure 1. The wheel graph W_9.

The following theorem discusses the modified chromatic Schultz polynomial of wheel graphs.

Theorem 5. *Let W_n be a wheel graph on n vertices. Then, we have*

$$S^*_{\chi^-}(W_n,x) = \begin{cases} \frac{1}{8}(9n^2-44n+35)x^2+\frac{1}{2}(13n-13)x+\frac{1}{2}(5n+13); & \text{if } n \text{ is odd};\\ \frac{1}{8}(7n^2-6n-88)x^2+(8n+3)x+\frac{5}{2}(n+8); & \text{if } n \text{ is even.}\end{cases}$$

Proof. Let $V=\{v_1,v_2,\dots,v_n\}$ be the vertex set of W_n, where the rim vertices are labelled consecutively from v_1 to v_{n-1} and v_n is the central vertex. We note that the diameter of W_n is 2. Hence, the power of the variable x varies from 0 to 2 in the modified Schultz polynomial of W_n. Here, we need to consider the following two cases:

Note that if n is odd and $\chi(W_n)=4$ if n is even. Let c_1,c_2,c_3,c_4 be the four colours we use for colouring W_n.

Case-1: Let n be odd. In this case, $\chi(W_n)=3$. Let c_1,c_2,c_3 be the three colours we use for colouring the vertices of W_n. Then, with respect to a χ^--colouring, the vertices $v_1,v_3,v_5,\dots v_{n-2}$ get the colour c_1, the vertices $v_2,v_4,v_6,\dots v_{n-1}$ get the colour c_2 and v_n gets the colour c_3. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 1.

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1,c_1)	$\frac{n-1}{2}$
	(c_2,c_2)	$\frac{n-1}{2}$
	(c_3,c_3)	1
1	(c_1,c_2)	$n-1$
	(c_1,c_3)	$\frac{n-1}{2}$
	(c_2,c_3)	$\frac{n-1}{2}$
2	(c_1,c_1)	$\frac{(n-3)(n-1)}{8}$
	(c_2,c_2)	$\frac{(n-3)(n-1)}{8}$
	(c_1,c_2)	$\frac{(n-5)(n-1)}{4}$

Table 1

In Table 1, the possible distances between different pairs of vertices are written in the first column, the different colour pairs with respect to each distance is written in the second column and the number of corresponding colour pairs with respect to each distance is written in the third column.

From Table 1, we have the modified chromatic Schultz polynomial of the wheel graph W_n when number of vertices n is odd, is given by

$$\begin{aligned} S^*_{\chi^-}(W_n,x) &= [1(\frac{n-1}{2})+4(\frac{n-1}{2})+9]x^0+[2(n-1)+3(\frac{n-1}{2})+6(\frac{n-1}{2})]x^1 \\ &\quad +[1(\frac{(n-3)(n-1)}{8})+4(\frac{(n-3)(n-1)}{8})+2(\frac{(n-5)(n-1)}{8})]x^2 \\ &= \frac{1}{8}(9n^2-44n+35)x^2+\frac{13}{2}(n-1)x+\frac{1}{2}(5n+13) \end{aligned}$$

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1,c_1)	$\frac{n-2}{2}$
	(c_2,c_2)	$\frac{n-2}{2}$
	(c_3,c_3)	1
	(c_4,c_4)	1
1	(c_1,c_2)	$n-3$
	(c_1,c_4)	$\frac{n-2}{2}$
	(c_2,c_4)	$\frac{n-2}{2}$
	(c_1,c_3)	1
	(c_2,c_3)	1
	(c_3,c_4)	1
2	(c_1,c_1)	$\frac{(n-4)(n-2)}{8}$
	(c_2,c_2)	$\frac{(n-4)(n-2)}{8}$
	(c_1,c_2)	$\frac{(n-4)(n-2)}{8}$
	(c_2,c_3)	$\frac{n-4}{2}$
	(c_1,c_3)	$\frac{n-4}{2}$

Table 2

Case-2: Let n be even. In this case, $\chi(W_n)=4$. Let c_1,c_2,c_3,c_4 be the four colours we use for colouring the vertices of W_n. Then, with respect to a given χ^--colouring, the vertices $v_1, v_3, v_5, \dots v_{n-3}$ get the colour c_1, the vertices $v_2, v_4, v_6, \dots v_{n-2}$ get the colour c_2, the vertex v_3 gets the colour c_3 and the vertex v_4 gets the colour c_4. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 2.

From Table 2, we have the modified chromatic Schultz polynomial of the Wheel graph W_n when the number of vertices, n is even, is given by

$$\begin{aligned} S^*_{\chi^-}(W_n,x) &= [1(\frac{n-2}{2})+4(\frac{n-2}{2})+9(1)+16(1)]x^0+[2(n-3)+4(\frac{n-2}{2})+ \\ &\quad 8(\frac{n-2}{2})+3(1)+6(1)+12(1)]x^1+[1\frac{(n-4)(n-2)}{8} \\ &\quad +4\frac{(n-4)(n-2)}{8}+2\frac{(n-4)(n-2)}{8}+6(\frac{n-4}{2})+3(\frac{n-4}{2})]x^2 \\ &= \frac{1}{8}(7n^2-6n-88)x^2+(8n+3)x+\frac{5}{2}(n+8) \end{aligned}$$

Therefore,

$$S^*_{\chi^-}(W_n,x) = \begin{cases} \frac{1}{8}(9n^2-44n+35)x^2+\frac{1}{2}(13n-13)x+\frac{1}{2}(5n+13); & \text{if } n \text{ is odd;} \\ \frac{1}{8}(7n^2-6n-88)x^2+(8n+3)x+\frac{5}{2}(n+8); & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof. □

Since χ^+-colouring can be obtained by reversing the colouring pattern, the modified chromatic Schultz polynomial of a wheel graph with respect to its χ^+-colouring can be determined as follows:

Theorem 6. *Let W_n be a wheel with n vertices. Then, we have*

$$S^*_{\chi^+}(W_n,x) = \begin{cases} \frac{1}{8}(25n^2-144n+99)x^2+\frac{1}{2}(17n-17)x+\frac{1}{2}(13n-11); & \textit{if } n \textit{ is odd;} \\ \frac{1}{8}(26n^2-100n-40)x^2+\frac{1}{2}(31n-54)x+\frac{1}{2}(25n-40); & \textit{if } n \textit{ is even.} \end{cases}$$

2.2 The Modified Chromatic Schultz Polynomial of Helm Graphs

A helm graph consists of wheel graphs consisting of n rim vertices and each rim vertex has an extra (pendant) vertex attached to it. Therefore, a helm graph consists of $2n+1$ vertices where n is the number of rim vertices. The helm graph on 17 vertices is depicted in Figure 2.

When n is odd, H_n is 4-colourable and when n is even, H_n is 3-colourable. The modified chromatic Schultz polynomial of the helm graph with respect to χ^- colouring, can be determined as n the following theorem.

Theorem 7. *Let H_n be a helm graph with $2n+1$ vertices. Then, we have*

$$S^*_{\chi^-}(H_n,x) = \begin{cases} \frac{1}{2}(n^2-3n)x^4+\frac{1}{2}(5n^2-15n-24)x^3 \\ +\frac{1}{8}(25n^2+128n-241)x^2+(11n+11)x+\frac{5}{2}(3n+1); & \textit{if } n \textit{ is odd;} \\ \frac{1}{2}(n^2-3n)x^4+\frac{1}{2}(27n)x^3+\frac{1}{4}(25n^2-74n+96)x^2+ \\ 11nx+\frac{1}{2}(15n+2); & \textit{if } n \textit{ is even.} \end{cases}$$

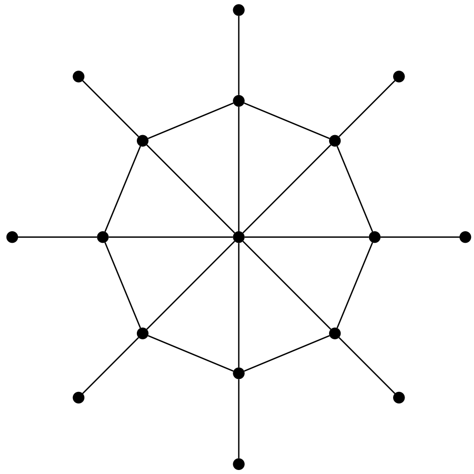

Figure 2. The helm graph H_8.

Proof. Let $V = \{v_1, v_2, \dots, v_{2n+1}$ be the vertex set of H_n, where the rim vertices are labelled consecutively from v_1 to v_n, the corresponding extra vertices are labelled from v_{n+1} to v_{2n} and v_{2n+1} is the central vertex. We note that the diameter of H_n is 4. Hence, the power of the variable x varies from 0 to 4 in the modified Schultz polynomial of H_n. Here, we need to consider the following two cases:

Note that $\chi(H_n) = 4$ if the number of rim vertices n is odd and $\chi(H_n) = 3$ if n is even. Let c_1, c_2, c_3, c_4 be the four colours we use for colouring H_n.

Case-1: Let n be odd. In this case, $\chi(H_n) = 4$. Let c_1, c_2, c_3, c_4 be the four colours we use for colouring the vertices of H_n. Then, with respect to a χ^--colouring, the vertices $v_n, v_{n+1}, v_{n+2}, \dots v_{2n}$ and v_{2n+1} get the colour c_1, the vertices $v_1, v_3, v_5, \dots v_{n-2}$ get the colour c_2, $v_2, v_4, v_6, \dots v_{n-1}$ get the colour c_3, and v_n gets the colour c_4. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 3.

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1, c_1)	$n+1$
	(c_2, c_2)	$\frac{n-1}{2}$
	(c_3, c_3)	$\frac{n-1}{2}$
	(c_4, c_4)	1
1	(c_1, c_2)	$n-1$
	(c_1, c_3)	$n-1$
	(c_1, c_4)	2
	(c_2, c_3)	$n-2$
	(c_2, c_4)	1
	(c_3, c_4)	1
2	(c_1, c_1)	n
	(c_2, c_2)	$\frac{(n-1)(n+1)}{8}$
	(c_3, c_3)	$\frac{(n-1)(n+1)}{8}$
	(c_1, c_2)	$n-1$
	(c_1, c_3)	$n-1$
	(c_2, c_3)	$\frac{(n-1)(n+1)}{4}$

	(c_1, c_4)	2
	(c_2, c_4)	$\frac{n-3}{2}$
	(c_3, c_4)	$\frac{n-3}{2}$
3	(c_1, c_1)	n
	(c_1, c_2)	$\frac{n(n-5)}{2}$
	(c_1, c_3)	$\frac{n(n-5)}{2}$
	(c_1, c_4)	$n-3$
4	(c_1, c_1)	$\frac{n(n-3)}{2}$

Table 3

In Table 3, the possible distances between different pairs of vertices are written in the first column, the different colour pairs with respect to each distance is written in the second column and the number of corresponding colour pairs with respect to each distance is written in the third column.

From Table 3, we have the chromatic Schultz polynomial of the helm graph H_n when number of rim vertices n is odd, is given by

$$\begin{aligned}
S^*_{\chi^-}(H_n, x) &= [n+1+13(\frac{n-1}{2})+16]x^0 + [5(n-1)+6(n-2)+28]x^1 \\
&\quad +[n+13\frac{(n^2-1)}{8}+5(n-1)+6\frac{(n^2-1)}{4}+8+20\frac{n-3}{2}]x^2 + \\
&\quad [n+5\frac{n(n-5)}{2}+4(n-3)]x^3+[1(\frac{(n-3)n}{2})]x^4 \\
&= \frac{1}{2}(n^2-3n)x^4+\frac{1}{2}(5n^2-15n-24)x^3+\frac{1}{8}(25n^2+128n-241)x^2 \\
&\quad +(11n+11)x+\frac{5}{2}(3n+1).
\end{aligned}$$

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1, c_1)	$n+1$
	(c_2, c_2)	$\frac{n}{2}$
	(c_3, c_3)	$\frac{n}{2}$
1	(c_1, c_2)	n
	(c_1, c_3)	n
	(c_2, c_3)	n
2	(c_1, c_1)	n
	(c_2, c_2)	$\frac{n(n-2)}{4}$
	(c_1, c_2)	n
	(c_1, c_3)	n
	(c_2, c_3)	$\frac{(n-4)(n-2)}{2}$
	(c_3, c_3)	$\frac{n(n-2)}{4}$
3	(c_1, c_1)	n
	(c_1, c_2)	$\frac{5n}{2}$
	(c_1, c_3)	$\frac{5n}{2}$
4	(c_1, c_1)	$\frac{n(n-3)}{2}$

Table 4

Case-2: Let n be even. In this case, $\chi(H_n) = 3$. Let c_1, c_2, c_3 be the three colours we use for colouring the vertices of H_n. Then, with respect to a χ^- colouring, the vertices $v_{n+1}, v_{n+2}, v_{n+3}, \ldots v_{2n}$ and v_{2n+1} get the colour c_1, the vertices $v_1, v_3, v_5, \ldots v_{n-1}$ get the colour c_2, and the vertices $v_2, v_4, v_6, \ldots v_n$ the colour c_3. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 4.

From Table 4, we have the modified chromatic Schultz polynomial of the helm graph H_n when the number of rim vertices n is even, is given by

$$\begin{aligned}
S^*_{\chi^-}(H_n, x) &= [1(n+1) + 4(\frac{n}{2}) + 9(\frac{n}{2})]x^0 + [2(n) + 3(n) + 6(n)]x^1 \\
&\quad + [1(n) + 4\frac{n(n-2)}{4} + 2(n) + 3(n) + 6\frac{(n-4)(n-2)}{2} + 9\frac{n(n-2)}{4}]x^2 \\
&\quad + [1(n) + 2(\frac{5n}{2}) + 3(\frac{5n}{2}]x^3 + [\frac{(n-3)n}{2}]x^4 \\
&= \frac{1}{2}(n^2 - 3n)x^4 + \frac{27}{2}nx^3 + \frac{1}{4}(25n^2 - 74n + 96)x^2 + 11nx + \frac{1}{2}(15n + 2).
\end{aligned}$$

Therefore,

$$S^*_{\chi^-}(H_n, x) = \begin{cases} \frac{1}{2}(n^2 - 3n)x^4 + \frac{1}{2}(5n^2 - 15n - 24)x^3 \\ +\frac{1}{8}(25n^2 + 128n - 241)x^2 + (11n + 11)x + \frac{5}{2}(3n + 1); & \text{if } n \text{ is odd;} \\ \frac{1}{2}(n^2 - 3n)x^4 + \frac{1}{2}(27n)x^3 + \frac{1}{4}(25n^2 - 74n + 96)x^2 + \\ (11n)x + \frac{1}{2}(15n + 2); & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof. □

Using a similar argument, we get the modified chromatic Schultz polynomial of a helm graph, with regard to a χ^+-colouring, as follows:

Theorem 8. *Let H_n be a helm graph with n rim vertices. Then, with respect to χ^+ colouring, we have*

$$S^*_{\chi^+}(H_n, x) = \begin{cases} 8(n^2 - 3n)x^4 + (10n^2 - 30n - 12)x^3 + \frac{1}{8}(19n^2 + \\ 308n - 181)x^2 + (26n - 19)x + \frac{1}{2}(45n + 20); & \textit{if } n \textit{ is odd;} \\ \frac{9}{2}(n^2 - 3n)x^4 + \frac{1}{2}(63n)x^3 + \frac{1}{4}(9n^2 + 38n + 32)x^2 + \\ 11nx + \frac{1}{2}(23n + 18); & \textit{if } n \textit{ is even.} \end{cases}$$

2.3 The Modified Chromatic Schultz Polynomial of Closed Helm Graphs

A *closed helm graph* consists of helm graphs with each attached extra vertex connected to its neighbouring extra vertices by edges. Therefore, a closed helm graph H^*_n consists of $2n+1$ vertices where n is the number of rim vertices. When n is odd, H^*_n is 4-colourable and when n is even, H^*_n is 3-colourable. The closed helm graph on 17 vertices is depicted in Figure 3.

The modified chromatic Schultz polynomial of the helm graph with respect to χ^- colouring, can be determined as in the following theorem.

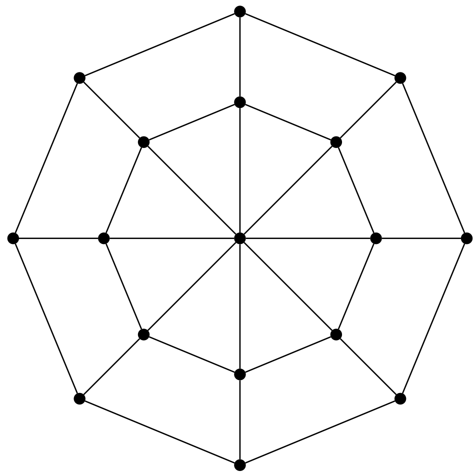

Figure 3. The closed helm graph H_8^*.

Theorem 9. *Let H_n^* be a closed helm graph with $2n+1$ vertices. Then, we have*

$$S^*_{\chi^-}(H_n^*,x)=\begin{cases}\frac{1}{8}(5n^2-4n-177)x^4+\frac{1}{4}(9n^2-2n-3)x^3\\ +\frac{1}{4}(2n^2+56n+10)x^2+\frac{1}{2}(24n+42)x+(5n+29); & \text{if } n \text{ is odd;}\\ \\ \frac{1}{4}(n^2-10n+24)x^4+(11n)x^3+\frac{1}{8}(2n^2+92n)x^2\\ +\frac{1}{2}(21n)x+(5n+9); & \text{if } n \text{ is even.}\end{cases}$$

Proof. Let $V=\{v_1,v_2,\ldots,v_{2n+1}\}$ be the vertex set of H_n^*, where the rim vertices are labelled consecutively from v_1 to v_n, the corresponding extra vertices are labelled from v_{n+1} to v_{2n} and v_{2n+1} is the central vertex. We note that the diameter of H_n^* is 4. Hence, the power of the variable x varies from 0 to 4 in the modified Schultz polynomial of H_n^*. Here, we need to consider the following two cases:

Note that $\chi(H_n^*)=4$ if the number of rim vertices n is odd and $\chi(H_n^*)=3$ if n is even. Let c_1,c_2,c_3,c_4 be the four colours we use for colouring H_n^*.

Case-1: Let n be odd. In this case, $\chi(H_n^*)=4$. Let c_1,c_2,c_3,c_4 be the four colours we use for colouring the vertices of H_n^*. Then, with respect to a χ^--colouring, the vertices $v_1,v_3,v_5,\ldots v_{n-2}$ and the vertices $v_{n+2},v_{n+4},v_{n+6},\ldots v_{2n}$ get the colour c_1, the vertices $v_2,v_4,v_6,\ldots v_{n-1}$ and the vertices $v_{n+3},vn+5,v_{n+7},\ldots v_{2n-1}$ get the colour c_2, the vertices v_n and v_{n+1} gets the colour c_3 and the vertex v_{2n+1} gets the colour c_4. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 5.

In Table 5, the possible distances between different pairs of vertices are written in the first column, the different colour pairs with respect to each distance is written in the second column and the number of corresponding colour pairs with respect to each distance is written in the third column.

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1,c_1)	$n-1$
	(c_2,c_2)	$n-1$
	(c_3,c_3)	2
	(c_4,c_4)	1

1	(c_1, c_2)	$3n-6$
	(c_1, c_3)	3
	(c_1, c_4)	$\frac{n-1}{2}$
	(c_2, c_3)	3
	(c_2, c_4)	$\frac{n-1}{2}$
	(c_3, c_4)	1
2	(c_1, c_1)	$\frac{3n-7}{2}$
	(c_2, c_2)	$\frac{3n-7}{2}$
	(c_3, c_3)	1
	(c_1, c_2)	$1+\frac{(n-5)(n-3)}{4}$
	(c_1, c_3)	$\frac{n-1}{2}$
	(c_2, c_3)	$\frac{(n-1)}{2}$
	(c_1, c_4)	$\frac{n-1}{2}$
	(c_2, c_4)	$\frac{n-1}{2}$
	(c_3, c_4)	1
3	(c_1, c_1)	$(\frac{n-3}{2})^2$
	(c_2, c_2)	$(\frac{n-3}{2})^2$
	(c_1, c_2)	$\frac{(n-3)(n+1)}{2}$
	(c_1, c_3)	$n-1$
	(c_2, c_3)	$n-1$
4	(c_1, c_1)	$\frac{(n-5)(n-3)}{8}$
	(c_2, c_2)	$\frac{(n-5)(n-3)}{8}$
	(c_1, c_3)	$\frac{n-7}{2}$
	(c_2, c_3)	$\frac{n-7}{2}$

Table 5

From Table 5, we have the modified chromatic Schultz polynomial of the closed helm graph H_n^* when number of rim vertices n is odd, is given by

$$\begin{aligned} S^*_{\chi^-}(H_n^*, x) &= [5(n-1)+34]x^0 + [2(3n-6)+9+12(\frac{n-1}{2})+30]x^1 \\ &\quad +[5(\frac{3n-7}{2})+9+2(\frac{n^2-8n+19}{4})+41(\frac{n-1}{2})+24]x^2 \\ &\quad +[5(\frac{n-3}{2})^2+2\frac{(n^2-2n-3)}{2}+9(n-1)]x^3 + \\ &\quad [5(\frac{(n^2-8n-15)}{4})+9(\frac{n-7}{2})]x^4 \\ &= \frac{1}{8}(5n^2-4n-177)x^4+\frac{1}{4}(9n^2-2n-3)x^3+\frac{1}{4}(2n^2+56n+10)x^2 \\ &\quad +\frac{1}{2}(24n+42)x+(5n+29). \end{aligned}$$

Case-2: Let n be even. In this case, $\chi(H_n^*) = 3$. Then, with respect to a χ^- colouring, the vertices the vertices $v_1, v_3, v_5, \ldots v_{n-1}$ and the vertices $v_{n+2}, v_{n+4}, v_{n+6}$, $\ldots v_{2n}$ get the colour c_1, the vertices $v_2, v_4, v_6, \ldots v_n$ and the vertices $v_{n+1}, v_{n+3}, v_{n+5}$, $\ldots v_{2n-1}$ get the colour c_2, and the vertex v_{2n+1} get the colour c_3. The possible colour pairs and their numbers in G in terms of the distances between them are as given in Table 6.

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1,c_1)	n
	(c_2,c_2)	n
	(c_3,c_3)	1
1	(c_1,c_2)	$3n$
	(c_1,c_3)	$\frac{n}{2}$
	(c_2,c_3)	$\frac{n}{2}$
2	(c_1,c_1)	$\frac{3n}{2}$
	(c_2,c_2)	$\frac{3n}{2}$
	(c_1,c_2)	$\frac{(n-4)(n+2)}{8}$
	(c_1,c_3)	$\frac{n}{2}$
	(c_2,c_3)	$\frac{n}{2}$
3	(c_1,c_1)	n
	(c_2,c_2)	n
	(c_1,c_2)	$3n$
4	(c_1,c_1)	$\frac{(n-6)(n-4)}{4}$
	(c_2,c_2)	$\frac{(n-6)(n-4)}{4}$

Table 6

From Table 6, we have the modified chromatic Schultz polynomial of H_n^*, when the number of rim vertices n is even, is given by

$$\begin{aligned} S^*_{\chi^-}(H_n^*,x) &= [5n+9]x^0+[6n+9(\frac{n}{2})]x^1+[5(\frac{3n}{2})+2(\frac{(n-4)(n+2)}{8})+9(\frac{n}{2})]x^2 \\ &\quad +11nx^3+[5(\frac{(n-6)(n-4)}{4})]x^4 \\ &= \frac{1}{4}(n^2-10n+24)x^4+11nx^3+\frac{1}{8}(2n^2+92n)x^2+\frac{21}{2}nx+(5n+9). \end{aligned}$$

Therefore,

$$S^*_{\chi^-}(H_n^*,x)=\begin{cases} \frac{1}{8}(5n^2-4n-177)x^4+\frac{1}{4}(9n^2-2n-3)x^3 \\ +\frac{1}{4}(2n^2+56n+10)x^2+\frac{1}{2}(24n+42)x+(5n+29); & \text{if } n \text{ is odd;} \\ \\ \frac{1}{4}(n^2-10n+24)x^4+(11n)x^3+\frac{1}{8}(2n^2+92n)x^2 \\ +\frac{1}{2}(21n)x+(5n+9); & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof. □

Theorem 10. *Let H_n^* be a closed helm graph with n rim vertices. Then, with respect to χ^+ colouring, we have*

$$S^*_{\chi^+}(H_n^*,x)=\begin{cases} \frac{1}{8}(25n^2-144n-73)x^4+\frac{1}{4}(49n^2-216n+97)x^3+ \\ \frac{1}{4}(12^2+12n-28)x^2+\frac{1}{2}(79n-63)x+(25n-16); & \textit{if } n \textit{ is odd;} \\ \\ \frac{13}{4}(n^2-10n+24)x^4+(31n)x^3 \\ +\frac{1}{4}(3n^2+82n-24)x^2+\frac{1}{2}(41n)x+(13n+1); & \textit{if } n \textit{ is even.} \end{cases}$$

2.4 The Modified Chromatic Schultz Polynomial of Sunflower Graphs

A sunflower graph SF_n is a graph obtained by replacing each edge of the rim of a wheel graph W_n by a triangle such that two triangles share a common vertex if and only if the corresponding edges in W_n are adjacent in W_n. Therefore, a sunflower graph consists of $2n+1$ vertices where n is the number of rim vertices. The sunflower graph on 17 vertices is depicted in Figure 4.

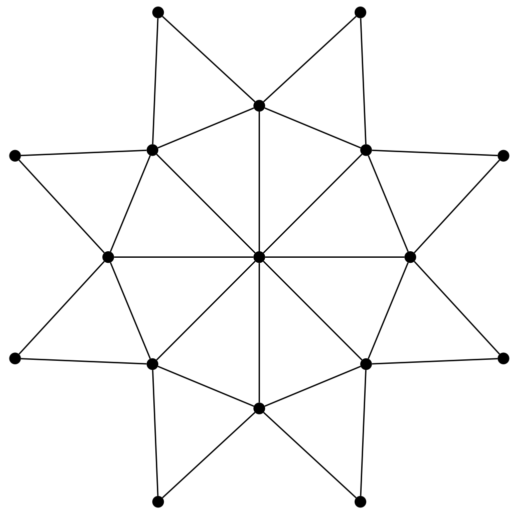

Figure 4. The sunflower graph SF_8.

When n is odd, SF_n is 4-colourable and when n is even, SF_n is 3-colourable. The modified chromatic Schultz polynomial of the sunflower graph with respect to χ^- colouring, can be determined as n the following theorem.

Theorem 11. *Let SF_n be a sunflower graph with $2n+1$ vertices. Then, we have*

$$S^*_{\chi^-}(SF_n,x) = \begin{cases} \frac{1}{2}(n^2-5n)x^4 + (15n-6)x^3 + \frac{1}{8}(24n^2+168n)x^2 \\ +\frac{1}{2}(27n+25)x + \frac{1}{2}(15n+5); & \text{if } n \text{ is odd;} \\ \\ \frac{1}{2}(n^2-5n)x^4 + \frac{3}{2}(n^2-6n)x^3 + \frac{1}{8}(19n^2+34n)x^2 \\ +\frac{21}{2}nx + \frac{1}{2}(15n+1); & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $V = \{v_1, v_2, \ldots, v_{2n+1}$ be the vertex set of SF_n, where the rim vertices of W_n are labelled consecutively from v_1 to v_n, the corresponding extra vertices of the triangle are labelled from v_{n+1} to v_{2n} and the central vertex is v_{2n+1}. We note that the diameter of SF_n is 4. Hence, the power of the variable x varies from 0 to 4 in the modified chromatic Schultz polynomial of SF_n. Here, we need to consider the following two cases:

Note that $\chi(SF_n) = 4$ if the number of rim vertices n is odd and $\chi(SF_n) = 3$ if n is even. Let c_1, c_2, c_3, c_4 be the four colours we use for colouring SF_n.

Case-1: Let n be odd. In this case, $\chi(SF_n) = 4$. Let c_1, c_2, c_3, c_4 be the four colours we use for colouring the vertices of SF_n. Then, with respect to a χ^--colouring, the vertices $v_{n+1}, v_{n+2}, v_{n+3}, \ldots v_{2n}$ and v_{2n+1} get the colour c_1, the vertices $v_1, v_3, v_5, \ldots v_{n-2}$ get the colour c_2, $v_2, v_4, v_6, \ldots v_{n-1}$ get the colour c_3, and v_n gets the colour c_4. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 7.

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1,c_1)	$n+1$
	(c_2,c_2)	$\frac{n-1}{2}$
	(c_3,c_3)	$\frac{n-1}{2}$
	(c_4,c_4)	1
1	(c_1,c_2)	$\frac{3n-3}{2}$
	(c_1,c_3)	$\frac{3n-3}{2}$
	(c_1,c_4)	3
	(c_2,c_3)	$n-2$
	(c_2,c_4)	1
	(c_3,c_4)	1
2	(c_1,c_1)	$2n$
	(c_1,c_2)	$n-1$
	(c_1,c_3)	$n-1$
	(c_1,c_4)	2
	(c_2,c_2)	$\frac{(n-3)(n-1)}{8}$
	(c_2,c_3)	$(\frac{(n-3)}{2})^2$
	(c_1,c_4)	$\frac{n-3}{2}$
	(c_2,c_4)	$\frac{(n-3)(n-1)}{8}$
	(c_3,c_4)	$\frac{n-3}{2}$
3	(c_1,c_1)	n
	(c_1,c_2)	$2n+2$
	(c_1,c_3)	$2n+2$
	(c_1,c_4)	$n-4$
4	(c_1,c_1)	$\frac{(n-5)n)}{2}$

Table 7

In Table 7, the possible distances between different pairs of vertices are written in the first column, the different colour pairs with respect to each distance is written in the second column and the number of corresponding colour pairs with respect to each distance is written in the third column.

From Table 7, we have the modified chromatic Schultz polynomial of the Sunflower graph SF_n when number of rim vertices n is odd, is given by

$$\begin{aligned} S_{\chi^-}(SF_n,x) &= [1(n+1)+4(\frac{n-1}{2})+9(\frac{n-1}{2})+16(1)]x^0 \\ &+[2(\frac{3n-3}{2})+3(\frac{3n-3}{2})+4(3)+6(n-2)+8(1)+12(1)]x^1 \\ &+[1(2n)+2(n-1)+3(n-1)+4(2)+4(\frac{(n-3)(n-1)}{2})+6(\frac{n-3}{2})^2 \\ &+4(\frac{n-3}{2})+8(\frac{(n-3)(n-1)}{4})+12(\frac{n-3}{2})]x^2 \\ &+[1(n)+2(2n+2)+3(2n+2)+4(n-4)]x^3+[1\frac{(n-5)n}{2}]x^4 \\ &= \frac{1}{2}(n^2-5n)x^4+(15n-6)x^3+\frac{1}{8}(24n^2+168)x^2 \\ &+\frac{1}{2}(27n+25)x+\frac{1}{2}(15n+5). \end{aligned}$$

Case-2: Let n be even. In this case, $\chi(SF_n) = 3$. Let c_1, c_2, c_3 be the three colours we use for colouring the vertices of SF_n. Then, with respect to a χ^- colouring, the vertices $v_{n+1}, v_{n+2}, v_{n+3}, \ldots v_{2n}$ and v_{2n+1} get the colour c_1, the vertices $v_1, v_3, v_5, \ldots v_{n-1}$ get the colour c_2, and the vertices $v_2, v_4, v_6, \ldots v_n$ the colour c_3. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 8

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1,c_1)	$n+1$
	(c_2,c_2)	$\frac{n}{2}$
	(c_3,c_3)	$\frac{n}{2}$
1	(c_1,c_2)	$\frac{3n}{2}$
	(c_1,c_3)	$\frac{3n}{2}$
	(c_2,c_3)	n
2	(c_1,c_1)	$2n$
	(c_1,c_2)	n
	(c_1,c_3)	n
	(c_2,c_2)	$\frac{n(n+2)}{8}$
	(c_2,c_3)	$\frac{n(n-2)}{8}$
	(c_3,c_3)	$\frac{n(n-2)}{8}$
3	(c_1,c_1)	n
	(c_1,c_2)	$\frac{(n-4)n}{2}$
	(c_1,c_3)	$\frac{(n-4)n}{2}$
4	(c_1,c_1)	$\frac{(n-5)n}{2}$

Table 8

From Table 8, we have the modified chromatic Schultz polynomial of the sunflower graph SF_n when the number of rim vertices n is even, is given by

$$
\begin{aligned}
S^*_{\chi^-}(SF_n,x) &= [1(n+1)+4(\frac{n}{2})+9(\frac{n}{2}]x^0+[2(\frac{3n}{2})+3(\frac{3n}{2})+6(n)]x^1 \\
&\quad +[1(2n)+2(n)+3(n)+4(\frac{n(n+2)}{8})+6(\frac{n(n-2)}{8})+9(\frac{n(n-2)}{8})]x^2 \\
&\quad +[n+5(\frac{(n-4)n}{2})]x^3+[\frac{(n-5)n}{2}]x^4 \\
&= \frac{1}{2}(n^2-5n)x^4+\frac{3}{2}(n^2-6n)x^3+\frac{1}{8}(19n^2+34n)x^2+\frac{21}{2}nx+\frac{1}{2}(15n+1).
\end{aligned}
$$

Therefore,

$$
S^*_{\chi^-}(SF_n,x) = \begin{cases} \frac{1}{2}(n^2-5n)x^4+(15n-6)x^3+\frac{1}{8}(24n^2+168n)x^2 \\ +\frac{1}{2}(27n+25)x+\frac{1}{2}(15n+5); & \text{if } n \text{ is odd;} \\ \\ \frac{1}{2}(n^2-5n)x^4+\frac{3}{2}(n^2-6n)x^3+\frac{1}{8}(19n^2+34n)x^2+ \\ \frac{1}{2}(21n)x+\frac{1}{2}(15n+1); & \text{if } n \text{ is even.} \end{cases}
$$

This completes the proof. □

Theorem 12. *Let SF_n be a sunflower graph with n rim vertices. Then, with respect to χ^+ colouring, we have*

$$S_{\chi^+}(SF_n, x) = \begin{cases} 8(n^2-5n)x^4 + (60n+24)x^3 \\ +\frac{1}{8}(18n^2+288n-24)x^2 + (36n-25)x + \frac{1}{2}(45n+20); & \text{if } n \text{ is odd;} \\ \frac{9}{2}(n^2-5n)x^4 + \frac{9}{2}(9n^2-2n)x^3 + \\ \frac{1}{8}(7n^2+218n)x^2 + \frac{1}{2}(33n)x + \frac{1}{2}(23n+18); & \text{if } n \text{ is even.} \end{cases}$$

2.5 The Modified Chromatic Schultz Polynomial of Flower Graphs

A flower graph Fl_n is a graph which is obtained by joining the pendant vertices of a helm graph H_n to its central vertex. Therefore, a flower graph consists of $2n+1$ vertices where n is the number of rim vertices. The flower graph on 17 vertices is depicted in Figure 5.

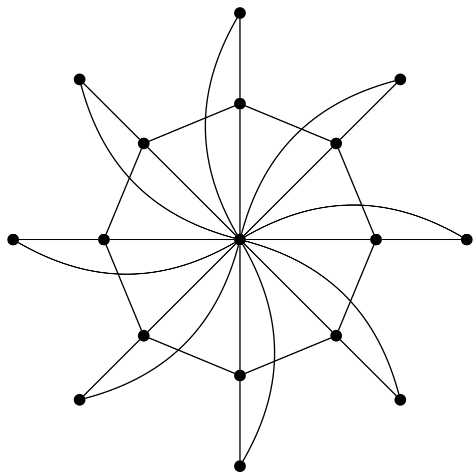

Figure 5. The flower graph Fl_8.

When n is odd, Fl_n is 5-colourable and when n is even, Fl_n is 4-colourable. The modified chromatic Schultz polynomial of the flower graph with respect to χ^- colouring, can be determined as n the following theorem.

Theorem 13. *Let Fl_n be a flower graph with $2n+1$ vertices. Then, we have*

$$S^*_{\chi^-}(Fl_n, x) = \begin{cases} \frac{3}{2}(2n^2-n-1)x^2 + (21n+22)x + \frac{1}{2}(15n+69); & \text{if } n \text{ is odd;} \\ 3(n^2-n)x^2 + \frac{1}{2}(45n)x + \frac{1}{2}(15n+32); & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $V = \{v_1, v_2, \ldots, v_{2n+1}$ be the vertex set of Fl_n, where the rim vertices of W_n are labelled consecutively from v_1 to v_n, the corresponding extra vertices are labelled from v_{n+1} to v_{2n} and the central vertex is v_{2n+1}. We note that the diameter of Fl_n is 2. Hence, the power of the variable x varies from 0 to 2 in the modified chromatic Schultz polynomial of Fl_n. Here, we need to consider the following two cases:

Note that $\chi(Fl_n) = 5$ if the number of rim vertices n is odd and $\chi(Fl_n) = 4$ if n is even. Let c_1, c_2, c_3, c_4, c_5 be the five colours we use for colouring Fl_n.

Case-1: Let n be odd. In this case, $\chi(Fl_n) = 5$. Let c_1, c_2, c_3, c_4, c_5 be the five colours we use for colouring the vertices of Fl_n. Then, with respect to a χ^--colouring, the vertices $v_{n+1}, v_{n+2}, v_{n+3}, \ldots v_{2n}$ get the colour c_1, the vertices $v_1, v_3, v_5, \ldots v_{n-2}$ get the colour c_2, $v_2, v_4, v_6, \ldots v_{n-1}$ get the colour c_3, v_n gets the colour c_4 and v_{2n+1} gets the colour c_5. The possible colour pairs and their numbers in G in terms of the distances between them are listed in the following table.

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1, c_1)	n
	(c_2, c_2)	$\frac{n-1}{2}$
	(c_3, c_3)	$\frac{n-1}{2}$
	(c_4, c_4)	1
	(c_5, c_5)	1
1	(c_1, c_2)	$\frac{n-1}{2}$
	(c_1, c_3)	$\frac{n-1}{2}$
	(c_1, c_4)	1
	(c_1, c_5)	1
	(c_2, c_3)	$n-2$
	(c_2, c_4)	1
	(c_2, c_5)	$\frac{n-1}{2}$
	(c_3, c_4)	1
	(c_3, c_5)	$\frac{n-1}{2}$
	(c_4, c_5)	1
2	(c_1, c_1)	$\frac{(n-1)n}{2}$
	(c_1, c_2)	$\frac{(n-1)^2}{2}$
	(c_1, c_3)	$\frac{(n-1)^2}{2}$
	(c_1, c_4)	$n-1$

Table 9

In Table 9, the possible distances between different pairs of vertices are written in the first column, the different colour pairs with respect to each distance is written in the second column and the number of corresponding colour pairs with respect to each distance is written in the third column.

From Table 9, we have the modified chromatic Schultz polynomial of the flower graph Fl_n when number of rim vertices n is odd, is given by

$$\begin{aligned} S_{\chi^{-*}}(Fl_n, x) &= [1(n) + 4(\frac{n-1}{2}) + 9(\frac{n-1}{2}) + 16(1) + 25(1)]x^0 \\ &\quad +[2(\frac{n-1}{2}) + 3(\frac{n-1}{2}) + 4(1) + 5(1) \\ &\quad +6(n-2) + 8(1) + 10(\frac{n-1}{2}) + 12(1) + 15(\frac{n-1}{2}) + 20(1)]x^1 \\ &\quad +[1(\frac{n(n-1)}{2} + 2\frac{(n-1)^2}{2} + 3(\frac{(n-1)^2}{2}) + 4(n-1)]x^2 \\ &= \frac{3}{2}(2n^2 - n - 1)x^2 + (21n + 12)x + \frac{1}{2}(15n + 69). \end{aligned}$$

Case-2: Let n be even. In this case, $\chi(Fl_n) = 4$. Let c_1, c_2, c_3, c_4 be the four colours we use for colouring the vertices of Fl_n. Then, with respect to a χ^- colouring, the vertices $v_{n+1}, v_{n+2}, v_{n+3}, \ldots v_{2n}$ get the colour c_1, the vertices $v_1, v_3, v_5, \ldots v_{n-1}$ get the colour c_2, the vertices $v_2, v_4, v_6, \ldots v_n$ the colour c_3 and the vertex v_{2n+1} gets the colour c_4. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 10.

Distance $d(u, v)$	Colour pairs	Number of pairs
0	(c_1, c_1)	n
	(c_2, c_2)	$\frac{n}{2}$
	(c_3, c_3)	$\frac{n}{2}$
	(c_4, c_4)	1
1	(c_1, c_2)	$\frac{n}{2}$
	(c_1, c_3)	$\frac{n}{2}$
	(c_1, c_4)	n
	(c_2, c_3)	n
	(c_2, c_4)	$\frac{n}{2}$
	(c_3, c_4)	$\frac{n}{2}$
2	(c_1, c_1)	$\frac{n(n-1)}{2}$
	(c_1, c_2)	$\frac{n(n-1)}{2}$
	(c_1, c_3)	$\frac{n(n-1)}{2}$

Table 10

From Table 10, we have the modified chromatic Schultz polynomial of the helm graph Fl_n when the number of rim vertices n is even, is given by

$$\begin{aligned} S^*_{\chi^-}(Fl_n, x) &= [1(n) + 4(\frac{n}{2}) + 9(\frac{n}{2} + 16(1)]x^0 \\ &\quad +[2(\frac{n}{2}) + 3(\frac{n}{2}) + 4(n) + 6(n) + 8(\frac{n}{2} + 12(\frac{n}{2})]x^1 \\ &\quad +[1(\frac{n(n-1)}{2}) + 2(\frac{n-1)^2}{2}) + 3(\frac{n-1)^2}{2})] \\ &= 3(n^2 - n)x^2 + \frac{45}{2}nx + \frac{1}{2}(15n + 32). \end{aligned}$$

Therefore,

$$S^*_{\chi^-}(Fl_n, x) = \begin{cases} \frac{3}{2}(2n^2 - n - 1)x^2 + (21n + 22)x + \frac{1}{2}(15n + 69); & \text{if } n \text{ is odd;} \\ 3(n^2 - n)x^2 + \frac{1}{2}(45n)x + \frac{1}{2}(15n + 32); & \text{if } n \text{ is even.} \end{cases}$$

This completes the proof. □

The following theorem provides the modified chromatic Schultz polynomial of a flower graph Fl_n with respect to its χ^+-colouring.

Theorem 14. *Let Fl_n be a flower graph with n rim vertices. Then, with respect to χ^+ colouring, we have*

$$S^*_{\chi^+}(Fl_n, x) = \begin{cases} \frac{1}{2}(60n^2 - 75n + 15)x^2 + \frac{1}{2}(18n + 46)x + \frac{1}{2}(75n - 15); & \textit{if } n \textit{ is odd;} \\ 20(n^2 - n)x^2 + \frac{1}{2}(39n)x + \frac{1}{2}(45n + 1); & \textit{if } n \textit{ is even.} \end{cases}$$

2.6 The Modified Chromatic Schultz Polynomial of Friendship Graphs

A *friendship graph* F_n consists of n triangles joined together by a single vertex. F_n consists of $2n+1$ vertices and $3n$ edges. F_n is 3-colourable. The friendship graph on 9 vertices is shown in Figure 6.

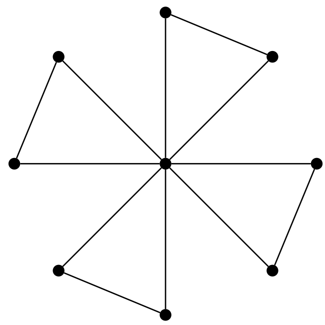

Figure 6. The friendship graph F_4.

The following theorem discusses the chromatic Schultz polynomial of friendship graphs:

Theorem 15. *Let F_n be a friendship graph with n triangles. Then, we have*

$$S_{\chi^-}(F_n, x) = \frac{9}{2}(n^2 - n)x^2 + (11n)x + (5n + 9)$$

Proof. Let $V = \{v_1, v_2, \ldots, v_{2n+1}\}$ be the vertex set of F_n. We note that the diameter of F_n is 2. Hence the power of the variable x varies from 0 to 2 in the modified chromatic Schultz polynomial of F_n. For Friendship graph, number of vertices is always odd. Now, $\chi(F_n) = 3$. Let c_1, c_2, c_3 be the three colours we use for colouring the vertices of F_n. Then, with respect to a χ^- colouring, one vertex each of the triangles are coloured c_1, the other is coloured c_2 and the common vertex is coloured c_3. The possible colour pairs and their numbers in F_n in terms of the distances between them are listed in Table 11.

Distance $d(u,v)$	Colour pairs	Number of pairs
0	(c_1, c_1)	n
	(c_2, c_2)	n
	(c_3, c_3)	1
1	(c_1, c_2)	n
	(c_1, c_3)	n
	(c_2, c_3)	n
2	(c_1, c_1)	$\frac{n(n-1)}{2}$
	(c_2, c_2)	$\frac{n(n-1)}{2}$
	(c_1, c_2)	$n(n-1)$

Table 11

From Table 11, the modified chromatic Schultz polynomial of F_n can be evaluated as follows:

$$S^*_{\chi^-}(F_n, x) = [1(n) + 4(n) + 9(1)]x^0 + [2(n) + 3(n) + 6(n)]x^1 +$$

$$[1(\frac{n(n-1)}{2}) + 4(\frac{n(n-1)}{2}) + 2(n(n-1))]x^2$$
$$= \frac{9}{2}(n^2 - n)x^2 + (11n)x + (5n + 9).$$

This completes the proof. □

In a similar way, the modified chromatic Schultz polynomial of a friendship graph with respect to its χ^+-colouring, can be determined as in the following theorem.

Theorem 16. *Let F_n be a friendship graph with n triangles. Then we have,*

$$S^*_{\chi^+}(F_n, x) = \frac{25}{2}(n^2 - n)x^2 + (11n)x + (13n + 1).$$

3 Conclusion

In this paper, we have discussed the modified chromatic Schultz polynomial of certain cycle related graphs. This polynomial can be determined for many other graph classes with finite diameter. Further investigations on the chromatic Schultz polynomial and modified chromatic Schultz polynomial of graph operations, graph products and graph powers are also promising. This study can be extended to other types of graph colouring such as injective colouring and equitable colouring. The concept can be extended to edge colouring and map colouring also. All these facts show a wide scope for further studies in this area.

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