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Generalized $(\theta, \phi)-$derivations and Jordan ideals in prime rings

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Abstract

In this paper we prove some basic results on Jordan ideals of rings that make the study of derivations on Jordan ideals of (semi)prime rings parallel to the study of derivations on ideals of (semi)prime rings. More precisely, we prove a number of commutativity theorems with generalized $(\theta, \phi)-$derivations that act on Jordan ideals of prime rings. Consequently, many known results of this subject are unified or extended.

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1 Introduction

For any ring R, an additive subgroup J of R is said to be the Jordan ideal (resp. Lie ideal) of R if $J \circ R \subseteq J$ (resp. $[J, R] \subseteq J$), where the symbol $x \circ y$ (resp. $[x, y]$) denotes the Jordan product $xy + yx$ (resp. Lie product $xy - yx$). It is well-known that if J is a nonzero Jordan ideal of R then for any $j \in J$, $2[j^2, R] \subseteq J$, $4j^2R \subseteq J$, $4Rj^2 \subseteq J$ (see [[3], proof of Lemma 3]) and $4jRj \subseteq J$ (see [[3], proof of Theorem 3]). Moreover, $2J[R, R] \subseteq J$ and $2[R, R]J \subseteq J$ (see [[19], Lemma 2.4]). It is well-known that if union of two proper subgroups G_1 and G_2 of a group G is whole of G, then either $G_1 = G$ or $G_2 = G$, provided $G_1 \nsubseteq G_2$ and $G_2 \nsubseteq G_1$. This fact is known as *Brauer's trick*. Recall that a ring R is said to be prime (resp. semiprime) if for any $a, b \in R$, $aRb = (0)$ (resp. $aRa = (0)$) implies $a = 0$ or $b = 0$ (resp. $a = 0$) and an additive mapping $d : R \to R$ is called a derivation of R if $d(xy) = d(x)y + xd(y)$ for any $x, y \in R$. For a fixed $\alpha \in R$, a mapping $d_\alpha : R \to R$ such that $x \mapsto [\alpha, x]$ is a derivation, which is known as the inner derivation induced by α. For some fixed $a, b \in R$, a mapping $F_{a,b} : R \to R$ such that $x \mapsto ax + xb$ is called the generalized inner derivation of R. Immediately it follows that, if $F_{a,b}$ is generalized inner derivation, then we have $F_{a,b}(xy) = F_{a,b}(x)y + xd_{-b}(y)$, where d_{-b} is inner derivation associated with the element $(-b)$. This observation of Brešar [5] gave rise to the notion of generalized derivation. Accordingly, if d is a derivation

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of R and $F : R \to R$ is an additive map such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, then F is called a generalized derivation of R. The familiar examples of these mappings are derivations, left multipliers (i.e., an additive mapping $T : R \to R$ such that $T(xy) = T(x)y$ for all $x, y \in R$). For any ring endomorphisms θ and ϕ, a $(\theta, \phi)-$derivation is an additive mapping $d : R \to R$ such that $d(xy) = d(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$. Such mappings appeared first time in the classic text [10] by Jacobson. Of course, a derivation is an $(1_R, 1_R)-$derivation, where 1_R is the identity map on R. A mapping $d_\alpha : R \to R$ such that $x \mapsto \alpha\theta(x) - \phi(x)\alpha$ is called the $(\theta, \phi)-$ inner derivation of R, where $\alpha \in R$ is a fixed element. Intuitively, a generalized $(\theta, \phi)-$ inner derivation of R is a mapping $F_{a,b} : R \to R$ such that $x \mapsto a\theta(x) + \phi(x)b$, where a, b are fixed elements of R. Similarly, if $F_{a,b}$ is a generalized $(\theta, \phi)-$inner derivation of R, then we have $F_{a,b}(xy) = F_{a,b}(x)\theta(y) + \phi(x)d_{-b}(y)$, where d_{-b} is $(\theta, \phi)-$inner derivation of R. This computation naturally extends the notion of generalized derivation to generalized $(\theta, \phi)-$derivation. More specifically, an additive mapping $F : R \to R$ is said to be a generalized $(\theta, \phi)-$derivation of R if $F(xy) = F(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$, where d is a $(\theta, \phi)-$ derivation of R. We denote the $(\theta, \phi)-$commutator and $(\theta, \phi)-$ anticommutator by $[x, y]_{\theta,\phi} = x\theta(y) - \phi(y)x$ and $(x \circ y)_{\theta,\phi} = x\theta(y) + \phi(y)x$ respectively. In order to prevent any confusion, note that a generalized $(\theta, \phi)-$derivation has also been used by many authors as generalized $(\alpha, \beta)-$derivation or generalized $(\sigma, \tau)-$derivation. We use the following commutator and anti-commutator identities without mentioning them specifically:

- $[x, yz] = y[x, z] + [x, y]z$,
- $[xy, z] = x[y, z] + [x, z]y$,
- $(x \circ yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$ and
- $(xy \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

The study of certain types of conditions (e.g. polynomial conditions and differential conditions) that finally imply commutativity of rings is very natural in the noncommutative ring theory. For instance the famous Wedderburn's theorem can be taken as the first precedent. In this line of investigation the initial results are mainly due to the work of Jacobson, Herstein and Posner (see [12]). Since then several authors investigated the commutativity of (semi)prime rings admitting various types of derivations which satisfy appropriate algebraic identities on suitable subsets of the rings. For example, we refer the reader to [1], [2], [6], [7], [13], [14], [15], [17], [18] with further references therein).

A classical result of Herstein [8] states that if d is a nonzero derivation of a 2-torsion free prime ring R such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Inspired by this, Bell and Daif [4] proved that: *If d is a nonzero derivation of a prime ring R and $d([x, y]) = 0$ for all $x, y \in I$, where I is nonzero ideal of R, then R is commutative.* In 2002, Ashraf and Rehman [2] extended this result by proving it for $(\sigma, \tau)-$derivations of 2-torsion free prime rings. Moreover, at the same time Ashraf and Rehman [1] studied the derivation d of prime ring R satisfying the conditions $d([x, y]) = [x, y]$ and $d(x \circ y) = (x \circ y)$, and obtained the commutativity of R. In [18], Rehman et al. obtained many informative results by studying generalized $(\alpha, \beta)-$derivations of prime rings that satisfy several conditions on Lie ideals. In the same direction, Marubayashi et al. [13] examined all these above mentioned conditions by replacing derivation and $(\sigma, \tau)-$derivation by generalized $(\alpha, \beta)-$derivation F. More precisely, they proved that every 2-torsion free prime ring is commutative if it satisfies any one of the identities: (i) $[F(x), x]_{\alpha,\beta} = 0$, (ii) $F([x, y]) = 0$, (iii) $F(x \circ y) = 0$, (iv) $F([x, y]) = [x, y]_{\alpha,\beta}$, (v) $F(x \circ y) = (x \circ y)_{\alpha,\beta}$, (vi) $F(xy) - \alpha(xy) \in Z(R)$, (vii) $F(x)F(y) - \alpha(xy) \in Z(R)$ for all $x, y \in R$. Recently, Dhara et al. [6] studied all these situations in semiprime rings admitting generalized $(\sigma, \tau)-$derivations.

On the other hand, Oukhtite and Mamouni [16] explored the commutativity of 2-torsion free prime ring R admitting a nonzero derivation d satisfying $d([x, y]) = 0$ on a nonzero Jordan ideal J of R. In addition, Oukhtite et al. [14] proved that a 2-torsion free prime ring R must be commutative if it admits a nonzero derivation d such that $d([x, y]) \in Z(R)$ for all $x, y \in J$, a nonzero Jordan ideal of R. Motivated by these commutativity theorems, we present some results which are the generalization and unification of the above-mentioned results. Precisely, we investigate the commutativity of 2-torsion free prime rings by taking the conditions: (i) $F([x, y]) = 0$, (ii) $F(x \circ y) = 0$, (iii) $F([x, y]) = [x, y]_{\theta,\phi}$, (iv) $F(x \circ y) = (x \circ y)_{\theta,\phi}$, (v)$F(xy) \pm \theta(xy) = 0$, (vi) $F(xy) \pm \phi(xy) \in Z(R)$, (vii) $[F(x), x]_{\theta,\phi} = 0$ for all $x, y \in J$; here F is a generalized $(\theta, \phi)-$derivation of R and J is a nonzero Jordan ideal of R.

The following lemmas are essential in the development of our main results.

Lemma 1. *[[7], Lemma 4] Let R be a 2-torsion free $*$-prime ring, J a nonzero $*$-Jordan ideal of R and d a nonzero (α, β)-derivation of R. If d commutes with $*$ and $d(J) = (0)$, then R is commutative.*

Lemma 2. *[[17], Lemma 3] Let R be a 2-torsion free $*$-prime ring and J a nonzero $*$-Jordan ideal of R. If $J \subseteq Z(R)$, then R is commutative.*

Lemma 3. *[[17], Theorem 1] Let $(R, *)$ be a 2-torsion free prime ring with involution. Let J be a nonzero $*$-Jordan ideal of R and d be a nonzero derivation centralizing on J. If R is $*$-prime, then R is commutative.*

Lemma 4. *[[19], Lemma 2.6] Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal of R. If $aJb = (0)$, then $a = 0$ or $b = 0$.*

Lemma 5. *[[19], Lemma 2.7] Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal of R. If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.*

The following lemma will play a key role throughout.

Lemma 6. *Let R be a ring and J be a Jordan ideal of R. Then*

1. $2R[j^2, i] \subseteq J$ *for all* $i, j \in J$,
2. $2[j^2, i]R \subseteq J$ *for all* $i, j \in J$,
3. $2R[j^2, i]R \subseteq J$ *for all* $i, j \in J$.

Proof. (1) Let $r \in R$ and $x \in J$. Then we know that $2[r, x^2] \in J$ (see [3], proof of Lemma 3). That means

$$2(rx^2 - x^2r) \in J. \tag{1.1}$$

For any $y \in J$, we replace r by ry in (1.1) and find $2(ryx^2 - x^2ry) \in J$. But

$$\begin{aligned} 2(ryx^2 - x^2ry) &= 2(ryx^2 - x^2ry + rx^2y - rx^2y) \\ &= 2(ryx^2 - rx^2y) + 2(rx^2y - x^2ry) \\ &= 2r[y, x^2] + 2[r, x^2]y. \end{aligned}$$

Since $2[r, x^2]y \in J$ for all $r \in R$ and $x, y \in J$, it follows that $2r[y, x^2] \in J$ for each $x, y \in J$ and $r \in R$. In other words, $2R[J^2, J] \subseteq J$.

(2) We replace r by yr in order to find $2(yrx^2 - x^2yr) \in J$. But

$$\begin{aligned} 2(yrx^2 - x^2yr) &= 2(yrx^2 - x^2yr + yx^2r - yx^2r) \\ &= 2(yrx^2 - yx^2r) + 2(yx^2r - x^2yr) \\ &= 2y[r, x^2] + 2[y, x^2]r. \end{aligned}$$

Since $2y[r, x^2] \in J$ for all $r \in R$ and $x, y \in J$, it follows that $2[y, x^2]r \in J$ for all $x, y \in J$ and $r \in R$. In other words $2[J^2, J]R \subseteq J$.
(3) We have $2[y, x^2]r \in J$ for all $x, y \in J$ and $r \in R$. For any $s \in R$, we compute $2((x^2y - yx^2)r)s + 2s(x^2y - yx^2)r \in J$. That is $R[2J^2, J]R \subseteq J$ and we are done. □

Lemma 7. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If $[J, [R, R]] = (0)$ (i.e., $J \subseteq C_R([R, R])$), then R is commutative.*

Proof. For any $x \in J$, we have $[x, [R, R]] = (0)$. Assume that R is noncommutative. Then $[R, R]$ is a noncommutative Lie ideal of R. By Lemma 2 of [11], if R is not a PI-ring, then $[R, R]$ and R satisfy the same GPIs. Thus, by assumption, we have $[x, R] = (0)$ implies $x \in Z(R)$ for all $x \in J$. In the light of Lemma 2, R is commutative, which is a contradiction. □

2 The results on generalized $(\theta, \phi)-$derivations

In everything that follows, R denotes a prime ring with char(R) $\neq 2$, J is a nonzero Jordan ideal and θ, ϕ are the automorphisms of R, unless otherwise mentioned.

Proposition 8. *If $[x, y]_{\theta,\phi} = 0$ or $(x \circ y)_{\theta,\phi} = 0$ for all $x, y \in J$, then R is commutative.*

Proof. Let us consider

$$[x, y]_{\theta,\phi} = x\theta(y) - \phi(y)x = 0, \tag{2.1}$$

for all $x, y \in J$. When replacing x by $4xz^2$ in (2.1), we get

$$xz^2\theta(y) - \phi(y)xz^2 = 0. \tag{2.2}$$

Post-multiply (2.1) by z^2 and subtract from (2.2), we obtain $x[z, \theta(y)] = 0$, where $x, y, z \in J$. With the aid of Lemma 4, we get $[\theta(y), z^2] = 0$. On substituting $y = x \circ r$, we get $[\theta(x \circ r), z^2] = 0$ for all $x, z \in J$ and $r \in R$. Explicitly we have

$$\theta(x)[\theta(r), z^2] + [\theta(r), z^2]\theta(x) = 0. \tag{2.3}$$

Replacing x by $2x[p, q]$ in (2.3), we obtain $\theta(x)[[\theta(p), \theta(q)], [\theta(r), z^2]] = 0$ for all $p, q, r \in R$ and $x, z \in J$. Invoking Lemma 4 gives $[[\theta(p), \theta(q)], [\theta(r), z^2]] = 0$. That is $[[R, R], [R, z^2]] = (0)$. For any $a, b, c \in R$, we have $[[a, b], [c, z^2]] = 0$. Replacing b by ba, we obtain $[a, b][a, [c, z^2]] = 0$. Taking br in place of b, we get $[a, b]R[a, [c, z^2]] = (0)$ for all $a, b, c \in R$ and $z \in J$. It implies that either R is commutative or $[R, [R, z^2]] = 0$. In the latter case we have $[r, [s, z^2]] = 0$ for all $z \in J$ and $r, s \in R$. Replacing s by z^2s, we have $[r, z^2][s, z^2] = 0$. It implies that $[r, z^2]R[r, z^2] = (0)$ for all $z \in J$ and $r \in R$. It forces that $[r, z^2] = 0$. Hence R is commutative (see [[14], proof of Lemma 5]). □

Theorem 9. *Let $d : R \to R$ be a $(\theta, \phi)-$derivation such that $d(x^2) = 0$ for all $x \in J$. Then, $d = 0$.*

Proof. By hypothesis, we have $d(x^2) = 0$, where $x \in J$. By linearizing, we get

$$d(xy + yx) = 0 \quad \text{for all } x, y \in J. \tag{2.4}$$

Now, we have two cases, as follows:
Case 1. Let $J \subseteq Z(R)$. For any $r \in R$ and $u \in J$, we have $2ru = u \circ r$. Hence, equation (2.4) gives that $2d(xy) = 0$ which implies $d(xy) = 0$. Substitute $2yr$ for y, we get $\phi(x)\phi(y)d(r) = 0$ for all $x, y \in J$ and $r \in R$. That is, $xJ\phi^{-1}(d(r)) = 0$. Hence, $\phi^{-1}(d(r)) = 0$ for all $r \in R$. It forces that $d = 0$.
Case 2. Suppose $J \nsubseteq Z(R)$. With the aid of Lemma 6, we substitute $y = 2r[u^2, v]s$ (where $r, s \in R$ and $u, v \in J$) in (2.4). Thus we have

$$d(xr[u^2, v]s) + d(r[u^2, v]sx) = 0.$$

Replacing s by sx in the last expression, we get $d(xr[u^2, v]s)\theta(x) + \phi(xr[u^2, v]s)d(x) + d(r[u^2, v]sx)\theta(x) + \phi(r[u^2, v]sx)d(x) = 0$ for all $x, y \in J$ and $r, s \in R$. It reduces to $\phi(x \circ r[u^2, v]s)d(x) = 0$. For any $z \in J$, we take zr in place of r and obtain $\phi(z)\phi(x \circ r[u^2, v]s)d(x) + \phi([x, z])\phi(r)\phi([u^2, v]s)d(x) = 0$. It implies $\phi([x, z])R\phi([u^2, v]s)d(x) = 0$ for all $x, y, z \in J$ and $r, s \in R$. That means, for each $x \in J$, either $[x, z] = 0$ or $\phi([u^2, v]s)d(x) = 0$. We set

$$G_1 = \{x \in J : [x, z] = 0 \quad \text{for all } z \in J\} \text{ and}$$
$$G_2 = \{x \in J : \phi([u^2, v]s)d(x) = 0 \quad \text{for all } u, v \in J,\ s \in R\}.$$

It is easy to see that G_1 and G_2 are additive subgroups of J and $J = G_1 \cup G_2$. By Brauer's trick, we have either $J = G_1$ or $J = G_2$. If $J = G_1$, that means $[x, z] = 0$ for all $x, z \in J$. It easily implies that $J \subseteq Z(R)$, which is a contradiction. On the other hand, let $I = G_2$ i.e., for any $s \in R$ and $x, u, v \in J$, we have $\phi([u^2, v]s)d(x) = 0$. That is $\phi([u^2, v])Rd(x) = (0)$ for all $x, u, v \in J$. Since R is a prime ring, either $[u^2, v] = 0$ for all $u, v \in J$ or $d(J) = (0)$. In view of [[14], proof of Lemma 5], a contradiction follows. Finally, we suppose that $d(x) = 0$ for all $x \in J$. It forces that $d = 0$, by Lemma 7. Thus we have $d = 0$ in each case. □

Remark 10. Similarly, we can prove that: *Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal of R. If R admits a nonzero $(\theta, \phi)-$derivation d such that $d([x, y]) = 0$ for all $x, y \in J$, then R is commutative.* Consequently, this result proves a complete form of Theorem 3 of [2].

Theorem 11. *Let $F : R \to R$ be a generalized (θ, ϕ)-derivation of R associated with a nonzero (θ, ϕ)-derivation d such that $F([x, y]) = 0$ for all $x, y \in J$. Then, R is commutative.*

Proof. By our hypothesis, we have $F(xy) - F(yx) = 0$ for all $x, y \in J$. That is

$$F(x)\theta(y) + \phi(x)d(y) - F(y)\theta(x) - \phi(y)d(x) = 0.$$

In view of Lemma 6, substitute $2r[u^2, v]s$ for y, where $u, v \in J$ and $r, s \in R$, then

$$F(x)\theta(r[u^2, v]s) + \phi(x)d(r[u^2, v]s) - F(r[u^2, v]s)\theta(x) - \phi(r[u^2, v]s)d(x) = 0. \tag{2.5}$$

Replace s by sx in (2.5), we get

$$\begin{aligned} F(x)\theta(r[u^2, v]s)\theta(x) + \phi(x)d(r[u^2, v]s)\theta(x) + \phi(x)\phi(r[u^2, v]s)d(x) - F(r[u^2, v]s) \\ \theta(x)\theta(x) - \phi(r[u^2, v]s)d(x)\theta(x) - \phi(r[u^2, v]s)\phi(x)d(x) = 0. \end{aligned} \tag{2.6}$$

Equation (2.5) reduces (2.6) to

$$\phi([x, r[u^2, v]s])d(x) = 0 \ \text{ for all } x, u, v \in J, \ r, s \in R. \tag{2.7}$$

Taking qr instead of r in (2.7), where $q \in R$, we get $\phi([x, q])\phi(r[u^2, v]s)d(x) = 0$. It implies that

$$[x, q]r[u^2, v]R\phi^{-1}d(x) = 0 \ \text{ for all } x, u, v \in J, \ r, q \in R.$$

Applying Brauer's trick, we get that either $[x, q]r[u^2, v] = 0$ for all $x, u, v \in J$ and $r, q \in R$ or $d(x) = 0$ for all $x \in J$. The second case forces R commutative. Now the first case gives $[x, q]R[u^2, v] = 0$ for all $x, u, v \in J$ and $q \in R$. It implies that either $J \subseteq Z(R)$ or $[u^2, v] = 0$ for all $u, v \in J$. In both of these cases, we get R is commutative (see Lemma 2 and [[14], proof of Lemma 5] respectively). □

Using similar techniques as we used in the proof of Theorem 11 with necessary variations, we can obtain the following result:

Theorem 12. *Let $F : R \to R$ be a generalized (θ, ϕ)-derivation of R associated with a nonzero (θ, ϕ)-derivation d such that $F(x \circ y) = (0)$ for all $x, y \in J$. Then, R is commutative.*

Theorem 13. *Let $F : R \to R$ be a generalized (θ, ϕ)-derivation of R associated with a nonzero (θ, ϕ)-derivation d such that $F([x, y]) = [x, y]_{\theta,\phi}$ for all $x, y \in J$. Then, R is commutative.*

Proof. By hypothesis, we have $F(xy) - F(yx) = x\theta(y) - \phi(y)x$ for all $x, y \in J$. Taking $y = 2r[u^2, v]s$ in particular, where $r, s \in R$ and $u, v \in J$, we find

$$F(x2r[u^2, v]s) - F(2r[u^2, v]sx) = x\theta(2r[u^2, v]s) - \phi(2r[u^2, v]s)x.$$

Replace s by sx, we obtain

$$F(x.2r[u^2, v]s)\theta(x) + \phi(x.2r[u^2, v]s)d(x) - F(2r[u^2, v]s.x)\theta(x) - \phi(2r[u^2, v]s.x)d(x) = x\theta(2r[u^2, v]s)\theta(x) - \phi(2r[u^2, v]s)\phi(x)x.$$

Our hypothesis reduces it to

$$\phi([x, r[u^2, v]s])d(x) = \phi(r[u^2, v]s)[x, x]_{\theta,\phi}. \tag{2.8}$$

Substituting pr instead of r in (2.8) and using it, we get $\phi([x, p])\phi(r[u^2, v]s)d(x) = 0$ for all $x, u, v \in J$ and $r, s, p \in R$. It implies that

$$[x, p]R[u^2, v]s\phi^{-1}(d(x)) = 0.$$

Since R is prime ring, we find that for each $x \in J$, either $[x, p] = 0$ for all $p \in R$ or $[u^2, v]s\phi^{-1}(d(x)) = 0$ for all $u, v \in J$ and $s \in R$. Application of Brauer's trick yields that either $J \subseteq Z(R)$ (and hence R is commutative by Lemma 2) or $[u^2, v]s\phi^{-1}(d(x)) = 0$. In the latter case we get that either $[u^2, v] = 0$ for all $u, v \in J$ or $d(x) = 0$ for all $x \in J$. The prior case implies that R is commutative (see the proof of Lemma 5 in [14]). Assume that $d(x) = 0$ for all $x \in J$. In view of Lemma 1, we get the commutativity of R. □

Using similar techniques as we used in the proof of Theorem 13 with necessary variations, we can obtain the following result:

Theorem 14. *Let $F : R \to R$ be a generalized (θ, ϕ)-derivation of R associated with a (θ, ϕ)-derivation d such that $F(x \circ y) = (x \circ y)_{\theta,\phi}$ for all $x, y \in J$. Then, R is commutative.*

Theorem 15. *Let $F : R \to R$ be a generalized (θ, ϕ)-derivation of R associated with a nonzero (θ, ϕ)-derivation d such that $F(xy) \in Z(R)$ for all $x, y \in J$. Then, R is commutative.*

Proof. By our hypothesis, we have $F(xy) \in Z(R)$ for any $x, y \in J$. Let us put $y = 2r[u^2, v]s$, where $r, s \in R$ and $u, v \in J$, that is

$$F(x)\theta(r[u^2, v]s) + \phi(x)d(r[u^2, v]s) \in Z(R). \tag{2.9}$$

Replace s by sq, where $q \in R$, we get

$$F(x)\theta(r[u^2, v]s)\theta(q) + \phi(x)d(r[u^2, v]s)\theta(q) + \phi(x)\phi(r[u^2, v]s)d(q) \in Z(R).$$

Commuting with $\theta(q)$ and using (2.9), we get

$$[\phi(x)\phi(r[u^2, v]s)d(q), \theta(q)] = 0 \ \text{ for all } x, u, v \in J, \ r, s, q \in R. \tag{2.10}$$

Replace x by $2[m, n]x$ (2.10), where $m, n \in R$, we get

$$[\phi([m, n]), \theta(q)]\phi(r[u^2, v]s)d(q) = 0.$$

That is

$$[[m, n], \phi^{-1}(\theta(q))]R[u^2, v]s\phi^{-1}(d(q)) = (0)$$

for all $u, v \in J$ and $s, q, m, n \in R$. It implies that either $[[m, n], \phi^{-1}(\theta(q))] = 0$ or $[u^2, v]s\phi^{-1}(d(q)) = 0$. Using Brauer's trick, we get either $[\phi([m, n]), \theta(q)] = 0$ for all $m, n, q \in R$ or $[u^2, v]s\phi^{-1}(d(q)) = 0$ for all $u, v \in J$ and $s, q \in R$. Clearly first assertion implies commutativity of R. Thus we consider the latter case i.e., $[u^2, v]R\phi^{-1}(d(q)) = (0)$ for all $u, v \in J$ and $q \in R$. Since R is prime ring and d is a nonzero $(\theta, \phi)-$derivation, we get $[u^2, v] = 0$ for all $u, v \in J$. In view of [[14], proof of Lemma 5], R is commutative. □

By substituting $F + \theta$ and $F - \theta$ for F in Theorem 15, we have the following theorem:

Theorem 16. *Let $F : R \to R$ be a generalized (θ, ϕ)-derivation of R associated with a nonzero (θ, ϕ)-derivation d. If any one of the following holds,*

1. $F(xy) + \theta(xy) \in Z(R)$,
2. $F(xy) - \theta(xy) \in Z(R)$,

for all $x, y \in J$, then R is commutative.

Theorem 17. *Let $F : R \to R$ be a generalized (θ, ϕ)-derivation of R associated with a nonzero (θ, ϕ)-derivation d of R such that $[F(x), x]_{\theta,\phi} = 0$ for all $x \in J$. Then, R is commutative.*

Proof. By hypothesis, we have

$$[F(x), x]_{\theta,\phi} = 0 \ \text{ for all } x \in J. \tag{2.11}$$

It implies that

$$[F(x), y]_{\theta,\phi} + [F(y), x]_{\theta,\phi} = 0 \ \text{ for all } x \in J. \tag{2.12}$$

In view of Lemma 6, we take $2r[u^2, v]x$ in place of y in (2.12) in order to obtain

$$\begin{aligned}[F(x), 2r[u^2, v]]_{\theta,\phi}\theta(x) + \phi(2r[u^2, v])[F(x), x]_{\theta,\phi} + [F(2r[u^2, v])\theta(x), x]_{\theta,\phi} \\ +[\phi(2r[u^2, v])d(x), x]_{\theta,\phi} = 0.\end{aligned} \tag{2.13}$$

Re-writing it as

$$[F(x), 2r[u^2, v]]_{\theta,\phi}\theta(x) + \phi(2r[u^2, v])[F(x), x]_{\theta,\phi} + \Lambda(x, u, v, r) = 0$$

for all $x, u, v \in J,\ r \in R$, where

$$\begin{aligned}\Lambda(x, u, v, r) &= [F(2r[u^2, v])\theta(x), x]_{\theta,\phi} + [\phi(2r[u^2, v])d(x), x]_{\theta,\phi}\\ &= F(2r[u^2, v])\theta(x)\theta(x) - \phi(x)F(2r[u^2, v])\theta(x) + \phi(2r[u^2, v])d(x)\theta(x)\\ &\quad -\phi(2r[u^2, v])\phi(x)d(x) + \phi(2r[u^2, v])\phi(x)d(x) - \phi(x)\phi(2r[u^2, v])d(x)\\ &= [F(2r[u^2, v]), x]_{\theta,\phi}\theta(x) + \phi(2r[u^2, v])[d(x), x]_{\theta,\phi} + \phi([2r[u^2, v], x])d(x).\end{aligned}$$

Combining the last relation with (2.13) and using (2.11) and (2.12), we obtain

$$\phi(r[u^2, v])[d(x), x]_{\theta,\phi} + [\phi(r[u^2, v]), \phi(x)]d(x) = 0 \text{ for all } x, u, v \in J, r \in R. \tag{2.14}$$

Taking sr instead of r in (2.14), where $s \in R$, we obtain $[\phi(s), \phi(x)]R\phi([u^2, v])d(x) = 0$ for all $x, u, v \in J$ and $s \in R$. And hence primeness of R forces that for each $x \in J$, either $[\phi(s), \phi(x)] = 0$ or $\phi([u^2, v])d(x) = 0$. By Brauer's trick, we obtain that either $J \subseteq Z(R)$ or $\phi([u^2, v])d(x) = 0$ for all $x, u, v \in J$. The first case implies that R is commutative and we are done. Let us assume that $\phi([u^2, v])d(x) = 0$. Replacing v by $2[p, q]v$, where $p, q \in R$, we get $\phi([u^2, [p, q]])Jd(x) = 0$ for all $x, u \in J$. By Lemma 4, it follows that either $u^2 \in Z([R, R])$ or $d(J) = 0$. In light of lemma 2, latter case implies that R is commutative. Now, in case $u^2 \in Z([R, R])$, we get $u^2 \in Z(R)$ and hence R is commutative. □

3 The results on generalized $(1_R, \phi)-$derivations

Theorem 18. *Let $F : R \to R$ be a generalized $(1_R, \phi)$-derivation associated with an $(1_R, \phi)-$derivation d. If for any $0 \neq \alpha \in R$, $\alpha(F(x)F(y) \pm xy) = 0$ for all $x, y \in J$, then either R is commutative or there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ and $\lambda^2 = \pm 1$.*

Proof. By hypothesis, we have

$$\alpha(F(x)F(y) \pm xy) = 0 \tag{3.1}$$

for all $x, y \in J$. Replace y by $4yz^2$, where $z \in J$. We get $\alpha(F(x)F(y) \pm xy)z^2 + \alpha F(x)\phi(y)d(z^2) = 0$. Our hypothesis reduces it to $\alpha F(x)\phi(y)d(z^2) = 0$ for all $x, y, z \in J$. That is, $\phi^{-1}(\alpha F(x))J\phi^{-1}(d(z^2)) = (0)$ for all $x, z \in J$. By Lemma 4, we have $\alpha F(x) = 0$ or $d(z^2) = 0$ for all $x, z \in J$. Firstly, let $\alpha F(J) = (0)$. Eq. (3.1) implies that $\alpha xy = 0$ for all $x, y \in J$. Since J is nonzero, we have $\alpha = 0$, which is a contradiction. Thus, we must have $d(z^2) = 0$ for all $z \in J$. In the light of Theorem 9, we obtain $d = 0$. Consequently $F(xy) = F(x)y$ for all $x, y \in R$. By Lemma 2 of [9], there exists $\lambda \in Q_{mr}(R_C)$ such that $F(x) = \lambda x$ for all $x \in R$. Since $2R[J^2, J]R \subseteq J$ (by Lemma 6), we replace x by $2r[u^2, v]y$ in (3.1) in order to obtain

$$\alpha(F(r[u^2, v])yF(y) \pm r[u^2, v]y^2) = 0 \text{ for all } u, v, y \in J \text{ and } r \in R. \tag{3.2}$$

By substituting $x = 2r[u^2, v]$ in (3.1), where $u, v \in J$ and $r \in R$, we obtain

$$\alpha(F(r[u^2, v])F(y) \pm r[u^2, v]y) = 0.$$

Post-multiplying the above relation by y, we get

$$\alpha(F(r[u^2, v])F(y)y \pm r[u^2, v]y^2) = 0. \tag{3.3}$$

Subtract (3.2) from (3.3), we have $\alpha F(r)[u^2, v][F(y), y] = 0$ for any $u, v, y \in J$ and $r \in R$. Replace r by rs, where $s \in R$, we get $\alpha F(r)s[u^2, v][F(y), y] = 0$. That is

$$\alpha F(r)R[u^2, v][F(y), y] = (0)$$

for all $u, v, y \in J$ and $r \in R$. Since R is prime ring, it follows that either $\alpha F(r) = 0$ or $[u^2, v][F(y), y] = 0$. The first case is not possible, thus we have $[u^2, v][F(y), y] = 0$ where $u, v, y \in J$. Replace v by $2[r, s]v$ in the last expression and using it, we find $[u^2, [r, s]]J[F(y), y] = (0)$ for all $u, y \in J$ and $r, s \in R$. Lemma 4 forces $[F(y), y] = 0$ or $[u^2, [r, s]] = 0$ for all $u, y \in J$ and $r, s \in R$. If $[u^2, [r, s]] = 0$ for all $u \in J$ and $r, s \in R$, then by Lemma 7, we find that $u^2 \in Z(R)$. Hence R is commutative, as we have already seen in the proof of Theorem 11.

Next, we consider $[F(y), y] = 0$ for all $y \in J$. It gives $[\lambda y, y] = 0$ for all $y \in J$. Linearizing w.r.t.y, we find $[\lambda, x]y + [\lambda, y]x = 0$ for all $x, y \in J$. Changing y by $2y[r, s]$, we find $[\lambda, x]y[r, s] + y[\lambda, [r, s]]x + [\lambda, y][r, s]x = 0$ for all $x, y \in J$ and $r, s \in R$. It implies

$$y[\lambda, [r, s]]x + [\lambda, y][[r, s], x] = 0. \tag{3.4}$$

Taking $2py^2$ in place of y in the last expression, where $p \in R$, we may infer that

$$p(2y^2)[\lambda, [r, s]] + p[\lambda, 2y^2][[r, s], x] + [\lambda, p]2y^2[[r, s], x] = 0.$$

Equation (3.1) reduces it to $[\lambda, p]2y^2[[r, s], x] = 0$ for all $x, y \in J$ and $r, s, p \in R$. It implies that $[\lambda, p]Ry^2[[r, s], x] = 0$. In view of our assumption, it yields that $[\lambda, p] = 0$ for all $p \in R$. It is a well known fact of theory of differential identities that a prime ring R and U (the Utumi quotient ring of R) satisfies the same GPI. Hence the first case implies that $\lambda \in C$, while it is easy to check that the latter case forces R to be commutative. □

Theorem 19. *Let $F : R \to R$ be a generalized $(1_R, \phi)$-derivation associated with an $(1_R, \phi)-$derivation d of R. If for any $0 \neq \alpha \in R$, $\alpha(F(x)F(y) \pm yx) = 0$ for all $x, y \in J$, then R is commutative or there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ and $\lambda^2 = \pm 1$.*

Proof. By the hypothesis, we have

$$\alpha(F(x)F(y) \pm yx) = 0 \quad \text{for all } x, y \in J. \tag{3.5}$$

In view of Lemma 6, we replace y by $2r[u^2, v]s$ (where $r, s \in R$ and $u, v \in J$) in (3.5) and get

$$\alpha(F(x)F(r[u^2, v]s) \pm r[u^2, v]sx) = 0. \tag{3.6}$$

Replacing s by sx, we obtain

$$\alpha(F(x)F(r[u^2, v]s)x + F(x)\phi(r[u^2, v]s)d(x) \pm r[u^2, v]sx^2) = 0.$$

Thus (3.6) reduces it to

$$\alpha(F(x)\phi(r)\phi([u^2, v]s)d(x)) = 0 \quad \text{for all } x, u, v \in J,\ r, s \in R.$$

That is $\alpha F(x)R\phi([u^2, v]s)d(x) = (0)$. It implies that for each $x \in J$, either $\alpha F(x) = 0$ or $\phi([u^2, v]s)d(x) = 0$. Applying Brauer's trick, we find that either $\alpha F(x) = 0$ for all $x \in J$ or $\phi([u^2, v]s)d(x) = 0$ for all $x, u, v \in J$ and $s \in R$. Now onwards, we split the proof into two parts.

First we assume that $\alpha F(x) = 0$ for all $x \in J$. In this view (3.5) implies $\alpha yx = 0$ for all $x, y \in J$. By Lemma 4, we get $\alpha = 0$, which is a contradiction. Thus, we have

$$\phi([u^2, v]s)d(x) = 0 \quad \text{for all } x, u, v \in J,\ s \in R.$$

It forces that either $d(x) = 0$ for all $x \in J$ or $[u^2, v] = 0$ for all $u, v \in J$. The latter subcase implies that R is commutative (see the proof of Lemma 5 in [14]). We now consider $d(x) = 0$ for all $x \in J$. In view of Lemma 1, it follows that either $d = 0$ or R is commutative.

Let us assume that $d = 0$. Consequently $F(xy) = F(x)y$ for all $x, y \in R$. By Lemma 2 of [9], there exists $\lambda \in Q_{mr}(R_C)$ such that $F(x) = \lambda x$ for all $x \in R$.

When replacing x by $4x^2r$ in (3.5), where $r \in R$, we get

$$\alpha(F(2x^2)rF(y) \pm 2yx^2r) = 0. \tag{3.7}$$

Replace x by $2x^2$ in (3.5), we find $\alpha(F(2x^2)F(y) \pm 2yx^2) = 0$. Post-multiply by r, we get

$$\alpha(F(2x^2)F(y)r \pm 2yx^2r) = 0. \tag{3.8}$$

Eq. (3.7) together with Eq. (3.8) gives $\alpha F(x^2)[F(y), r] = 0$, where $x, y \in J$ and $r \in R$. It easily follows that $\alpha F(x^2)R[F(y), r] = (0)$. Since R is a prime ring, we have either $\alpha F(x^2) = 0$ or $[F(y), r] = 0$. Let us consider $\alpha F(x^2) = 0$ for all $x \in J$. When linearizing, we obtain $\alpha F(x \circ y) = 0$ for any $x, y \in J$. Putting $4uy^2$ for y (where $u \in J$) in the last relation, we obtain

$$\begin{aligned} 0 &= \alpha F((x \circ u)y^2) - \alpha F(u[x, y^2]) \\ &= \alpha F((x \circ u))y^2 - \alpha F(u)[x, y^2] \\ &= -\alpha F(u)[x, y^2]. \end{aligned}$$

Replace u by $4uru$ in above expression, we obtain $\alpha F(u)Ru[x, y^2] = (0)$ for all $u, x, y \in J$. It follows that either $\alpha F(u) = 0$ or $u[x, y^2] = 0$. But according to our assumption $\alpha F(u) = 0$ is not the case, hence we have $u[x, y^2] = 0$ implies $[x, y^2] = 0$ for all $x, y \in J$. As above, it implies that R is commutative.

In the latter case, we assume that $[F(y), r] = 0$ for all $y \in J$ and $r \in R$. Thus by our hypothesis, we find

$$\alpha(F(y)F(x) \pm yx) = 0 \quad \text{for all } x, y \in J.$$

By repeating the same reasoning as in Theorem 18, there exists some $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ and $\lambda^2 = \pm 1$. □

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