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New descriptions of certain classes of graceful graphs

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Abstract

The aim of this paper is to bring new descriptions and characterizations of graceful labellings of certain graphs by using methods and tools such as graph chessboard, labelling sequence and labelling relation. These methods and tools bring new insights to the study of graceful graphs, among them the extra value of visualization. By their application new descriptions of graceful labellings of sunlet graphs and wheels are presented. These classes of graphs are known to be graceful, however the results presented in this paper bring their new characterisations.

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1 Introduction

The basis of the study of graph labellings was laid out in the late 1960s. Interest in graph labellings began with a Kotzig-Ringel conjecture and a paper by Rosa [10]. The most extensive source of regularly updated information on graph labellings is "Dynamic Survey of Graph Labeling" by Galian [2], where one can found a huge number of results, methods and techniques on graph labellings.

The subject of study in this text are β-labellings, which were introduced together with other types of graph labellings by Rosa [9] in 1965. Later Golomb [4] named β-labelling as *graceful labelling*. A graph with m edges has a graceful labelling and it is said to be graceful, when its vertices can be assigned the labels from the set $\{0, 1, ..., m\}$ such that the absolute values of the differences in vertex labels of edges form the set $\{1, ..., m\}$.

This paper brings new results in the area of graceful labellings of graphs which have been achieved by tools such as graph chessboard, labelling sequence and labelling relation. The main source of inspiration for this text has been the book "Vertex Labellings of Simple Graphs" by the authors Haviar and Ivaška [5], where the mentioned tools are described. The labelling sequences were introduced in 1976 by Sheppard in [11] while the graph chessboards and labelling relations were introduced and studied in [5]. We also used a computer program called *Graph processor* from [6], which turned out to be very helpful for finding and describing graceful labellings of graphs during our investigations.

The basic terms from graph theory needed in the paper are introduced in Section 2. Here we also present the basic facts on the graph labellings and especially graceful labellings. Then we introduce the tools that we used to achieve our results, namely the graph chessboards, labelling sequences and labelling relations.

In this text we focus on two classes of graphs: *sunlet graphs* and *wheel graphs*. In both cases the graceful labellings of these graphs have been known since 1979 due to Frucht [1]. It might sound interesting this time that the author in his paper called the sunlet graphs by *coronas*. When studying the sunlet graphs by our tools and our methods, we distinguish four subclasses according to the length of the cycle of these graphs: $n \equiv 0 \pmod 4$, $n \equiv 1 \pmod 4$, $n \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$. As we shall see in Section 3, all four cases have something in common, yet they are different in details. For the wheel graphs we will similarly distinguish two subcases, for even and odd lengths of their cycle. In both classes of graphs the pattern which represents the graceful labellings of these graphs is very well indicated in the graph chessboard, where one can immediately recognize that the corresponding graph labelling is graceful. For the sunlet graphs we show and prove formulas for the labelling sequence and the labelling relation. For the wheel graphs we define specific labelling sequences, which correspond to the graceful labellings of these graphs. Our methods are frequently illustrated by figures and examples.

2 Preliminaries

Here we mention certain terms from graph theory. We present graph labellings, especially graceful labellings (called in the original terminology β-labellings), which we use in our paper. Moreover we introduce labelling sequences, labelling relations and simple chessboard as our tools to describe graceful graphs. These basic preliminaries, concepts and definitions are taken primarily from [5].

Simple graphs are in graph theory well-known as finite undirected graphs without loops and multiple edges. In this work we use only these graphs. As usual, for the number of vertices of a graph G we use the term *order* of G, and for the number of edges in G we use the term *size* of G.

2.1 Graph labellings

Definition 2.1. A **vertex labelling** (or only labelling) f of a graph $G = (V, E)$ is a one-to-one mapping of its vertex set V(G) into the set of non-negative integers assigning to the vertices so-called **vertex labels**.

Definition 2.2. By the **label of an edge** uv in the labelling f we mean the number $|f(u) - f(v)|$, where $f(u), f(v)$ are the labels of the vertices u,v, respectively.

In this text we will denote $f(V_G)$ the set of all vertex labels and $f(E_G)$ the set of all edge labels in the labelling f of the graph G.

We know several types of graph labellings, e.g. $\alpha, \beta, \sigma, \rho$ defined by Rosa in his seminal paper [10] in 1967, and further γ, δ, p, q introduced also by Rosa in his dissertation thesis [9] in 1965. There exists a hierarchy of labellings $\alpha, \beta, \sigma, \rho$:

α-labelling
β-labelling
σ-labelling
ρ-labelling.

Each labelling of a given graph is at the same time also the next lower labelling. For instance, every σ-labelling is also ρ-labelling, but a σ-labelling need not be β-labelling or α-labelling.

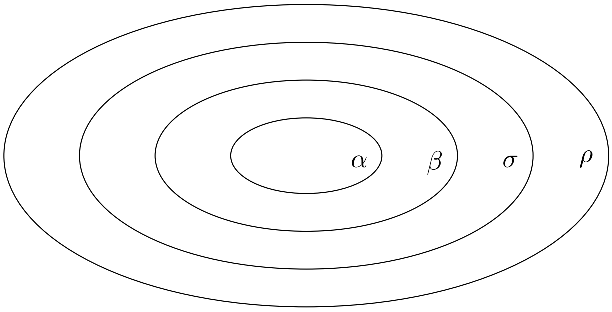


Figure 1. Visualization of a Rosa hierarchy

For this hierarchy of labellings we shall be using the term *Rosa hierarchy* as in [5]. In this text we study only β-labelling, which is called **graceful labelling**, therefore we will present only definition of this labelling.

Definition 2.3. A **graceful labelling** (or β-labelling) of a graph $G = (V, E)$ of size m is a vertex labelling with the following properties:

1. $f(V_G) \subseteq \{0, 1, \ldots, m\}$, and
2. $f(E_G) = \{1, 2, \ldots, m\}$.

Hence a graceful labelling of a graph of size m has vertex labels among the numbers $0, 1, ..., m$ such that the induced edge labels are different and cover all values $1, 2, ..., m$. When a graph has a graceful labelling then we say that graph is *graceful*. While the concept of a β-labelling was introduced by Rosa [10] in 1967, in 1972 Golomb [4] called such labelling “graceful” and this term was popularized by mathematician Martin Gardner [3]. In Figure 2 we can see some graceful graphs.

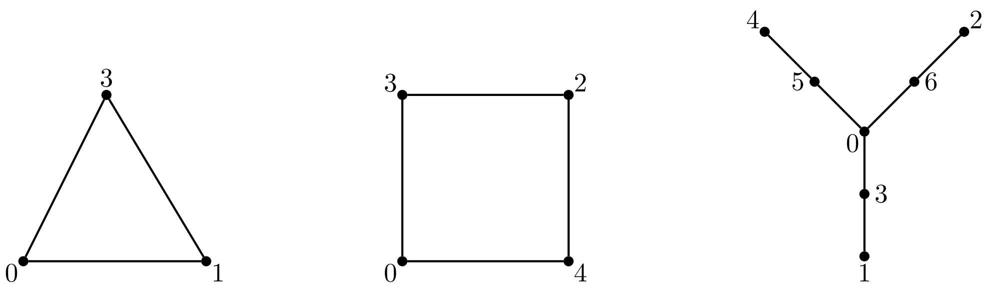


Figure 2. Some graceful graphs

2.2 Labelling sequences and relations

Each graceful graph can be represented by a sequence of non-negative integers. It was shown by Sheppard in his article [11], where he introduced a new concept of a *labelling sequence* as follows:

Definition 2.4. ([11], [5]) For a positive integer m, a **labelling sequence** is the sequence of non-negative integers $(j_1, j_2, \ldots, j_m)$, denoted (j_i), where

$$0 \leq j_i \leq m - i \quad \text{for all } i \in \{1, 2, \ldots, m\}. \tag{LS}$$

Sheppard also proved that there is a one-to-one correspondence between graceful labellings of graphs (without isolated vertices) and labelling sequences. Therefore we can understand labelling sequences as a tool to encode graceful labellings of graphs. The connection is described in the following theorem.

Theorem 2.5. ([11], [5]) There exists a one-to-one correspondence between graphs of size m having a graceful labelling f and between labelling sequences (j_i) of m terms. The correspondence is given by

$$j_i = \min\{f(u), f(v)\}, \quad i \in \{1, 2, \dots, m\},$$

where u, v are the end-vertices of the edge labelled i.

Now we introduce definition of a labelling relation. It is an another tool to describe gracefully labelled graphs, which is closely related to term of labelling sequence. The concept of a *labelling relation* was introduced and studied by Haviar and Ivaška in [5]:

Definition 2.6. ([5]) Let $L = (j_1, j_2, \dots, j_m)$ be a labelling sequence. Then the relation $A(L) = \{[j_i, j_i + i], i \in \{1, 2, \dots, m\}\}$ will be called a **labelling relation** assigned to the labelling sequence L.

To visualize a labelling relation and also a labelling sequence we shall use a *labelling table* (see Figure 3). A table is formed by heading with the numbers $1, 2, \dots, m$ and two rows. The first row contains the numbers from the labelling sequence and the second row contains the sums of numbers from the heading and the first row. The pairs from first and second row in each column are then the elements of the labelling relation (and also edges of the graph).

1	2	3	...	m
j_1	j_2	j_3	...	j_m
$j_1 + 1$	$j_2 + 2$	$j_3 + 3$	...	$j_m + m$

Figure 3. Displaying a labelling table

Example 2.7. In Figure 4 we can see the labelling table assigned to the labelling sequence (5,4,3,2,1,0) and its corresponding graceful graph whose edges are the elements of the labelling relation.

1	2	3	4	5	6
5	4	3	2	1	0
6	6	6	6	6	6

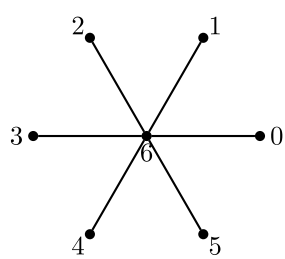


Figure 4. Example of a labelling table and its corresponding graceful graph

2.3 Simple graph chessboards

Every labelled simple graph with n vertices can be represented by a *chessboard*, i.e. a table with n rows and n columns, where every edge of graph uv is represented by a pair of dots with coordinates (u, v) or (v, u). This idea of visualization of vertex labellings of graphs by chessboard and also other independent discoveries of similar ideas are described in [5].

There exist several types of graph chessboards like simple chessboard, double chessboard, M-chessboard, dual chessboard and twin chessboard, which were discovered by Haviar and Ivaška and presented in [5]. In this text we use only the idea of a simple chessboard, which is a useful tool in visualization of labelled graphs. Consider a graph G

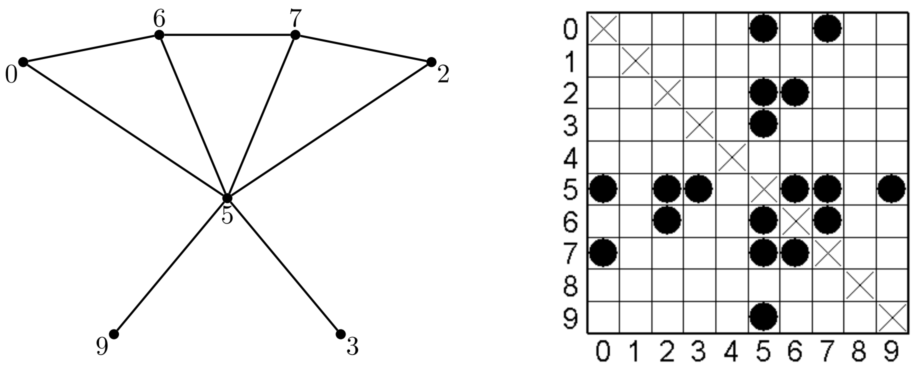


Figure 5. Example of a graph and its corresponding simple chessboard

of size m whose vertices are labelled by different numbers from the set $\{0, 1, 2, ..., m\}$ and consider a table with $m + 1$ rows and $m + 1$ columns. Rows are numbered by $0, 1, ..., m$ from the top to the bottom and columns are numbered by $0, 1, ..., m$ from the left to the right as we can see in Figure 5. The cell with coordinates $[i, j]$ of the table will mean the cell in the i-th row and the j-th column. The *r-th diagonal* in the table is the set of all cells with coordinates $[i, j]$ where $i - j = r$ and $i \geq j$. The *main diagonal* is 0-th diagonal in the table and other diagonals are *associated.* A *simple chessboard* of size m is

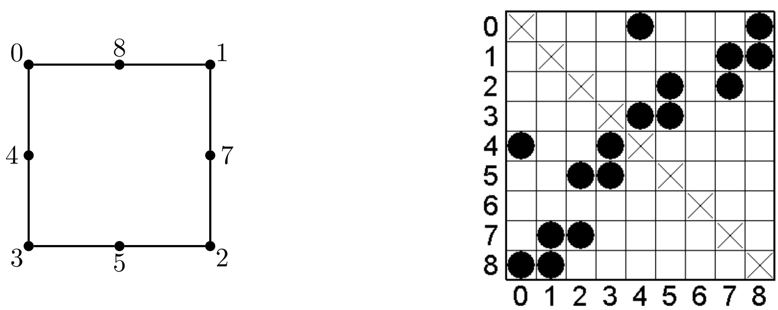


Figure 6. Gracefully labelled graph and its graceful simple chessboard

a table, which is assigned to a labelled graph of size m in the following way: every edge uv in the graph is represented by a pair of dots in the cells with coordinates $[u, v]$ and $[v, u]$. It follows that the simple chessboards are symmetric about the main diagonal. An illustration of the simple chessboard of a graph is in Figure 5.

If there is exactly one dot on each of the associate diagonals, then a simple chessboard will be called *graceful* as it clearly encodes a graceful graph. We can see a gracefully labelled graph and its graceful simple chessboard in Figure 6.

3 Sunlet graphs

Here we present new characterizations of sunlet graphs. We describe their graceful labellings via labelling sequences, labelling relations and simple graph chessboards. Our method is similar to that used in [5, Chapter 4].

Definition 3.1. ([7]) The **sunlet graph** is the graph on 2n vertices obtained by attaching n pendant edges to a cycle graph C_n. We will denote it by SG_n.

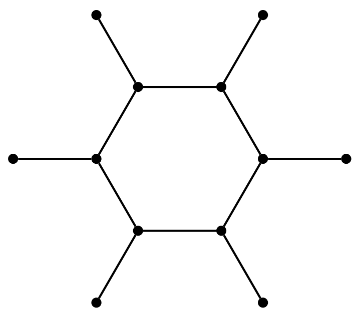


Figure 7. The sunlet graph SG_6

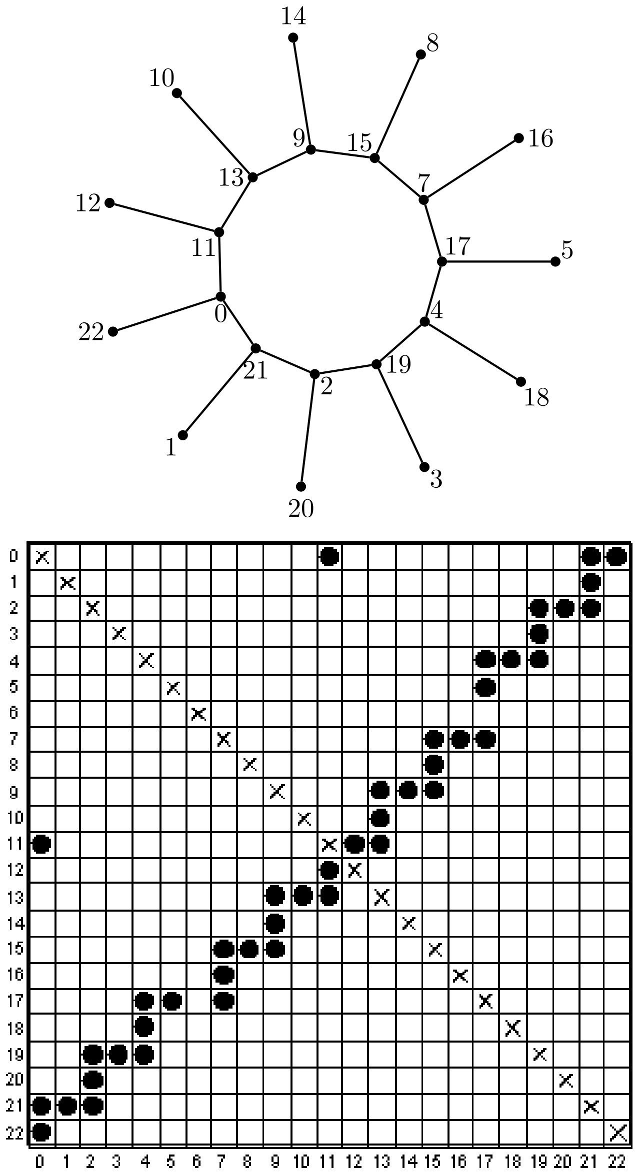
In Figure 7 we see the sunlet graph SG_6. These graphs are known to be graceful since 1979 due to Roberto Frucht [1]. In this paper the author called these graphs as *coronas*, in the book [2] they are also called *crown graphs*. We will use the name *sunlet graphs*. We shall describe graceful labellings of these graphs via labelling sequences, labelling relations and simple chessboards. We distinguish four subclasses of sunlet graphs according to the length of their cycle: $n \equiv 0 \pmod 4$, $n \equiv 1 \pmod 4$, $n \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$. All four cases have something in common, yet they are different in details.

Before presenting our descriptions of sunlet graphs via simple chessboards, labelling sequences and labelling relations, we illustrate these descriptions on an example.

Example 3.2. The sequence $(11, 11, 10, 9, 9, 9, 8, 7, 7, 7, 0, 5, 4, 4, 4, 3, 2, 2, 2, 1, 0, 0)$ is a labelling sequence of the sunlet fan graph SG_n of size m where we have $n = 11$ and $m = 22$. Corresponding graph diagram, labelling relation and graph chessboard are in Figure 8. We can divide this labelling sequence into three parts: the first cascade of the labelling sequence (from the number n to the number $\lfloor \frac{n}{2} \rfloor + 2$), number 0 and the second cascade (from the number $\lfloor \frac{n}{2} \rfloor$ to the number 0). So number zero is between two decreasing cascades of numbers.

As we see in Figure 8 in the simple chessboard below its main diagonal, these cascade parts of the labelling sequence (or these parts in the labelling relation) are represented by two decreasing cascade roads of dots in the chessboard (the beginning of the first one is on the first diagonal and the end of the second one is on the m-th diagonal). The number zero in the labelling sequence is represented by the isolated dot in the position $[n, 0]$ below its main diagonal in the graph chessboard.

Definition 3.3. The graph chessboard described in Example 3.2 will be called a **cascade graph chessboard of type 3**.

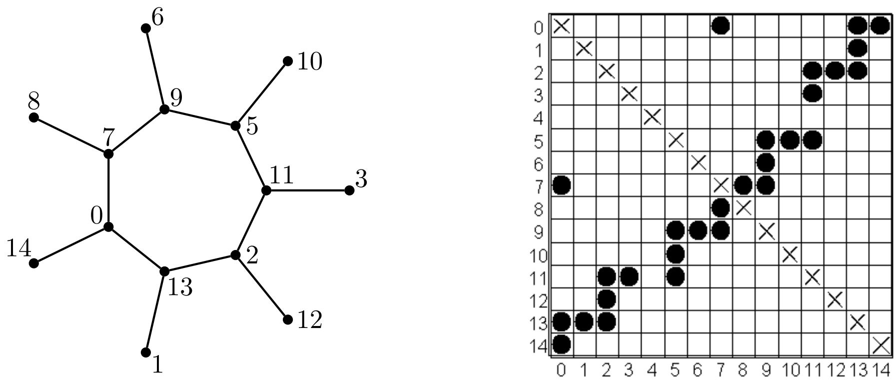


1	2	3	4	5	6	7	8	9	10	11
11	11	10	9	9	9	8	7	7	7	0
12	13	13	13	14	15	15	15	16	17	11

12	13	14	15	16	17	18	19	20	21	22
5	4	4	4	3	2	2	2	1	0	0
17	17	18	19	19	19	20	21	21	21	22

Figure 8. The representations of the sunlet graph SG_{11}

We see an example of a cascade graph chessboard of type 3 also in Figure 9 together with the corresponding graph diagram and the labelling relation. The corresponding labelling sequence is $(7, 7, 6, 5, 5, 5, 0, 3, 2, 2, 2, 1, 0, 0)$.



1	2	3	4	5	6	7	8	9	10	11	12	13	14
7	7	6	5	5	5	0	3	2	2	2	1	0	0
8	9	9	9	10	11	7	11	11	12	13	13	13	14

Figure 9. The representations of the sunlet graph SG_7

We prove our results only in the case $n \equiv 3 \pmod 4$ that follows. In all other three cases we only formulate the results, their proofs can be done analogously. We illustrate each case on an example.

Theorem 3.4. Let G be a graph of size $m = 2n$ for $n \equiv 3 \pmod 4$. Then the following are equivalent:

(1) G is the sunlet graph SG_n.

(2) There is a graceful labelling of G with a cascade graph chessboard of type 3.

(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} n - 2\left\lfloor \frac{i+1}{4} \right\rfloor, & \text{if } i \leq n-1 \wedge i \equiv 0,1,2 \pmod 4, \\ n + 1 - \left\lceil \frac{i}{2} \right\rceil, & \text{if } i \leq n-1 \wedge i \equiv 3 \pmod 4, \\ 0, & \text{if } i = n, \\ 2\left\lceil \frac{i-2}{4} \right\rceil - (i \mod n), & \text{if } i \geq n+1 \wedge i \equiv 0,1,3 \pmod 4, \\ n - \frac{i}{2}, & \text{if } i \geq n+1 \wedge i \equiv 2 \pmod 4. \end{cases} \tag{LSSG3}$$

(4) There exists a labelling sequence L of G with labelling relation $A(L)$ of the form

$$\{[n-2\left\lfloor\frac{i+1}{4}\right\rfloor, n-2\left\lfloor\frac{i+1}{4}\right\rfloor+i] \mid i \leq n-1 \wedge i \equiv 0,1,2 \pmod 4\} \cup$$
$$\{[n+1-\left\lceil\frac{i}{2}\right\rceil, n+1-\left\lceil\frac{i}{2}\right\rceil+i] \mid i \leq n-1 \wedge i \equiv 3 \pmod 4\} \cup$$
$$\{[0,n]\} \cup \{[2\left\lceil\frac{i-2}{4}\right\rceil - i \pmod n, 2\left\lceil\frac{i-2}{4}\right\rceil - i \pmod n + i] \mid$$
$$i \geq n+1 \wedge i \equiv 0,1,3 \pmod 4\} \cup$$
$$\{n-\frac{i}{2}, n+\frac{i}{2}] \mid i \geq n+1 \wedge i \equiv 2 \pmod 4\}.$$

Proof. (1) ⇒ (2) Let G be the sunlet fan graph SG_n of size $m = 2n$ for $n \equiv 3 \pmod 4$. We label graph according to Frucht [1] and then we construct to G a simple graph chessboard of size m as a cascade graph chessboard of type 3. (We have seen an illustration of such cascade graph chessboard of type 3 for $n = 11$ in Figure 8 and for $n = 7$ in Figure 9.) The first cascade road of our constructed graph chessboard represents the part of the sunlet graph from the edge $\{m, 0\}$ anticlockwise to the edge $\{m - \left\lfloor\frac{n}{2}\right\rfloor, \left\lfloor\frac{n}{2}\right\rfloor\}$, where the dots of the road in which the direction is changing represent the edges in the cycle and the dots between these dots represent the pendant edges. In the same way this pattern is valid for the second cascade road, which represents the part of the sunlet graph from the edge $\{m - \left\lfloor\frac{n}{2}\right\rfloor, \left\lfloor\frac{n}{2}\right\rfloor + 2\}$ anticlockwise to the edge $\{n+1, n\}$. The isolated dot, which represents the edge $\{n, 0\}$ in the cycle, connects these two cascade roads.

Hence we have been able to represent the edges of the given sunlet fan graph SG_n as the dots in a simple graph chessboard in the way that every diagonal of the chessboard contains exactly one dot. This determines a graceful labelling of the graph. We have showed (2).

(2) ⇒ (3) Assume we have a graceful labelling of G with a cascade graph chessboard of type 3. The first road in the chessboard starting on the first diagonal is represented in the corresponding labelling sequence by the members described in the first two cases of the formula (LSSG3). More specifically, the vertical columns of the dots of this road correspond to the integers j_i from the labelling sequence, which have the form $n-2\left\lfloor\frac{i+1}{4}\right\rfloor$ and the single dots connecting these vertical columns correspond to the integers j_i from the labelling sequence, which have the form $n+1-\left\lceil\frac{i}{2}\right\rceil$. The isolated dot corresponds to number 0 for $i = n$ in the labelling sequence. The second road then corresponds to the members described in the last two cases of the formula (LSSG3). More precisely, the vertical columns of the dots of this road correspond to the integers j_i from the labelling sequence, which have the form $2\left\lceil\frac{i-2}{4}\right\rceil - (i \mod n)$ and the single dots connecting these vertical columns correspond to the integers j_i from the labelling sequence, which have the form $n - \frac{i}{2}$.

Therefore the labelling sequence corresponding to the cascade graph chessboard of type 3 of G is given by the formula (LSSG3).

(3) ⇒ (4) Assume we have a labelling sequence of G which satisfies the formula (LSSG3). It can be seen that when we make the corresponding labelling relation $A(L)$, it has the form described in (4). Indeed, the non-negative integers j_i from the labelling sequence are gradually paired in the labelling relation $A(L)$ with the sums of the integers j_i with the numbers $1, 2, 3, \ldots, 2n$. It follows that the integers j_i from the labelling sequence of the form $n-2\left\lfloor\frac{i+1}{4}\right\rfloor$ correspond to the pairs $\{[n-2\left\lfloor\frac{i+1}{4}\right\rfloor, n-2\left\lfloor\frac{i+1}{4}\right\rfloor+i] \mid i \leq n-1 \wedge i \equiv 0,1,2 \pmod 4\}$ in $A(L)$. The next j_i from the labelling sequence, which have the form $n+1-\left\lceil\frac{i}{2}\right\rceil$

correspond to the pairs $\{[n+1-\lceil \frac{i}{2} \rceil, n+1-\lceil \frac{i}{2} \rceil + i] \mid i \leq n-1 \wedge i \equiv 3 \pmod 4\}$. The number 0 from the labelling sequence corresponds to the pair $\{[0,n]\}$. The integers j_i from the labelling sequence, which have the form $2\lceil \frac{i-2}{4} \rceil - (i \mod n)$ correspond to the pairs $\{[2\lceil \frac{i-2}{4} \rceil - (i \mod n), 2\lceil \frac{i-2}{4} \rceil - (i \mod n) + i] \mid i \geq n+1 \wedge i \equiv 0,1,3 \pmod 4\}$. Finally, the integers j_i from the labelling sequence, which have the form $n - \frac{i}{2}$ correspond to the pairs $\{[n-\frac{i}{2}, n-\frac{i}{2}+i] \mid i \geq n+1 \wedge i \equiv 2 \pmod 4\}$ in $A(L)$.
(4) $\Rightarrow$ (1) Let L be a labelling sequence of G with the labelling relation $A(L)$ of the form as in (4). We know that the edges of G are the pairs in $A(L)$. We explain this implication with a help of Example 3.2, where we can see in Figure 8 the table of the labelling relation satisfying our statement in (4). We shall look on the pairs in the labelling relation $A(L)$ as the edges of the graph G. In the first part of $A(L)$ $(i = 1, 2, ..., n-1)$ we can note that the pairs on the even positions $(i = 2, 4, 6, ..., n-1)$ as edges of G form a path, and the pairs on the odd positions $(i = 1, 3, 5, ..., n-2)$ correspond to the pendant edges of this path (see the graph diagram in Fig. 8). It holds the other way round for the second part of the relation $A(L)$. So the pairs on the odd positions as edges of G form a path, while the pairs on the even positions correspond to the pendant edges of this path. The edge $\{0,n\}$ connects the edges $\{n, n+1\}$ and $\{0,m\}$, so it connects these two parts of the graph G, each of which is formed by a path with the pendant edges. We can also note that these two parts of the graph G have a common vertex because the expression $n - 2\lfloor \frac{i+1}{4} \rfloor + i$ for $i = n-1$ is equivalent to the expression $n - \lceil \frac{i}{2} \rceil + i$ for $i = n+1$. (In Figure 8 it is the number 17.) It means that the paths of both parts of G are connected, so they form a cycle of length n in G. In this cycle every vertex has a pendant edge. So we get that the graph G is a sunlet graph SG_n. □

We now present our result and its illustration by an example in the case $n \equiv 0 \pmod 4$.

Example 3.5. The sequence $(8, 7, 7, 7, 6, 5, 5, 5, 0, 3, 2, 2, 2, 1, 0, 0)$ is a labelling sequence of the sunlet fan graph SG_8 of size 16. Corresponding graph diagram, labelling relation and graph chessboard are in Figure 10. We can note that this labelling sequence has an analogous structure as the labelling sequence for the graph SG_{11} in Figure 8 and so it consists of two decreasing cascades of numbers and the number 0 between them. The structure of the labelling sequence is different only in small detail that the first cascade of numbers in this case starts by single n followed by the triplet of numbers $n-1$, and it does not start by two occurrences of n as in the previous case $n \equiv 3 \pmod 4$.

Definition 3.6. The graph chessboard with pattern as in Figure 10 will be called a **cascade graph chessboard of type 0**.

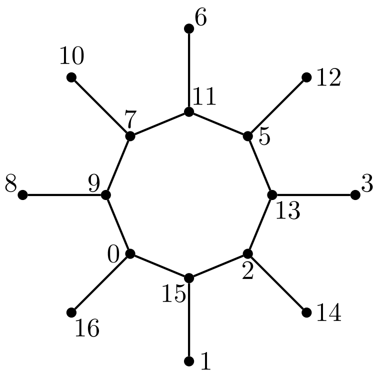
Theorem 3.7. Let G be a graph of size $m = 2n$ for $n \equiv 0 \pmod 4$. Then the following are equivalent:

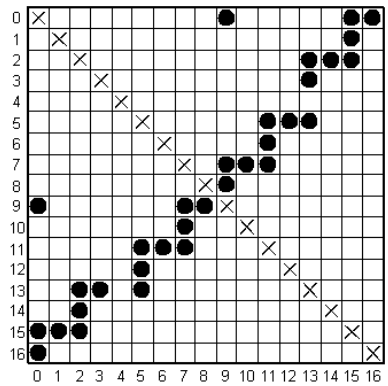
(1) G is the sunlet graph SG_n.

(2) There is a graceful labelling of G with a cascade graph chessboard of type 0.

(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} n+1-2\lfloor \frac{i+3}{4} \rfloor, & \text{if } i \leq n \wedge i \equiv 0,2,3 \pmod 4, \\ n - \lfloor \frac{i}{2} \rfloor, & \text{if } i \leq n \wedge i \equiv 1 \pmod 4, \\ 0, & \text{if } i = n+1, \\ 2\lceil \frac{i}{4} \rceil - (i \bmod n) - 1, & \text{if } i \geq n+2 \wedge i \equiv 1,2,3 \pmod 4, \\ n - \frac{i}{2}, & \text{if } i \geq n+2 \wedge i \equiv 0 \pmod 4. \end{cases} \qquad (LSSG0)$$





1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
8	7	7	7	6	5	5	5	0	3	2	2	2	1	0	0
9	9	10	11	11	11	12	13	9	13	13	14	15	15	15	16

Figure 10. The representations of the sunlet graph SG_8

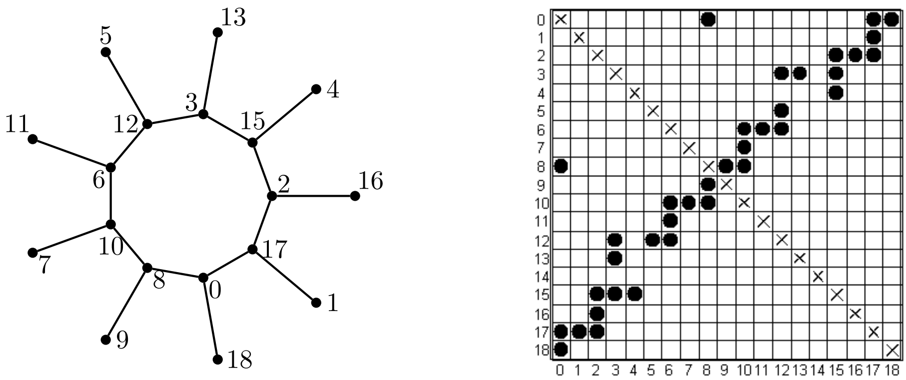
(4) There exists a labelling sequence L of G with labelling relation $A(L)$ of the form

$$\{[n+1-2\left\lfloor\frac{i+3}{4}\right\rfloor, n+1-2\left\lfloor\frac{i+3}{4}\right\rfloor+i] \mid i \le n \wedge i \equiv 0,2,3 \pmod 4\} \cup$$
$$\{[n-\left\lceil\frac{i}{2}\right\rceil, n-\left\lceil\frac{i}{2}\right\rceil+i] \mid i \le n \wedge i \equiv 1 \pmod 4\} \cup \{[0,n+1]\} \cup$$
$$\{[2\left\lceil\frac{i}{4}\right\rceil - i \pmod n - 1, 2\left\lceil\frac{i}{4}\right\rceil - i \pmod n - 1 + i] \mid$$
$$i \ge n+2 \wedge i \equiv 1,2,3 \pmod 4\} \cup$$
$$\{[n-\frac{i}{2}, n+\frac{i}{2}] \mid i \ge n+2 \wedge i \equiv 0 \pmod 4\}.$$

We now present our result and its illustration by an example in the case $n \equiv 1 \pmod 4$.

Example 3.8. The sequence $(8,8,7,6,6,6,5,0,3,3,4,3,2,2,2,1,0,0)$ is a labelling sequence of the sunlet fan graph SG_9 of size 18. Corresponding graph diagram, labelling relation and graph chessboard are in Figure 11. Unlike the first two cases, there is not only number 0 between two decreasing cascades of numbers, but there is also pair of integers, which are not part of two cascades. It is better seen in the chessboard. Therefore one more formula describing these two numbers is added in the third condition of Theorem 3.10. The structure of the labelling sequence is different compared to the previous case in small detail again. The first cascade of numbers starts by pair of integers $n-1$ and ends by single integer $\lceil \frac{n}{2} \rceil$. The second cascade of numbers has the same form for all four cases.

Definition 3.9. The graph chessboard with pattern as in Figure 11 will be called a **cascade graph chessboard of type 1**.



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
8	8	7	6	6	6	5	0	3	3	4	3	2	2	2	1	0	0
9	10	10	10	11	12	12	8	12	13	15	15	15	16	17	17	17	18

Figure 11. The representations of the sunlet graph SG_9

Theorem 3.10. Let G be a graph of size $m = 2n$ for $n \equiv 1 \pmod 4$. Then the following are equivalent:

(1) G is the sunlet fan graph SG_n.

(2) There is a graceful labelling of G with a cascade graph chessboard of type 1.

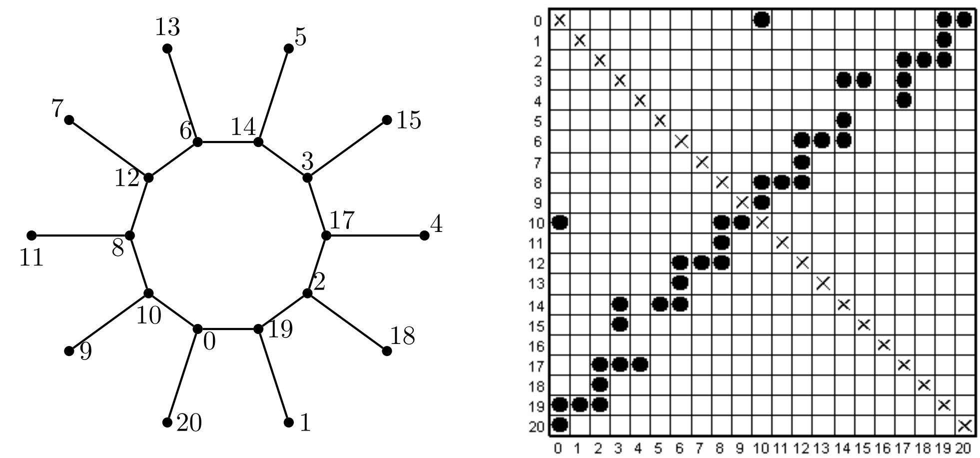
(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} n-1-2\left\lfloor \frac{i+1}{4} \right\rfloor, & \text{if } i \le n-2 \land i \equiv 0,1,2 \pmod 4, \\ n - \left\lceil \frac{i}{2} \right\rceil, & \text{if } i \le n-2 \land i \equiv 3 \pmod 4, \\ 0, & \text{if } i = n-1, \\ n-2-2\left\lfloor \frac{i+1}{4} \right\rfloor, & \text{if } i = n, n+1, \\ 2\left\lceil \frac{i-2}{4} \right\rceil - (i \bmod n), & \text{if } i \ge n+2 \land i \equiv 0,1,3 \pmod 4, \\ n - \frac{i}{2}, & \text{if } i \ge n+2 \land i \equiv 2 \pmod 4. \end{cases} \tag{LSSG1}$$

(4) There exists a labelling sequence L of G with labelling relation $A(L)$ of the form

$$\{[n-1-2\left\lfloor \frac{i+1}{4} \right\rfloor, n-1-2\left\lfloor \frac{i+1}{4} \right\rfloor + i] \mid i \le n-2 \land i \equiv 0,1,2 \pmod 4\} \cup$$
$$\{[n - \left\lceil \frac{i}{2} \right\rceil, n - \left\lceil \frac{i}{2} \right\rceil + i] \mid i \le n-2 \land i \equiv 3 \pmod 4\} \cup \{[0, n+1]\} \cup$$
$$\{[n-2-2\left\lfloor \frac{i+1}{4} \right\rfloor, n-2-2\left\lfloor \frac{i+1}{4} \right\rfloor + i] \mid i = n, n+1\} \cup$$
$$\{[2\left\lceil \frac{i-2}{4} \right\rceil - i \pmod n, 2\left\lceil \frac{i-2}{4} \right\rceil - i \pmod n + i] \mid$$
$$i \ge n+2 \land i \equiv 0,1,3 \pmod 4\} \cup$$
$$\{n - \frac{i}{2}, n + \frac{i}{2}] \mid i \ge n+2 \land i \equiv 2 \pmod 4\}.$$

Example 3.11. The sequence $(9, 8, 8, 8, 7, 6, 6, 6, 5, 0, 3, 3, 4, 3, 2, 2, 2, 1, 0, 0)$ is a labelling sequence of the sunlet fan graph SG_{10} of size 20. Corresponding graph diagram, labelling relation and graph chessboard are in Figure 12. A form of the labelling sequence for this case is very similar to the form of the labelling sequence in the previous case. They are different only in the first cascade of numbers. Here it starts by a single integer followed by a triplet of the same integers.



1	2	3	4	5	6	7	8	9	10
9	8	8	8	7	6	6	6	5	0
10	10	11	12	12	12	13	14	14	10

11	12	13	14	15	16	17	18	19	20
3	3	4	3	2	2	2	1	0	0
14	15	17	17	17	18	19	19	19	20

Figure 12. The representations of the sunlet graph SG_{10}

Definition 3.12. The graph chessboard with pattern as in Figure 12 will be called a **cascade graph chessboard of type 2**.

Theorem 3.13. Let G be a graph of size $m = 2n$ for $n \equiv 2 \pmod 4$. Then the following are equivalent:

(1) G is the sunlet fan graph SG_n.

(2) There is a graceful labelling of G with a cascade graph chessboard of type 2.

(3) There exists a labelling sequence $L = (j_1, j_2, ..., j_m)$ of G such that

$$j_i = \begin{cases} n - 2\left\lfloor \frac{i+3}{4} \right\rfloor, & \text{if } i \le n-1 \wedge i \equiv 0,2,3 \pmod 4, \\ n - \frac{i+1}{2}, & \text{if } i \le n-1 \wedge i \equiv 1 \pmod 4, \\ 0, & \text{if } i = n, \\ n - 1 - 2\left\lfloor \frac{i+3}{4} \right\rfloor, & \text{if } i = n+1, n+2, \\ 2\left\lceil \frac{i}{4} \right\rceil - (i \mod n) - 1, & \text{if } i \ge n+3 \wedge i \equiv 1,2,3 \pmod 4, \\ n - \frac{i}{2}, & \text{if } i \ge n+3 \wedge i \equiv 0 \pmod 4. \end{cases} \qquad (LSSG2)$$

(4) There exists a labelling sequence L of G with labelling relation $A(L)$ of the form

$$\{[n-2\left\lfloor\frac{i+3}{4}\right\rfloor, n-2\left\lfloor\frac{i+3}{4}\right\rfloor+i] \mid i \le n-1 \wedge i \equiv 0,2,3 \pmod 4\}\cup$$
$$\{[n-\frac{i+1}{2}, n-\frac{i+1}{2}+i] \mid i \le n-1 \wedge i \equiv 1 \pmod 4\} \cup \{[0,n]\}\cup$$
$$\{[n-1-2\left\lfloor\frac{i+3}{4}\right\rfloor, n-1-2\left\lfloor\frac{i+3}{4}\right\rfloor+i] \mid i = n+1, n+2\}\cup$$
$$\{[2\left\lceil\frac{i}{4}\right\rceil-(i \mod n)-1, 2\left\lceil\frac{i}{4}\right\rceil-(i \mod n)-1+i] \mid$$
$$i \ge n+3 \wedge i \equiv 1,2,3 \pmod 4\}\cup$$
$$\{n-\frac{i}{2}, n+\frac{i}{2}] \mid i \ge n+3 \wedge i \equiv 0 \pmod 4\}.$$

4 Wheels

In this section we shall show our method in class of graphs, which are called *wheels.*

Definition 4.1. A **wheel graph** (or shortly a **wheel**) is a graph obtained by connecting a single vertex to all vertices of a cycle. We will denote the wheel consisting of n vertices and $2(n-1)$ edges by W_n.

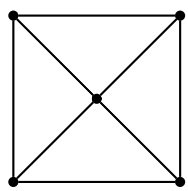


Figure 13. The wheel graph W_5

Some authors use the symbol W_n to denote the wheel with $n+1$ vertices, but we will use the above defined notation in this text. In Figure 13 we can see the wheel graph W_5. We will distinguish two cases of wheels for even n and odd n. For both cases we describe graceful labelling via example and theorem. We use graceful labelling from Frucht [1] from 1979 for our methods. The proof of the theorem will be presented only for even n, because for odd n it can be done analogously.

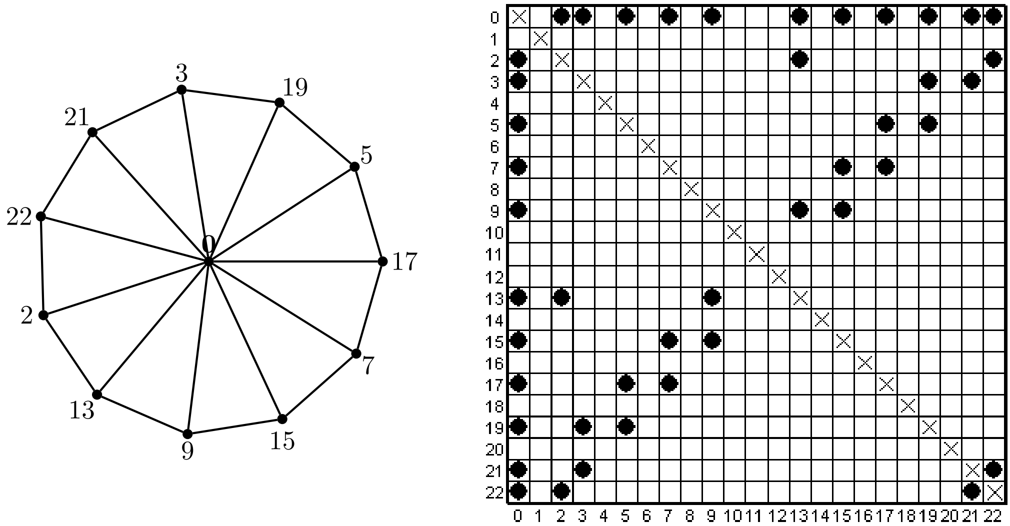
Now in an example we shall show and describe the labelling sequence for the wheel SG_{12}, which corresponds to graceful labelling of this graph. We explain here the pattern of graceful labelling for wheels with even n.

Example 4.2. The sequence $(21, 0, 0, 9, 0, 9, 0, 7, 0, 7, 2, 5, 0, 5, 0, 3, 0, 3, 0, 2, 0, 0)$ is a labelling sequence of the wheel graph SG_n of size m where we have $n = 12$ and $m = 22$.

Corresponding graph diagram, labelling relation and graph chessboard are in Figure 14.

We can divide this labelling sequence into three parts: the first triplet of numbers $(m-1, 0, 0)$, "regular sequence with exception", and the last triplet of numbers $(2, 0, 0)$. The regular sequence with exception is the following sequence of numbers:

$$(n-3, 0, n-3, 0, n-1, 0, n-1, 0, ..., 3, 0, 3, 0),$$



1	2	3	4	5	6	7	8	9	10	11
21	0	0	9	0	9	0	7	0	7	2
22	2	3	13	5	15	7	15	9	17	13

12	13	14	15	16	17	18	19	20	21	22
5	0	5	0	3	0	3	0	2	0	0
17	13	19	15	19	17	21	21	22	21	22

Figure 14. The representations of the wheel graph W_{12}

in our case

$$(9, 0, 9, 0, 7, 0, 7, 2, 5, 0, 5, 0, 3, 0, 3, 0),$$

where the exception is on the position $i = n - 1$ of the labelling sequence (the position $i = n-4$ of the "regular sequence with exception"): here the number 0 would be expected by the pattern of this sequence, but we can see the number 2 instead 0 there.

As we see in Figure 14 in the simple chessboard below its main diagonal, the dots in the first column of the chessboard represent the $(n - 1)$ zeros in the labelling sequence. Numbers $(m - 1)$ and 2 from the first and the last triplet of numbers are represented by the dots in m-th row. Other dots in the chessboard are the dots "creating steps" which start on the position with coordinates $[m - 1, 3]$. They represent the regular sequence. The exception - number 2 - is represented by the dot, which is out of steps on the position with coordinates $[n + 1, 2]$.

In Figure 14 we can also note a pattern of the graceful labelling in the wheel graph W_{12}. The central vertex connected to all vertices of the cycle is always labelled by 0. Then let number 21 be on the first position in the cycle, number 3 on the second and we proceed clockwise using the following pattern: on the positions $1, 3, 5, 7, 9$ are numbers $21, 19, 17, 15, 13$ while on the positions $2, 4, 6, 8$ are numbers $3, 5, 7, 9$. The remaining vertices are labelled by m (in this case 22) and by 2.

Definition 4.3. The labelling sequence described in Example 4.2 will be called an **even wheel labelling sequence** and the graph chessboard described in Example 4.2 will be called an **even wheel graph chessboard**.

Theorem 4.4. Let G be a graph of size $m = 2(n-1)$ for even $n \geq 4$. Then the following are equivalent:

(1) G is the wheel graph W_n.

(2) There is a graceful labelling of G with an even wheel graph chessboard.

(3) There exists an even wheel labelling sequence L of G with corresponding labelling relation $A(L)$.

Proof. (1) ⇒ (2) Let G be the wheel graph W_n. We shall label our graph according to labelling from Frucht [1], which is graceful and we construct an even wheel graph chessboard (see Figure 14). Let the central vertex of our graph with degree $n-1$ be labelled by 0. The vertices of the cycle in our graph will be labelled in the following way. We start to label the graph with an arbitrary vertex of the cycle because the positions of all vertices of the cycle are equal. So we label any vertex of the cycle by $(m-1)$. We will say that the vertex with labelling $(m-1)$ is on the first position in the cycle and other vertices in the clockwise direction from this vertex have positions $2, 3, 4, \dots$ up to $(n-1)$. Then the vertices on the positions $3, 5, 7, \dots, n-3$ are labelled gradually by numbers $m-3, m-5, m-7, \dots, m-(n-3)$. The vertices on the positions $2, 4, 6, \dots, n-4$ are labelled gradually by numbers $3, 5, 7, \dots, n-3$. Two unlabelled vertices remain. The vertex, which is adjacent to the vertex with labelling $m-1$ will be labelled by m and the vertex, which is adjacent to the vertex with labelling $n-3$ will be labelled by 2. Now our labelling is done and we shall show that the corresponding graph chessboard of this labelling is an even wheel graph chessboard.

There are $(n-1)$ dots in the first column of the chessboard, which represent all edges connected to the central vertex. The dots creating the increasing steps in the chessboard together with the three dots out of the steps with coordinates $[n+1, 2], [m, 2], [m, m-1]$ represent the cycle in the graph. Hence we get an even wheel graph chessboard (see Figure 14). Since each diagonal of the chessboard contains exactly one dot, this confirms that the applied labelling is graceful.

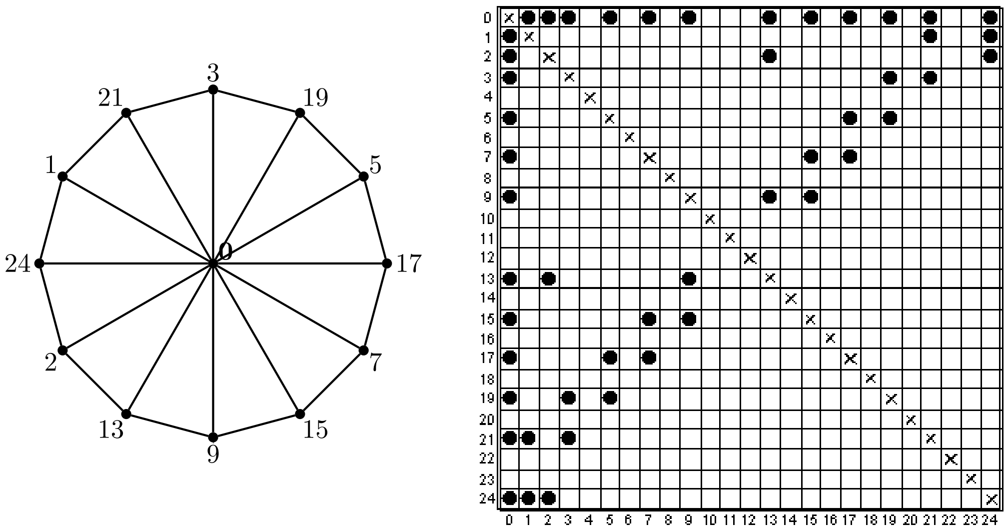
(2) ⇒ (3) Assume we have a graceful labelling of the graph G with an even wheel graph chessboard. The dots which are on the positions with coordinates $[m, m-1], [2, 0], [3, 0]$ are represented in the corresponding labelling sequence by numbers $(m-1, 0, 0)$, what is the first triplet in the even wheel labelling sequence. The dots which create the increasing steps and start on the position with coordinates $[m-1, 3]$, together with the dots in the first column of the chessboard excepting the first two and the last two dots of this column, and the dot with coordinates $[n+1, 2]$ are represented in the labelling sequence by the sequence $(n-3, 0, n-3, 0, n-1, 0, n-1, 0, \dots, 3, 0, 3, 0)$, what is exactly the regular sequence with exception (see the description of the labelling sequence in Example 4.2). The last dots with coordinates $[m, 2], [m-1, 0], [m, 0]$ are represented in the labelling sequence by the numbers $(2, 0, 0)$, what represents the last triplet in the even wheel labelling sequence.

(3) ⇒ (1) Let L be an even wheel labelling sequence with corresponding labelling relation $A(L)$. We shall verify that corresponding graph G to this labelling sequence is the wheel graph W_n. The wheel graph connects in a certain natural way a star S_{n-1} and a cycle with $(n-1)$ edges. When we look at our labelling sequence L (for example in Fig. 14), we can find there exactly $(n-1)$ zeros. It means that the pairs containing zero in the corresponding labelling relation represent the star S_{n-1}. Other pairs represent the cycle. Indeed, consider the pair $(m-1, m)$ in $A(L)$. In the relation $A(L)$ there are also the

pairs $(m-1,3)$, $(3,m-3)$, etc., and this way we get back to the pair $(m-1,m)$. So we get in G the cycle with $(n-1)$ edges. Every vertex in the cycle is also connected with the vertex labelled by zero. Hence we get the wheel graph W_n. □

Now we present an example and the theorem for the wheels with odd n.

Example 4.5. The sequence $(0,0,0,9,0,9,0,7,0,7,2,5,0,5,0,3,0,3,0,1,0,2,1,0)$ is the labelling sequence of the wheel graph W_n of size m where we have odd $n=13$ and $m=24$. Corresponding graph diagram, labelling relation and graph chessboard are in Figure 15. We can divide this labelling sequence also into three parts: the first triplet



1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	9	0	9	0	7	0	7	2	5
1	2	3	13	5	15	7	15	9	17	13	17

13	14	15	16	17	18	19	20	21	22	23	24
0	5	0	3	0	3	0	1	0	2	1	0
13	19	15	19	17	21	19	21	21	24	24	24

Figure 15. The representations of the wheel graph W_{13}

of zeros, regular sequence with exception and the last triplet of numbers $(2,1,0)$. The regular sequence with exception is the following sequence of numbers: $(m-3,0,m-3,0,m-1,0,m-1,0,...,3,0,3,0,1,0)$. So a small difference in the regular sequence with exception, compared to the previous case, is in adding the numbers 1 and 0 to the end of this sequence.

As we see in Figure 15 in the simple chessboard below its main diagonal, the zeros in the labelling sequence are represented by the dots in the first column of the chessboard. Numbers 1 and 2 from the last triplet of numbers are represented by the dots in the $(m+1)$-th row. Other dots in the chessboard are the dots in the "increasing steps", which start on the position with coordinates $(m-1,3)$. They represent the regular sequence. The exception - number 2 - is represented by the dot, which is out of the steps on the position $(n+1, 2)$.

In Figure 15 we can also note the pattern of graceful labelling in the wheel graph W_{13}. The central vertex connected to all vertices of the cycle is always labelled by 0. Then let number 1 is on the first position in the cycle, number 21 on the second and we continue in the clockwise direction: so on the positions $1, 3, 5, 7, 9$ are numbers $1, 3, 5, 7, 9$ and on the positions $2, 4, 6, 8, 10$ are numbers $21, 19, 17, 15, 13$. Remaining vertices are labelled by m (in this case 24) and by 2.

Definition 4.6. The labelling sequence described in Example 4.5 will be called an **odd wheel labelling sequence** and the graph chessboard described in Example 4.5 will be called an **odd wheel graph chessboard**.

Theorem 4.7. Let G be a graph of size $m = 2(n-1)$ for odd $n \geq 5$. Then the following are equivalent:

(1) G is the wheel graph W_n.

(2) There is a graceful labelling of G with an odd wheel graph chessboard.

(3) There exists an odd labelling sequence L of G with corresponding labelling relation $A(L)$.

Proof. Analogous as the proof of Theorem 4.4. □

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