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The Numerical Solutions of linear and Non-Linear Volterra Integral Equations of the Second Kind using Variational Iteration Method

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Abstract

This article provides approximate solutions to some linear and nonlinear Volterra-Integral equations of the second kind by using the Variational Iteration Method (VIM). Conversion Volterra's integral equation to an initial value problem or Volterra integro-differential equation is considered. The convergence of the method is also considered to provide rapidly convergent successive approximations to the exact solution if such a closed form solution exists. A comparison of the approximate solutions of this method with the Adomian decomposition method and an exact solution will be demonstrated through numerical examples to shows that the method is reliable, accurate and readily implemented.

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1 Introduction

Several authors in engineering and physical sciences have studied and used different numerical methods to solve Volterra Integral equations. In recent years, many of these numerical methods gave reliable and accurate solutions. [9] applied the two-step Laplace decomposition method for solving nonlinear Volterra integral equations. [8] used the homotopy analysis method for solving linear integral equations. [11] implemented a new modified of Adomian decomposition method by the Taylor expansion of the components apart from the zeroth of the Adomian series solution for Volterra integral equation of the second kind. [10] employed the Taylor collocation method to approximate solutions and convergence analysis for the Volterra-Fredholm integral equations, and [1] combined Laplace transform with analytical methods for solving Volterra integral equations with a convolution kernel. [6] studied the reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations. [7] constructed the numerical solution of nonlinear Volterra-Fredholm integral equations by variational iteration method.[12] used modified variational iteration method for the numerical solutions of some non-linear Fredholm integro-differential equations of the second kind. [5] studied recent advances in reliable methods for solving Volterra-Fredholm integral and integro-differential equations. [3] implemented the usage of the homotopy analysis method for solving fractional Volterra-Fredholm integro-differential equation of the second kind. [2] introduced the approximate solutions using the Adomian decomposition method and its modification for solving Fredholm integral equations. [4] employed modified the Adomian decomposition method to solve fuzzy Volterra-Fredholm integral equations. [14] used iterative methods to solve two-dimensional nonlinear Volterra-Fredholm integro-differential equations.

In this article, we consider linear Volterra integral equation of the second kind of the form

$$y(x) = f(x) + \lambda \int_a^{x)} k(x,t)y(t)dt, \tag{1.1}$$

and nonlinear Volterra integral equation of the second kind is represented by the form

$$y(x) = f(x) + \int_a^x k(x,t)F(y(t))dt, \tag{1.2}$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real valued functions, λ is a parameter and $F(y(x))$ is a nonlinear function of $y(x)$ and the unknown function $y(x)$ appears inside and outside the integral sign.

The structure of this article is organized as follows: In the second section we present linear and nonlinear Volterra integral equations of the second kind were solved by variational iteration method which uses a few numbers of iterations. Section 3 presents our numerical examples and graphical results will demonstrate the efficiency of the method and will be shown that the method is accurate and readily implemented compared to some exact solutions. Finally, the conclusion will be in Section 4.

2 VIM for solving Volterra integral equations

To use the variational iteration method for solving Volterra integral equations, it is necessary to convert the integral equation to an equivalent initial value problem or to an equivalent integro-differential equation.

To convert Equation (1.1) to equivalent initial value problems [13] we achieved simply by differentiating both sides of Volterra equation with respect to x as many times as we need to get rid of the integral sign and come out with a differential equation. The conversion of Volterra equations requires the use of Leibnitz rule for differentiating the integral at the right hand side. The initial conditions can be obtained by substituting $x = 0$ into $y(x)$ and its derivatives.

2.1 Linear Volterra integral equations:

For the purpose of illustration of the methodology to the variational iteration method, we begin by considering a nonlinear differential equation of the formal form

$$L(y) + N(y) = g(x), \tag{2.1}$$

where L and N are linear and nonlinear operators respectively, $g(x)$ is a known analytical function and y is an unknown function to be determined. He [13] introduced method where a correction function for Equation (2.1) can be written as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)(Ly_n(\xi) + N\tilde{y}(\xi) - g(\xi))d\xi, \tag{2.2}$$

Where λ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function, and $\tilde{y_n}$ is a restricted value that means it behaves like a constant, hence $\delta\tilde{y_n} = 0$, where δ is the variational derivative.

For the complete use of the variational iteration method, we should follow two steps, first, we determine the Lagrange multiplier $\lambda(\xi)$ that will be identified optimally, second, we substitute the result into Equation (2.2) where the restrictions should be omitted.

Taking the variation of Equation (2.2) with respect to the independent variable y we find

$$\frac{\delta y_{n+1}}{\delta y_n} = 1 + \frac{\delta}{\delta y_n}\left(\int_0^x \lambda(\xi)(Ly_n(\xi) + N\tilde{y_n}(\xi) - g(\xi))d\xi\right), \tag{2.3}$$

integration by parts is usually used for the determination of the Lagrange multiplier $\lambda(\xi)$. In other words, we can use

$$\begin{aligned}
\int_0^x \lambda(\xi)y_n'(\xi)d\xi &= \lambda(\xi)y_n(\xi) - \int_0^x \lambda'(\xi)y_n(\xi)d\xi \\
\int_0^x \lambda(\xi)y_n''(\xi)d\xi &= \lambda(\xi)y_n'(\xi) - \lambda'(\xi)y_n(\xi) + \int_0^x \lambda''(\xi)y_n(\xi)d\xi \\
\int_0^x \lambda(\xi)y_n'''(\xi)d\xi &= \lambda(\xi)y_n''(\xi) - \lambda'(\xi)y_n'(\xi) + \lambda''(\xi)y_n(\xi) - \int_0^x \lambda'''(\xi)y_n(\xi)d\xi \\
\int_0^x \lambda(\xi)y_n^{iv}(\xi)d\xi &= \lambda(\xi)y_n'''(\xi) - \lambda'(\xi)y_n''(\xi) + \lambda''(\xi)y_n'(\xi) - \lambda'''(\xi)y_n(\xi) + \int_0^x \lambda^{iv}(\xi)y_n(\xi)d\xi
\end{aligned} \tag{2.4}$$

and so on.

Having determined the Lagrange multiplier $\lambda(\xi)$, the successive approximations y_{n+1}, $n \geq 0$, of the solution $y(x)$ will be readily obtained upon using selective function $y_0(x)$. However, for fast convergence, the function $y(x)$ should be selected by using the initial conditions as follows:

$$\begin{aligned}
&y_0(x) = y(0), \text{ for first order } y_n', \\
&y_0(x) = y(0) + xy_0', \text{ for second order } y_n'', \\
&y_0(x) = y(0) + xy_n' + \frac{1}{2!}x^2y_0'', \text{ for third order } y_n''',
\end{aligned}$$

and so on. Consequently, the solution

$$y(x) = \lim_{n\longrightarrow\infty} y_n(x). \tag{2.5}$$

The determination of the Lagrange multiplier plays a major role in the determination of the solution of the problem. In what follows, we write generally iteration formulae that show ODE, its corresponding Lagrange multiplier,and its correction functional respectively:

$$y^{(n)} + f(y(\xi), y'(\xi), \ldots, y^{(n)}(\xi)) = 0, \lambda = (-1)^n\frac{1}{(n-1)!}(\xi - x)^{n-1}$$

$$y_{n+1} = y_n + (-1)^n\int_0^x \frac{1}{(n-1)!}(\xi - x)^{n-1}[y_n''' + f(y_n, \ldots, y_n^{(n)})]d\xi, \tag{2.6}$$

for $n \geq 1$

2.2 Nonlinear Volterra integral equation:

For solving Equation (1.2) by variational iteration method [15], first we differentiate once from both sides of Equation (1.2), with respect to x:

$$y'(x) = f'(x) + k(x,x)F(y(x)) + \int_0^x \frac{\partial k(x,t)}{\partial x} F(y(t))dt, \tag{2.7}$$

now we apply variational iteration method to Equation (2.7). According to this method correction functional can be written in the following form:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda \left(y_n'(s) - f'(s) - k(s,s)F(\tilde{y}_n(s)) - \int_0^s \frac{\partial k(s,t)}{\partial s} F(\tilde{y}_n(t)dt) \right) ds, \tag{2.8}$$

to make the above correction functional stationary with respect to y_n, we have:

$$\begin{aligned} \delta y_{n+1}(x) &= \delta y_n(x) + \delta \int_0^x \lambda \left(y_n'(s) - f'(s) - k(s,s)F(\tilde{y}_n(s)) - \int_0^s \frac{\partial k(s,t)}{\partial s} F(\tilde{y}_n(t)dt) \right) \\ &= \delta y_n(x) + \int_0^x \lambda, (s)\delta(y_n'(s))ds = \delta y_n(x) + \lambda(x)\delta y_n(x) + \int_0^x \lambda'(s)\delta y_n(s)ds = 0, \end{aligned} \tag{2.9}$$

from the above relation for any δy_n, we obtain the Euler-Lagrange equation:

$$\lambda'(s) = 0, \tag{2.10}$$

with the following natural boundary condition:

$$\lambda(x) + 1 = 0, \tag{2.11}$$

using equations (2.10) and (2.11), Lagrange multiplier can be identified optimally as follows:

$$\lambda(s) = 1, \tag{2.12}$$

substituting the identified Lagrange multiplier into Equation (2.8) we obtain the following iterative relation:

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(y_n'(s) - f'(s) - k(s,s)F(y_n(s)) - \int_0^s \frac{\partial k(s,t)}{\partial s} F(y_n(t)dt) \right) ds, \tag{2.13}$$

we can obtain the exact solution or an approximate solution to the Equation (1.2) by starting from $y_0(x)$. Also in some Volterra integral equations by differentiating from integral equation, for example when the kernel is independent of x, we obtain a differential equation then we solve it by using variational iteration method.

3 Illustrative examples

In this section we solve three examples of the linear and nonlinear of Volterra integral equations which have solved in [13]. Numerical results show that our proposed method has a high accuracy.

Example 1. Consider the following linear Volterra integral equation with the exact solution $y(x) = e^x$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 y(t)dt, \tag{3.1}$$

differentiate both sides of Equation (3.1) with respect to x by using Leibnitz rule gives the integro-differential equation

$$y'(x) = 1 + x + \int_0^x (x-t)y(t)dt, \qquad y(0) = 1, \tag{3.2}$$

applyig the variational iteration method to Equation (3.2) we get the correction functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi \left(y_n'(\xi) - 1 - \xi - \int_0^\xi (\xi - s)\tilde{y}_n(s)ds \right) d\xi, \tag{3.3}$$

we find the Lagrange multiplier

$$\lambda = -1, \tag{3.4}$$

substituting this value of the Lagrange multiplier into the functional Equation (3.3) gives the iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi \left(y_n'(\xi) - 1 - \xi - \int_0^\xi (\xi - s)y_n(s)ds \right) d\xi, \tag{3.5}$$

using the initial conditions to select $y_0(x) = y(0) = 1$ and use it into Equation 3.5 we get

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}, \\ y_2(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}, \\ y_n(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots + \frac{x^n}{n!}, \end{aligned} \tag{3.6}$$

which converges to the exact solution $y(x) = e^x$
Figure 1 shows the comparison between the exact solution and the approximate solution

Table 1. Numerical results of Example 1, N = 4.

x	$y_{Exact}(x)$	$y_{Appr.}(x)$	$E_4(y)$
0.1	1.105170918	1.105170918	———
0.2	1.221402758	1.221402758	———
0.3	1.349858808	1.349858808	———
0.4	1.491824698	1.491824698	———
0.5	1.648721271	1.648721270	1×10^{-9}
0.6	1.822118800	1.822118799	1×10^{-9}
0.7	2.013752707	2.013752699	8×10^{-9}
0.8	2.225540928	2.225540897	31×10^{-9}
0.9	2.459603111	2.459603007	104×10^{-9}
1.0	2.718281828	2.718281526	302×10^{-9}

obtained by the VIM. It is seen from Fig.1 the solution obtained by the proposed method nearly identical to the exact solution. In this example, the simplicity and accuracy of the proposed method is illustrated by computing the absolute error $E_4(x)$.
The accuracy of the result can be improved by introducing more terms of the approximate solutions. In Table 1, VIM solutions is compared with the exact solution of the

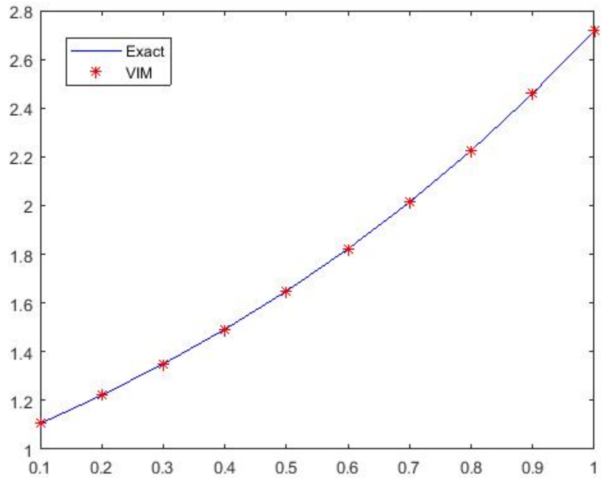

Figure 1. Comparison between exact and approximate solutions for Example 1

Volterra integral Equation (3.1). There is good agreement between exact and approximate solution obtained by proposed method. The table also shows the absolute error between the exact and approximate solutions.

Example 2. Consider the following nonlinear Volterra integral equation with the exact solution $y(x) = tan(x)$

$$y(x) = x + \int_0^x y^2(t)dt, \tag{3.7}$$

differentiate both sides of Equation (3.7) with respect to x by using Leibnitz rule gives the integro-differential equation

$$y'(x) = 1 + \int_0^x y^2(t), \qquad y(0) = x, \tag{3.8}$$

applyig the variational iteration method VIM to Equation (3.7) we get the correction functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi \left(y_n'(\xi) - 1 \int_0^\xi \tilde{y}_n^2(s)ds \right) d\xi, \tag{3.9}$$

we find the Lagrange multiplier

$$\lambda = -1, \tag{3.10}$$

substituting this value of the Lagrange multiplier into the functional (3.9) gives the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y_n'(\xi) - 1 - \int_0^\xi (y_n^2(s)ds \right) d\xi, \tag{3.11}$$

using the initial conditions to select $y_0(x) = y(0) = 1$ and use it into Equation (3.11) we get

$$y_0(x) \quad = \quad x,$$

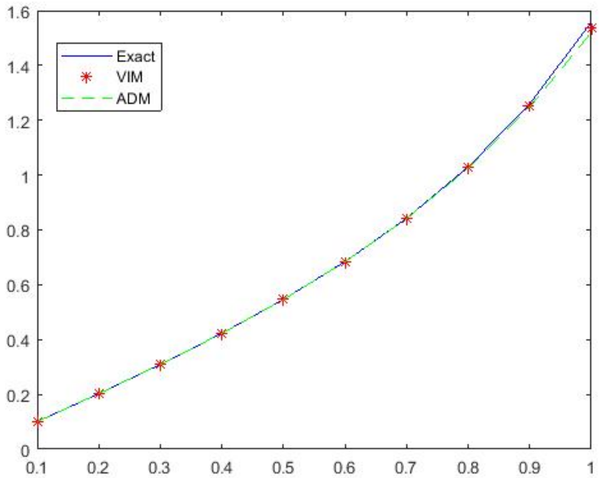

Figure 2. Comparison between exact and approximate solutions for Example 2

$$
\begin{aligned}
y_1(x) &= x + \frac{x^3}{3}, \\
y_2(x) &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{x^7}{63}, \\
y_3(x) &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{342x^9}{25515} + \frac{1206x^{11}}{467775} + \frac{4x^{13}}{12285} + \frac{x^{15}}{59535}, \quad (3.12)
\end{aligned}
$$

which converges to the exact solution $y(x) = tan(x)$

Table 2. Numerical results of Example 2, N = 4.

x	$y_{Exact}(x)$	VIM	ADM	$E_4(y_{VIM})$	$E_4(y_{ADM})$
0.1	0.100334672	0.100334672	0.100334672	—	—
0.2	0.202710035	0.202710031	0.202710024	4×10^{-9}	11×10^{-9}
0.3	0.309336249	0.309336071	0.309335802	178×10^{-9}	447×10^{-9}
0.4	0.422793218	0.422790712	0.422787088	2.506×10^{-6}	6.13×10^{-6}
0.5	0.546302489	0.546282438	0.546254960	20.015×10^{-6}	47.538×10^{-6}
0.6	0.684136808	0.684023632	0.683878765	113.176×10^{-6}	258.043×10^{-6}
0.7	0.842288380	0.841782292	0.841187184	506.088×10^{-6}	1.101196×10^{-3}
0.8	1.029638557	1.027714288	1.025675297	1.924269×10^{-3}	3.96326×10^{-3}
0.9	1.260158218	1.253633063	1.247544849	6.525155×10^{-3}	12.613369×10^{-3}
1.0	1.557407725	1.536959360	1.520634921	20.448365×10^{-3}	36.772804×10^{-3}

Figure 2 shows the comparison between the exact solution and the approximate solutions obtained by the VIM and ADM. It is seen from Figure 2 the solution obtained by the proposed method nearly identical to the exact solution. In this example, the simplicity and accuracy of the proposed method is illustrated by computing the absolute error $E_4(x)$.

The accuracy of the result can be improved by introducing more terms of the approximate solutions. In Table 2, VIM solutions is compared with ADM and the exact solution of the Volterra integral Equation (3.1). There is good agreement between exact and approximate solution obtained by proposed method. The table also shows the absolute error between the exact and approximate solutions. VIM is better than ADM and it has more accuracy.

Example 3. Consider the following linear Volterra integral equation with the exact solution $y(x) = x + cos(x)$

$$y(x) = 1 + x + \frac{x^3}{3!} - \int_0^x (x-t)y(t)dt, \tag{3.13}$$

differentiate both sides of Equation (3.13) with respect to x by using Leibnitz rule gives the integro-differential equation

$$y'(x) = 1 + \frac{x^2}{2} - \int_0^x y(t)dt, \qquad y(0) = 1, \tag{3.14}$$

we obtain the initial value problem by differentiating Equation (3.14) again

$$y''(x) = x - y(x), \qquad y(0) = 1, y'(0) = 1, \tag{3.15}$$

(a) Applyig the variational iteration method to Equation (3.14) we get the correction functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(y'_n(\xi) - 1 - \frac{\xi^2}{2} - \int_0^\xi \tilde{y}_n(s)ds\right)d\xi, \tag{3.16}$$

we find the Lagrange multiplier of the first order

$$\lambda = -1, \tag{3.17}$$

substituting this value of the Lagrange multiplier into the functional Equation (3.15) gives the iteration formula

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(y'_n(\xi) - 1 - \frac{\xi^2}{2} - \int_0^\xi (y_n(s)ds\right)d\xi, \tag{3.18}$$

using the initial conditions to select $y_0(x) = y(0) = 1$ and use it into Equation (3.18) we get

$$\begin{aligned}
y_0(x) &= 1,\\
y_1(x) &= 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!},\\
y_2(x) &= 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!},\\
y_3(x) &= 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!},\\
y_n(x) &= x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}\right),
\end{aligned} \tag{3.19}$$

which gives the exact solution $y(x) = x + cos(x)$

(b) Applying the variational iteration method to Equation (3.14) we get the correction functional

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left(y_n''(\xi) + \tilde{y}_n(\xi)\right)d\xi, \tag{3.20}$$

we find the Lagrange multiplier for second order

$$\lambda = \xi - x, \tag{3.21}$$

substituting this value of the Lagrange multiplier into the functional (3.20) gives the iteration formula

$$y_{n+1}(x) = y_n(x) + \int_0^x (\xi - x)\left(y_n''(\xi) + y_n(\xi) - \xi\right)d\xi, \tag{3.22}$$

using the initial conditions to select $y_0(x) = y(0) + xy_0' = 1 + x$ and use it into Equation (3.22) we get

$$\begin{aligned} y_0(x) &= 1 + x, \\ y_1(x) &= 1 + x - \frac{x^2}{2!}, \\ y_2(x) &= 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!}, \\ y_3(x) &= 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}, \\ y_n(x) &= x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}\right), \end{aligned} \tag{3.23}$$

which gives the exact solution $y(x) = x + cos(x)$.

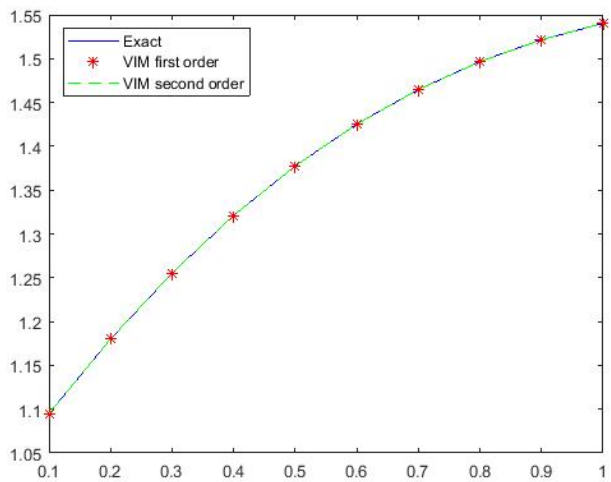

Figure 3. Comparison between exact and approximate solutions for Example 3

Figure 3 shows the comparison between the exact solution and the approximate solution obtained by the VIM of the first order and the second order respectively. It is seen from Figure 3 the solution obtained by the proposed method nearly identical to the exact solution. In this example, the simplicity and accuracy of the proposed method is illustrated by computing the absolute error $E_4(x)$ .The accuracy of the result can be improved by introducing more terms of the approximate solutions. In Table 3, VIM solutions are compared with the exact solution of the Volterra integral Equation (3.13). There is good agreement between the exact and approximate solution obtained by the proposed method. The table also shows the absolute error between the exact and approximate solutions and the approximate solution is obtained from the second order is accuracy more than that is obtained from first order with the same iterations.

Table 3. Numerical results of Example 3, N = 4.

x	$y_{Exact}(x)$	VIM 1^{st}order	VIM 2^{nd} order	$E_4(y)$ 1^{st}order	$E_4(y)$ 2^{nd} order
0.1	1.095004165	1.095004165	1.095004165	———	———
0.2	1.180066578	1.180066580	1.180066578	10×10^{-9}	———
0.3	1.255336489	1.255336531	1.255336488	42×10^{-9}	1×10^{-9}
0.4	1.321060994	1.321061303	1.321060978	309×10^{-9}	16×10^{-9}
0.5	1.377582562	1.377584015	1.377582465	1.453×10^{-6}	97×10^{-9}
0.6	1.425335615	1.425340754	1.425335520	5.139×10^{-6}	415×10^{-9}
0.7	1.464842187	1.464857105	1.464840765	14.918×10^{-6}	1.422×10^{-6}
0.8	1.496706709	1.496744188	1.496702578	37.479×10^{-6}	4.131×10^{-6}
0.9	1.521609968	1.521694288	1.521599388	84.32×10^{-6}	10.58×10^{-6}
1.0	1.540302306	1.540277778	1.540277778	173.884×10^{-6}	24.528×10^{-6}

4 Conclusion

In this article, the variational iteration method has been successfully employed to obtain the approximate and analytical solution of linear and nonlinear Volterra integral equation of the second kind. The results showed that the convergence, powerful and efficient of this technique was in a good agreement with the exact, analytical and approximate solutions for wide classes of problems. The solution is obtained by the our proposed method has high accuracy and also VIM better than Adomian decomposition method. The computations associated with the examples in this work were performed using Maple 17.

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