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A k-dimensional systems of fractional neutral functional differential equations involving $\psi-$Caputo fractional derivative

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Abstract

This paper is devoted to the study of the initial value problem for a class of $k-$dimensional systems of fractional neutral functional differential equations involving $\psi-$Caputo fractional derivative with respect to another function. Existence and uniqueness results for the problem are established by means of some standard fixed point theorems. Finally, we give an example to demonstrate our results.

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1 Introduction

Fractional calculus is a branch of mathematics which deals with non-integer order integrals and derivatives. Though the fractional calculus developed as a pure mathematical idea, now it has tremendous applications. Viscoelasticity, electro magnetism, electrical circuits, sound propagation, lateral and longitudinal control, fluid mechanics, edge detection, cardiac tissue electrode interface, earth system dynamics are some of them [1, 2, 3, 4, 5, 6]. There are numerous definitions for fractional derivatives and integrals. Nowadays many studies are being done in generalised fractional operators [1, 2], [7, 8, 9, 10, 11]. Recently, Almeida [8] used the idea of the fractional derivative in the Caputo sense to propose a new generalized fractional differential operator called $\psi-$Caputo fractional derivative with respect to another function ψ. There are many studies on the existence and uniqueness of different fractional differential equations involving $\psi-$Caputo fractional differential and integral operators [8, 18, 19]. Neutral differential equations have importance in many areas of applied Mathematics [12, 13, 14, 15, 16, 17].

The aim of this paper is to investigate the existence and uniqueness of solutions of Initial Value Problem for a class of $k-$dimensional systems of fractional neutral functional differential equation with bounded delay involving the Caputo-type fractional derivative of a function x with respect to another function ψ.

$$\begin{cases} {}^C D_{t_0}^{\alpha_1,\psi}(x_1(t) - g_1(t, x_t)) & = f_1(t, x_t), \\ {}^C D_{t_0}^{\alpha_2,\psi}(x_2(t) - g_2(t, x_t)) & = f_2(t, x_t), \\ \vdots \\ {}^C D_{t_0}^{\alpha_k,\psi}(x_k(t) - g_k(t, x_t)) & = f_k(t, x_t), \end{cases} \tag{1.1}$$

$$x_{1_{t_0}} = \phi_1, \quad x_{2_{t_0}} = \phi_2, \quad \cdots, \quad x_{k_{t_0}} = \phi_k,$$

where $a, r \in \mathbb{R}^+$, $t_0 \geq 0$ and $t \in (t_0, \infty)$, $0 < \alpha_i < 1$, for $i = 1, 2, \cdots, k$. ${}^C D_{t_0}^{\alpha_i,\psi}$ is the Caputo-type fractional derivative of a function x_i with respect to another function ψ. $f_i, g_i : ([t_0, \infty) \times C([-r, 0], \mathbb{R}^n) \times C([-r, 0], \mathbb{R}^n) \times \cdots \times C([-r, 0], \mathbb{R}^n)) \to \mathbb{R}^n$, $i = 1, 2, \cdots, k$ are $\mathbb{R}^n-$valued functions satisfying certain assumptions, which will be mentioned later. Consider $x_{it} = (x_{1_t}, x_{2_t}, \cdots, x_{k_t}) \in \mathbb{R}^n$ and $\phi_i \in C([-r, 0], \mathbb{R}^n)$ for $i = 1, 2, \cdots, k$. If $x_i \in C([t_0 - r, t_0 + a], \mathbb{R}^n)$ define x_{it} by $x_{it}(\theta) = x_i(t + \theta)$ for $\theta \in [-r, 0]$, for any $t \in [t_0, t_0+a]$. Let $\psi \in C^n[t_0, \infty)$ be a continuous increasing function such that $\psi'(x) \neq 0$, $\forall\, x \in [t_0, \infty)$.

In this paper, the first section deals with the introduction about the $\psi-$Caputo fractional differential equations and the problem is also given. In the second section we present essential definitions and results and in the third section we prove the existence and uniqueness results of the IVP(1.1) by means of Krasnoselskii's and Banach's fixed point theorems. In the last section, we give an example to demonstrate our results.

2 Preliminaries

Here we deal with fractional derivatives and fractional integrals with respect to another function.

Definition 1. [1] Let $\alpha > 0, I = [a, b]$ be a finite or infinite interval, f an integrable function defined on I and $\psi \in C^n(I)$ an increasing function such that $\psi'(t) \neq 0$, $\forall\, t \in I$. Fractional integrals and fractional derivatives of a function f with respect to another function ψ are defined as

$$I_a^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \left[\psi(t) - \psi(s)\right]^{\alpha-1} f(s) ds$$

and

$$\begin{aligned} D_a^{\alpha,\psi} f(t) &= \left[\frac{1}{\psi'(t)} \frac{d}{dt}\right]^n I_{a+}^{n-\alpha,\psi} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left[\frac{1}{\psi'(t)} \frac{d}{dt}\right]^n \int_a^t \psi'(s) \left[\psi(t) - \psi(s)\right]^{n-\alpha-1} f(s) ds \end{aligned}$$

where $n = [\alpha] + 1$.

For different choices of the function ψ, we get the Riemann-Liouville, the Hadamard and the Erdélyi-Kober fractional derivatives and fractional integrals, etc.

Definition 2. [8] Let $\alpha > 0, n \in \mathbb{N}, I$ be the interval $-\infty \leq a < b \leq \infty$, $f, \psi \in C^{(n)}(I)$ be two functions such that ψ is increasing and $\psi'(t) \neq 0 \quad \forall x \in I$. Then the $\psi-$Caputo fractional derivative of f of order α is given by

$$^{C}D_a^{\alpha,\psi} f(t) = I_a^{n-\alpha,\psi} \left[\frac{1}{\psi'(t)} \frac{d}{dt} \right]^n f(t)$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$.

To simplify the notation, we are using the abbreviated symbol

$$f_\psi^{[n]} f(t) = \left[\frac{1}{\psi'(t)} \frac{d}{dt} \right]^n f(t)$$

From the definition it is clear that, given $\alpha = m \in \mathbb{N}$, $\quad {}^{C}D_a^{\alpha,\psi} f(t) = f_\psi^{[m]}(t)$ and if $\alpha \notin \mathbb{N}$, then

$$^{C}D_a^{\alpha,\psi} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s) \left[\psi(t) - \psi(s) \right]^{n-\alpha-1} f_\psi^{[n]}(s) ds$$

In particular, if $0 < \alpha < 1$

$$^{C}D_a^{\alpha,\psi} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left[\psi(t) - \psi(s) \right]^{-\alpha} f'(s) ds$$

Lemma 3. *[8, 18, 19] Given a function $f \in C^n[a, b]$ and order $\alpha > 0$, we have for $n = [\alpha] + 1$:*

$$^{C}D_a^{\alpha,\psi} f(t) = D_a^{\alpha,\psi} \left[f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} \left[\psi(t) - \psi(a) \right]^k f_\psi^{[k]}(a) \right]$$

$$I_a^{\alpha,\psi}\, {}^{C}D_a^{\alpha,\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} \left[\psi(x) - \psi(a) \right]^k$$

Also $I_a^{\alpha,\psi}\, {}^{C}D_a^{\alpha,\psi} f(t) = f(t) - f(a)$, if $0 < \alpha < 1$.

Lemma 4. *(Krasnoselskii's Fixed Point Theorem)[20]*
Let X be a Banach space, let E be a bounded closed convex subset of X and let S, U be maps of E into X such that $Sx + Uy \in E$ for every pair $x, y \in E$. If S is a contraction and U is completely continuous, then the equation $Sx + Ux = x$ has a solution on E.

Lemma 5. *(Banach's Fixed Point Theorem)*
Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed point x^* in X.

Let $I \subset \mathbb{R}$ be any interval and $X = C(I, \mathbb{R}^n)$ with the norm $||x|| = \sup_{t \in I} |x(t)|$, where $|\cdot|$ as a suitable complete norm on $\mathbb{R}^n$. Let $(X^k = \underbrace{X \times X \times \cdots \times X}_{k}, ||\cdot||_*)$, where $||(x_1, x_2, \cdots, x_k)||_* = max\{||x_1||, ||x_2||, \cdots, ||x_k||\}$ is the norm on the corresponding product Banach space X^k.

3 Main Results

Consider the Initial Value Problem (1.1). Let δ and $\gamma \in \mathbb{R}$ be positive constants, $I_0 = [t_0, t_0 + \delta]$ and

$$A(\delta, \gamma) = \{(x_1, x_2, \cdots, x_k) : x_{i_{t_0}} = \phi_i \sup_{t_0 \le t \le t_0+\delta} |x_i(t) - \phi_i(0)| \le \gamma \quad \forall i = 1, 2, \cdots, k\}, \tag{3.1}$$

where $x_i \in C([t_0 - r, t_0 + \delta], \mathbb{R}^n)$. Before starting and proving the main results, we assume the following hypotheses.

(H1) $f_i(t, \phi_1, \phi_2, \cdots, \phi_k)$ is measurable with respect to t on I_0, $\forall\, i = 1, 2, \cdots, k$.

(H2) $f_i(t, \phi_1, \phi_2, \cdots, \phi_k)$ is continuous with respect to ϕ_j on $C([-r, 0], \mathbb{R}^n)$, $\forall\, i, j = 1, 2, \cdots, k$.

(H3) There exist $\alpha_{i_1} \in (0, \alpha_i)$ and a real valued function $m_i(t) \in L^{\frac{1}{\alpha_{i_1}}}(I_0)$, such that for any $(x_1, x_2, \cdots, x_k) \in A(\delta, \gamma)$, $\forall\, i = 1, 2, \cdots, k$

$$|f_i(t, x_t)| \le m_i(t), \quad t \in I_0, \tag{3.2}$$

(H4) For any $(x_1, x_2, \cdots, x_k) \in A(\delta, \gamma)$, $g_i(t, x_t) = g_{i_1}(t, x_t) + g_{i_2}(t, x_t)$.

(H5) g_{i_1} is continuous and $|g_{i_1}(t, x_t) - g_{i_1}(t, y_t)| \le l_i||x - y||_*$, where $l_i \in (0, 1)$, $\forall\, x = (x_1, x_2, \cdots, x_k), y = (y_1, y_2, \cdots, y_k) \in A(\delta, \gamma), t \in I_0$, $i = 1, 2, \cdots, k$.

(H6) g_{i_2} is completely continuous and for any bounded set $\Lambda \in A(\delta, \gamma)$ the set $\{t \to g_{i_2}(t, x_t) : (x_1, x_2, \cdots, x_k) \in \Lambda\}$, is equicontinuous on $\underbrace{C(I_0, \mathbb{R}^n) \times C(I_0, \mathbb{R}^n) \times \cdots \times C(I_0, \mathbb{R}^n)}_{k} \quad \forall\, i = 1, 2, \cdots, k$.

(H7) $\psi \in C^1([t_0, \infty])$ is a continuous increasing function with $|\psi(t) - \psi(s)| \le N|t - s|, N \in (0, 1)$ and $|\psi\prime(s)| < K, K$ be any positive integer.

Lemma 6. *If there exist $\delta \in (0, a)$ and $\gamma \in (0, \infty)$ such that $(H1) - (H3)$ are satisfied, then for $t \in (t_0, t_0 + \delta]$, IVP (1.1) is equivalent to the following equation:*

$$\begin{cases} x_i(t) = & \phi_i(0) - g_i(t_0, \phi_1, \phi_2, \cdots, \phi_k) + g_i(t, x_t) \\ & + \frac{1}{\Gamma(\alpha_i)} \int_{t_0}^{t} \psi\prime(s) \left[\psi(t) - \psi(s)\right]^{\alpha_i - 1} f_i(s, x_s) ds, \; t \in I_0 \\ x_{i_{t_0}} = \phi_i \end{cases} \tag{3.3}$$

for $i = 1, 2, \cdots, k$ and $t \in I_0$.

Proof. From the conditions $(H1)$ and $(H2)$, it is obvious that $f_i(t, x_t)$ is Lebesgue measurable on I_0. A direct calculation using $(H7)$ gives that

$$\left(\psi\prime(s) \left[\psi(t) - \psi(s)\right]^{\alpha_i - 1}\right) \in L^{\frac{1}{1-\alpha_{i_1}}}([t_0, t]) \; t \in I_0$$

In the light of Holder's inequality and $(H3)$, we obtain that $\left(\psi\prime(s) \left[\psi(t) - \psi(s)\right]^{\alpha - 1}\right) f_i(s, x_s)$ is Lebesgue integrable with respect to $s \in [t_0, t]$

$\forall\, t \in I_0$, $i = 1, 2, \cdots, k$ and $(x_1, x_2, \cdots, x_k) \in A(\delta, \gamma)$ and

$$\int_{t_0}^{t} \left(\psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1}\right) f_i(s, x_s)ds \leq \\ \left\|\psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1}\right\|_{L^{\frac{1}{1-\alpha_{i1}}}(I_0)} \|m_i\|_{L^{\frac{1}{\alpha_{i1}}}(I_0)}, \quad (3.4)$$

where

$$\|F\|_{L^p(J)} = \left(\int_J |f(t)|^p dt\right)^{\frac{1}{p}},$$

for any p integrable function $F : J \to \mathbb{R}$.

According to the definition of fractional integral of a function f with respect to another function ψ and Caputo derivative of order α_i, it is easy to see that if x_i is a solution of the IVP (1.1), then x_i is a solution of equation (3.3).

On the other hand, if equation (3.3) is satisfied then $\forall\, t \in (t_0, t_0 + \delta]$, we have:

$$\begin{aligned}
&{}^C D_{t_0}^{\alpha_i,\psi}\left(x_i(t) - g_i(t, x_t)\right) = \\
&{}^C D_{t_0}^{\alpha_i,\psi}\left(\phi_i(0) - g_i(t_0, \phi_1, \phi_2, \cdots, \phi_k) + \frac{1}{\Gamma(\alpha_i)}\int_{t_0}^{t} \psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1} f_i(s, x_s)ds\right) \\
&= {}^C D_{t_0}^{\alpha_i,\psi}\left(\frac{1}{\Gamma(\alpha_i)}\int_{t_0}^{t} \psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1} f_i(s, x_s)ds\right)
\end{aligned}$$

$$\begin{aligned}
{}^C D_{t_0}^{\alpha_i,\psi}\left(x_i(t) - g_i(t, x_t)\right) &= {}^C D_{t_0}^{\alpha_i,\psi} I_{t_0}^{\alpha_i,\psi} f_i(t, x_t) \\
&= D_{t_0}^{\alpha_i,\psi}\left[I_{t_0}^{\alpha_i,\psi} f_i(t, x_t) - \sum_{k=0}^{n-1}\frac{\left[I^{\alpha_i,\psi} f_i(t, x_t)\right]^{[k]}(t_0)}{k!}\left(\psi(t)-\psi(t_0)\right)^k\right] \\
&= D_{t_0}^{\alpha_i,\psi}\ I_{t_0}^{\alpha_i,\psi} f_i(t, x_t) - \sum_{k=0}^{n-1}\frac{\left[I^{\alpha_i,\psi} f_i(t, x_t)\right]^{[k]}(t_0)}{\Gamma(k-\alpha_i+1)}\left(\psi(t)-\psi(t_0)\right)^{k-\alpha_i}) \\
&= f_i(t, x_t) - \left[I^{\alpha_i,\psi} f_i(t, x_t)\right]_{t=t_0}\frac{\left(\psi(t)-\psi(t_0)\right)^{-\alpha_i}}{\Gamma(1-\alpha_i)} = f_i(t, x_t)
\end{aligned}$$

since $\left[I^{\alpha_i,\psi} f_i(t, x_t)\right]_{t=t_0} = 0$.

Hence we get ${}^C D_{t_0}^{\alpha_i,\psi}\left(x_i(t) - g_i(t, x_t)\right) = f_i(t, x_t), \quad t \in (t_0, t_0 + \delta]$.
And this completes the proof. □

Theorem 7. *If there are $\delta \in (0, a)$ and $\gamma \in (0, \infty)$ satisfying the assumptions $(H1) - (H7)$, then IVP(1.1) has at least one solution on $[t_0, t_0 + \eta]$ for $\eta \in \mathbb{R}^+$.*

Proof. According to $(H4)$, equation (3.3) is equivalent to the following equation:

$$\begin{cases}
x_i(t) &= \phi_i(0) - g_{i_1}(t_0, \phi_1, \phi_2, \cdots, \phi_k) - g_{i_2}(t_0, \phi_1, \phi_2, \cdots, \phi_k) \\
&\quad + g_{i_1}(t, x_t) + g_{i_2}(t, x_t) \\
&\quad + \frac{1}{\Gamma(\alpha_i)}\int_{t_0}^{t} \psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1} f_i(s, x_s)ds, \quad t \in I_0 \\
x_{i_{t_0}} &= \phi_i \quad i = 1, 2, \cdots, k
\end{cases}$$

Let $(\tilde{\phi_1}, \tilde{\phi_2}, \cdots, \tilde{\phi_k}) \in A(\delta, \gamma)$ be defined as

$$\tilde{\phi}_{i_{t_0}} = \phi_i, \quad \tilde{\phi}_i(t_0 + t) = \phi_i(0) \quad \forall\, t \in [0, \delta],\ i = 1, 2, \cdots, k.$$

If $x = (x_1, x_2, \cdots, x_k)$ is a solution of the IVP(1.1), let $x_i(t_0 + t) = \tilde{\phi}_i(t_0 + t) + y_i(t)$, $t \in [-r, \delta], i = 1, 2, \cdots, k$.

Then we have, $x_{i_{t_0+t}} = \tilde{\phi}_{i_{t_0+t}} + y_{i_t}$, $t \in [0, \delta], i = 1, 2, \cdots, k$.
Thus

$$\begin{aligned} y_i(t) = &-g_{i_1}(t_0, \phi_1, \phi_2, \cdots, \phi_k) - g_{i_2}(t_0, \phi_1, \phi_2, \cdots, \phi_k) \\ &+g_{i_1}(t_0 + t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}}) \\ &+ g_{i_2}(t_0 + t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}}) \\ &+ \frac{1}{\Gamma(\alpha_i)} \int_0^t \psi'(s + t_0) \left[\psi(t + t_0) - \psi(s + t_0)\right]^{\alpha_i - 1} \\ &\quad f_i(t_0 + s, y_{1_s} + \tilde{\phi}_{1_{t_0+s}}, y_{2_s} + \tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s} + \tilde{\phi}_{k_{t_0+s}})ds, \end{aligned} \tag{3.5}$$

$t \in [0, \delta]$, $i = 1, 2, \cdots, k$.

Since g_{i_1}, g_{i_2} are continuous and x_{i_t} is continuous in t for all $1 = 1, 2, \cdots, k$, there exists $\delta' > 0$ such that:

$$|g_{i_1}(t_0 + t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}}) - g_{i_1}(t_0, \phi_1, \phi_2, \cdots, \phi_k)| < \frac{\gamma}{3} \tag{3.6}$$

$$|g_{i_2}(t_0 + t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}}) - g_{i_2}(t_0, \phi_1, \phi_2, \cdots, \phi_k)| < \frac{\gamma}{3} \tag{3.7}$$

for $0 < t < \delta'$ and $i = 1, 2, \cdots, k$.

Choose

$$\eta = min\left\{ \delta, \delta', \left(\frac{\gamma\Gamma(\alpha_i)(1 + \beta_i)^{(1-\alpha_{i_1})}}{3M_i NK}\right)^{\frac{1}{(1+\beta_i)(1-\alpha_{i_1})}} \right\}, \tag{3.8}$$

where $\beta_i = \frac{\alpha_i - 1}{1 - \alpha_{i_1}} \in (-1, 0)$ and $M_i = ||m_i||_{L^{\frac{1}{\alpha_{i_1}}}(I_0)}, i = 1, 2, \cdots, k$

Define $E(\eta, \gamma)$ as follows:

$$E(\eta, \gamma) = \\ \Big\{(y_1, y_2, \cdots, y_k) : y_i \in C([-r, \eta], \mathbb{R}^n)/y_i(s) = 0\, for\, s \in [-r, 0],\ ||y_i|| \le r, i = 1, 2, \cdots, k\Big\}.$$

Then $E(\eta, \gamma)$ is a closed, bounded and convex subset of $C([-r, \eta], \mathbb{R}^n) \times C([-r, \eta], \mathbb{R}^n) \times \cdots \times C([-r, \eta], \mathbb{R}^n)$.

On $E(\eta, \gamma)$, we define the operators S and U on $E(\eta, \gamma)$ by:

$$S(y_1, y_2, \cdots, y_k)(t) = \begin{pmatrix} S_1(y_1, y_2, \cdots, y_k)(t) \\ S_2(y_1, y_2, \cdots, y_k)(t) \\ \vdots \\ S_k(y_1, y_2, \cdots, y_k)(t) \end{pmatrix}$$

$$U(y_1, y_2, \cdots, y_k)(t) = \begin{pmatrix} U_1(y_1, y_2, \cdots, y_k)(t) \\ U_2(y_1, y_2, \cdots, y_k)(t) \\ \vdots \\ U_k(y_1, y_2, \cdots, y_k)(t) \end{pmatrix}$$

$$S_i(y_1, y_2, \cdots, y_k)(t) = \begin{cases} 0 & t \in [-r, 0] \\ -g_{i_1}(t_0, \phi_1, \phi_2, \cdots, \phi_k) & \\ +g_{i_1}(t_0 + t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}}) & t \in [0, \eta] \end{cases}$$

$$U_i(y_1, y_2, \cdots, y_k)(t) = \begin{cases} 0 & t \in [-r, 0] \\ -g_{i_2}(t_0, \phi_1, \phi_2, \cdots, \phi_k) & \\ +g_{i_2}(t_0 + t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}}) & \\ +\frac{1}{\Gamma(\alpha_i)} \int_0^t \psi'(s + t_0) \left(\psi(t + t_0) - \psi(s + t_0)\right)^{\alpha_i - 1} & \\ f_i(t_0 + s, y_{1_s} + \tilde{\phi}_{1_{t_0+s}}, y_{2_s} + \tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s} + \tilde{\phi}_{k_{t_0+s}})ds & t \in [0, \eta] \end{cases}$$

$for\ i = 1, 2, \cdots, k.$

It is easy to see that if the operator equation $y = Sy + Uy$ has a solution $y = (y_1, y_2, \cdots, y_k) \in E(\eta, \gamma)$ if and only if y_i is a solution of (3.5) $\forall i = 1, 2, \cdots, k$. Thus $x_i(t_0+t) = y_i(t)+\tilde{\phi}_i(t_0+t)$ is a solution of equation (1.1) on $[0, \eta]$. Therefore the existence of a solution of the IVP(1.1) is equivalent to the existence of a fixed point for the operator $S+U$ on $E(\eta, \gamma)$. Hence it is sufficient to show that $S+U$ has a fixed point in $E(\eta, \gamma)$.

The proof is divided into three steps.

Step I: $Sz + Uy \in E(\eta, \gamma)$ for every pair $z = (z_1, z_2, \cdots, z_k)$, $y = (y_1, y_2, \cdots, y_k) \in E(\eta, \gamma)$.

In fact, for every pair $z, y \in E(\eta, \gamma)$, $S_i z+U_i y \in C([-r, \eta], \mathbb{R}^n)$, $i = 1, 2, \cdots, k$, which implies $(Sz + Uy)(t) = 0$, $\forall t \in [-r, 0]$.

Now we have

$$|S_iz(t)-U_iy(t)|\leq$$
$$|-g_{i_1}(t_0,\phi_1,\phi_2,\cdots,\phi_k)+g_{i_1}(t_0+t,z_{1_t}+\tilde{\phi}_{1_{t_0+t}},z_{2_t}+\tilde{\phi}_{2_{t_0+t}},\cdots,z_{k_t}+\tilde{\phi}_{k_{t_0+t}})|$$
$$+|-g_{i_2}(t_0,\phi_1,\phi_2,\cdots,\phi_k)+g_{i_2}(t_0+t,y_{1_t}+\tilde{\phi}_{1_{t_0+t}},y_{2_t}+\tilde{\phi}_{2_{t_0+t}},\cdots,y_{k_t}+\tilde{\phi}_{k_{t_0+t}})|$$
$$+\frac{1}{\Gamma(\alpha_i)}\int_0^t|\psi'(s+t_0)\left[\psi(t+t_0)-\psi(s+t_0)\right]^{\alpha_i-1}$$
$$f_i(t_0+s,y_{1_s}+\tilde{\phi}_{1_{t_0+s}},y_{2_s}+\tilde{\phi}_{2_{t_0+s}},\cdots,y_{k_s}+\tilde{\phi}_{k_{t_0+s}})|ds$$

$$\leq\frac{2\gamma}{3}+\frac{1}{\Gamma(\alpha_i)}\left(\int_0^t|\psi'(s+t_0)\left[\psi(t+t_0)-\psi(s+t_0)\right]^{\alpha_i-1}|^{\frac{1}{1-\alpha_{i_1}}}ds\right)^{1-\alpha_{i_1}}$$
$$\left(\int_{t_0}^{t_0+t}(m_i(s))^{\frac{1}{\alpha_{i_1}}}ds\right)^{\alpha_{i_1}}$$
$$\leq\frac{2\gamma}{3}+\frac{M_iKN^{\alpha_i-1}}{\Gamma(\alpha_i)}\frac{\eta^{(1+\beta_i)(1-\alpha_{i_1})}}{(1+\beta_i)^{1-\alpha_{i_1}}}$$
$$\leq\gamma,\ \forall\, t\in[0,\eta]\ and\quad i=1,2,\cdots,k$$

Therefore

$$||S_iz+U_iy||=\sup_{t\in[0,\eta]}|(S_iz)(t)+(U_iy)(t)|\leq\gamma,\ \forall i=1,2,\cdots,k$$

which means that $Sz+Uy\in E(\eta,\gamma)$ for any $z,y\in E(\eta,\gamma)$.

Step II: To prove that S is a contraction on $E(\eta,\gamma)$.

Let $y'=(y'_1,y'_2,\cdots,y'_k),y''=(y''_1,y''_2,\cdots,y''_k)\in E(\eta,\gamma)$,

then, $(y'_{1_t}+\tilde{\phi}_{1_{t_0+t}},y'_{2_t}+\tilde{\phi}_{2_{t_0+t}},\cdots,y'_{k_t}+\tilde{\phi}_{k_{t_0+t}})\in A(\delta,\gamma)$ and

$(y''_{1_t}+\tilde{\phi}_{1_{t_0+t}},y''_{2_t}+\tilde{\phi}_{2_{t_0+t}},\cdots,y''_{k_t}+\tilde{\phi}_{k_{t_0+t}})\in A(\delta,\gamma)$.

Also by $(H5)$, we get that

$$\begin{aligned}&|S_iy'(t)-S_iy''(t)|\\&=|g_{i_1}(t_0+t,y'_{1_t}+\tilde{\phi}_{1_{t_0+t}},y'_{2_t}+\tilde{\phi}_{2_{t_0+t}},\cdots,y'_{k_t}+\tilde{\phi}_{k_{t_0+t}})\\&\qquad-g_{i_1}(t_0+t,y''_{1_t}+\tilde{\phi}_{1_{t_0+t}},y''_{2_t}+\tilde{\phi}_{2_{t_0+t}},\cdots,y''_{k_t}+\tilde{\phi}_{k_{t_0+t}})|\\&\leq l_i||y'-y''||_*\end{aligned}$$

which implies $||Sy'-Sy''||_*\leq l||y'-y''||_*$ where $l=max\{l_1,l_2,\cdots,l_k\}$

Since $0<l<1,S$ is a contraction on $E(\eta,\gamma)$.

Step III: Now we show that U is a completely continuous operator.

$$U_{i_1}(y_1, y_2, \cdots, y_k)(t) = \begin{cases} 0 & t \in [-r, 0], \\ -g_{i_2}(t_0, \phi_1, \phi_2, \cdots, \phi_k) & \\ +g_{i_2}(t_0 + t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}}) & t \in [0, \eta]. \end{cases}$$

and

$$U_{i_2}(y_1, y_2, \cdots, y_k)(t) = \begin{cases} 0 & t \in [-r, 0] \\ \dfrac{1}{\Gamma(\alpha_i)} \displaystyle\int_0^t \psi'(s + t_0) \left[\psi(t + t_0) - \psi(s + t_0)\right]^{\alpha_i - 1} & \\ f_i(t_0 + s, y_{1_s} + \tilde{\phi}_{1_{t_0+s}}, y_{2_s} + \tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s} + \tilde{\phi}_{k_{t_0+s}}) ds & t \in [0, \eta] \end{cases}$$

for $i = 1, 2, \cdots, k$

Clearly U= $\begin{pmatrix} U_{11} + U_{12} \\ U_{21} + U_{22} \\ \vdots \\ U_{k1} + U_{k2} \end{pmatrix}$

Since g_{i_2} is completely continuous for all $i = 1, 2, \cdots, k$, U_{i1} is continuous and also $\{U_{i1}(y) : y \in E(\eta, \gamma)\}$ is uniformly bounded. By using the condition (H6), it is easy to check that $\{U_{i1}(y) : y \in E(\eta, \gamma)\}$ is a completely continuous operator.

On the other hand

$$\begin{aligned} |U_{i_2} y(t)| &\leq \frac{1}{\Gamma(\alpha_i)} \int_0^t |\psi'(s + t_0) \left[\psi(t + t_0) - \psi(s + t_0)\right]^{\alpha_i - 1} \\ &\qquad f_i(t_0 + s, y_{1_s} + \tilde{\phi}_{1_{t_0+s}}, y_{2_s} + \tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s} + \tilde{\phi}_{k_{t_0+s}})| ds \\ &\leq \frac{1}{\Gamma(\alpha_i)} \left(\int_0^t |\psi'(s + t_0) \left[\psi(t + t_0) - \psi(s + t_0)\right]^{\alpha_i - 1} |^{\frac{1}{1 - \alpha_{i_1}}} \right)^{1 - \alpha_{i_1}} \\ &\qquad \left(\int_0^t (m_i(s))^{\frac{1}{\alpha_{i_1}}} ds \right)^{\alpha_{i_1}} \\ &\leq \frac{1}{\Gamma(\alpha_i)} \frac{\eta^{(1+\beta_i)(1-\alpha_{i_1})}}{(1 + \beta_i)^{1 - \alpha_{i_1}}} \frac{M_i K N^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad \forall t \in [0, \eta], \quad i - 1, 2, \cdots, k \end{aligned}$$

Hence $\{U_{i_2}(y) : y \in E(\eta, \gamma)\}$ is uniformly bounded.

Now we will prove that $\{U_{i_2} y : y \in E(\eta, \gamma)\}$ is equicontinuous.

For any $0 \leq t_1 < t_2 \leq \eta$ and $y \in E(\eta, \gamma)$, we get that

$$
\begin{aligned}
&|U_{i_2}y(t_2) - U_{i_2}y(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} |\psi'(s+t_0)\left[(\psi(t_2+t_0)-\psi(s+t_0))^{\alpha_i-1} - (\psi(t_1+t_0)-\psi(s+t_0))^{\alpha_i-1}\right] \\
&\qquad\qquad f_i(t_0+s, y_{1_s}+\tilde{\phi}_{1_{t_0+s)}}, y_{2_s}+\tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s}+\tilde{\phi}_{k_{t_0+s)}})|ds \\
&+ \frac{1}{\Gamma(\alpha_i)} \int_{t_1}^{t_2} |\psi'(s+t_0)\left[\psi(t_2+t_0)-\psi(s+t_0)\right]^{\alpha_i-1} \\
&\qquad\qquad f_i(t_0+s, y_{1_s}+\tilde{\phi}_{1_{t_0+s}}, y_{2_s}+\tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s}+\tilde{\phi}_{k_{t_0+s}})|ds \\
&\leq \frac{M_iKN^{\alpha_i-1}}{\Gamma(\alpha_i)}\left[\int_0^{t_1}(t_1-s)^{\beta_i}-(t_2-s)^{\beta_i}ds\right]^{\alpha_i-1} + \frac{M_iKN^{\alpha_i-1}}{\Gamma(\alpha_i)}\left[\int_{t_1}^{t_2}(t_2-s)^{\beta_i}ds\right]^{1-\alpha_{i_1}} \\
&\leq \frac{2M_iKN^{\alpha_i-1}}{\Gamma(\alpha_i)(\beta_i+1)^{1-\alpha_{i_1}}}(t_2-t_1)^{(1+\beta_i)(1-\alpha_{i_1})},
\end{aligned}
$$

which means that $\{U_{i2}y : y \in E(\eta,\gamma)\}$ is equicontinuous. Moreover, it is also clear that U_2 is continuous. So U_2 is a completely continuous operator. Then $U = U_1 + U_2$ is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $S+U$ has a fixed point on $E(\eta,\gamma)$ and hence the IVP(1.1) has a solution $x = (x_1, x_2, \cdots, x_k)$ where $x_i(t) = \phi_i(0) + y_i(t-t_0)$ for all $t \in [t_0, t_0+\eta], i = 1, 2, \cdots, k$. This completes the proof. □

In the case where $g_{i_1} \equiv 0, \forall\, i = 1, 2, \cdots, k$, we get the following result:

Corollary 8. *[16] Assume that there exist $\delta \in (0,a)$ and $\gamma \in (0,\infty)$ such that $(H1)$–$(H3)$ hold, $g_{i1}$is continuous for all $i = 1, 2, \cdots, k$ and*

$$
|g_i(t,x_t) - g_i(t,y_t))| \leq l_i||x-y||_*, \forall\, x = (x_1, x_2, \cdots, x_k), y = (y_1, y_2, \cdots, y_k) \in A(\delta,\gamma)
$$

and $t \in I_0$ where $l_i \in (0,1)$ is a constant for all $i = 1, 2, \cdots, k$. Then IVP (1.1) has at least one solution on $[t_0, t_0+\eta]$ for some positive number η.

In the case where $g_{i_2} \equiv 0$, $\forall\, i = 1, 2, \cdots, k$, we have the following result:

Corollary 9. *[16] Assume that there exist $\delta \in (0,a)$ and $\gamma \in (0,\infty)$ such that $(H1)$–$(H3)$ hold, g_i is completely continuous for all $i = 1, 2, \cdots, k$ and the family $\{t \to g_i(t,x_t) : (x_1, x_2, \cdots, x_k) \in \Lambda\}$ is equicontinuous on $C(I_0,\mathbb{R}^n) \times C(I_0,\mathbb{R}^n) \times \cdots \times C(I_0,\mathbb{R}^n)$ for all bounded sets Λ in $A(\delta,\gamma)$. Then IVP (1) has at least one solution on $[t_0, t_0+\eta]$ for some positive number η.*

Theorem 10. *Assume that the functions f and g are Lipschitz continuous with respect to the second variable, that is, there exist positive constants L_{i1} and L_{i2} such that*

$$
\|f_i(t,x_{it}) - f_i(t,x_{i2t})\| \leq L_{i1} \text{ and } \|g_i(t,x_{it}) - g_i(t,x_{i2t})\| \leq L_{i2}.
$$

Then there is a constant $h \in \mathbb{R}^+$ such that there exists a unique solution to the IVP(1.1) in the interval $[t_0, t_0+h] \subseteq [a,b]$ if $\left(\frac{L_{i1}}{\Gamma(\alpha_i+1)}(\psi(t_0+h)-\psi(t_0))_i^{\alpha} + L_{i2}\right) < 1$.

Proof. For $t \in I_0$, define the function F by

$$F_i(x,t) = \phi_i(0) - g_{i1}(t_0,\phi_1,\phi_2,\dots,\phi_k) - g_{i2}(t_0,\phi_1,\phi_2,\dots,\phi_k) + g_{i1}(t,x_t) + g_{i2}(t,x_t) + \frac{1}{\Gamma(\alpha_i)} \int_{t_0}^{t} \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha_i-1} f_i(s,x_s)ds.$$

Let $U = \{x_i \in C\left([t_0-r,t_0+a],\mathbb{R}^n\right) : {}^C D_{t_0}^{\alpha_i,\psi} x_i(t)$ exists and is continuous in $[t_0,t_0+h]\}$. It is enough to prove that $F_i : U \to U$ is a contraction.

Let us see that F_i is well defined, i.e., $F_i(U) \subseteq U$.

Given the function $x_i \in U$, we see that ${}^C D_{t_0}^{\alpha_i,\psi}(F_i(x_i)(t) - g_i(x_{it})) = f_i(t,x_{it})$ is continuous and

$$F_i(x_i)(t) = I_{t_0}^{\alpha_i,\psi} f_i(t,x_{it}) + g_i(t,x_it).$$

Now let $x_{i1}, x_{i2} \in U$ be arbitrary, then by assumptions H_1, H_2, we have

$$\|F_i(x_{i1}) - F_i(x_{i2})\| \le \|I_{t_0}^{\alpha_i,\psi}(f_i(t,x_{i1t}) - f_i(t,x_{i2t}))\| + \|g_i(t,x_{i1t}) - g_i(t,x_{i2t})\|$$

$$\le \left[\frac{L_{i1}}{\Gamma(\alpha_i+1)}\left(\psi(t_0+h)-\psi(t_0)\right)_i^{\alpha} + L_{i2}\right]\|x_{i1}-x_{i2}\|,$$

which proves that F_i is a contraction. By the Banach fixed point theorem, we get the result of the theorem. □

4 Example

Here we give an example to demonstrate our results.

Consider the 3-dimensional system of $\psi-$Caputo neutral fractional differential equations

$$\begin{cases} D^{\frac{1}{2},x} & \left(x_1(t) - \frac{e^{-3t}}{12\sqrt{6400+t^4}}(\sin x_1(t) + \cos x_2(t) + \sin x_3(t))\right) \\ & = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}(t+3)^{\frac{-3}{4}} \frac{\sin^4(x_1(t))}{1+|(x_2(t))|} \times \frac{(x_3(t))^2}{1+|x_3(t)|^3} \\ D^{\frac{1}{4},x} & \left(x_2(t) - \frac{1}{12\sqrt{3600+t^2}}\left(\cos x_1(t) + \frac{|x_2(t)|}{2+|x_2(t)|} + \frac{|x_3(t)|}{4+|x_3(t)|}\right)\right) \\ & = \frac{\Gamma(\frac{3}{8})}{\Gamma(\frac{1}{8})}(t+\frac{3}{2})^{\frac{-7}{8}} \frac{\cos^2(x_1(t))}{1+\sin^4(x_3(t))+(x_2(t))^2} \\ D^{\frac{1}{3},x} & (x_3(t) - \left(\frac{e^t}{18} + \frac{\cos^2 x_1(t)}{9} + \frac{25|x_2(t)|}{10+|x_2(t)|}\right) \\ & = \frac{\Gamma(\frac{5}{6})}{\sqrt{\pi}}(t+1)^{\frac{-1}{2}} \frac{|x_1(t))|}{1+(x_1(t))^2+6|x_2(t)|^5} \\ & \text{for } t \in (0,1) \end{cases}$$

$$x_{i_0} = t, i = 1,2,3,\ t \in [-1,0].$$

Define the maps

$$
\begin{aligned}
f_1(t,x_1,x_2,x_3) &= \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}(t+3)^{\frac{-3}{4}}\frac{\sin^4(x_1(t-1))}{1+|(x_2(t-1))|}\times\frac{(x_3(t-1))^2}{1+|x_3(t-1)|^3}\\
f_2(t,x_1,x_2,x_3) &= \frac{\Gamma(\frac{3}{8})}{\Gamma(\frac{1}{8})}(t+\frac{3}{2})^{\frac{-7}{8}}\frac{\cos^2(x_1(t-1))}{1+\sin^4(x_3(t-1))+(x_2(t-1))^2})\\
f_3(t,x_1,x_2,x_3) &= \frac{\Gamma(\frac{5}{6})}{\sqrt{\pi}}(t+1)^{\frac{-1}{2}}\frac{|x_1(t))|}{1+(x_1(t))^2+6|x_2(t)|^5}\\
g_1(t,x_1,x_2,x_3) &= \frac{e^{-3t}}{12\sqrt{6400+t^4}}(\sin x_1(t)+\cos x_2(t)+\sin x_3(t))\\
g_2(t,x_1,x_2,x_3) &= \frac{1}{12\sqrt{3600+t^2}}\left(\cos x_1(t)+\frac{|x_2(t)|}{2+|x_2(t)|}+\frac{|x_3(t)|}{4+|x_3(t)|}\right)\\
g_3(t,x_1,x_2,x_3) &= \frac{e^t}{18}+\frac{\cos^2 x_1(t)}{9}+\frac{25|x_2(t)|}{10+|x_2(t)|}
\end{aligned}
$$

with $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{3}$, $\psi(x) = x$ and if $m_1(t) = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}(t+3)^{\frac{-3}{4}}$, $m_2(t) = \frac{\Gamma(\frac{3}{8})}{\Gamma(\frac{1}{8})}(t+\frac{3}{2})^{\frac{-7}{8}}$, $m_3(t) = \frac{\Gamma(\frac{5}{6})}{\sqrt{\pi}}(t+1)^{\frac{-1}{2}}$, it is easy to check that $|f_1(t,x_{t_1},x_{t_2},x_{t_3})(t)| \le m_1(t)$, $|f_2(t,x_{t_1},x_{t_2},x_{t_3})(t)| \le m_2(t)$, $|f_3(t,x_{t_1},x_{t_2},x_{t_3})(t)| \le m_3(t)$.

Also $g_1(t,x_{t_1}x_{t_2},x_{t_3})$, $g_2(t,x_{t_1},x_{t_2},x_{t_3})(t)$ and $g_3(t,x_{t_1},x_{t_2},x_{t_3})(t)$ satisfy Lipschitz condition with $l_1 = \frac{1}{320}$, $l_2 = \frac{1}{240}$ and $l_3 = \frac{1}{18}$ respectively.

Thus, all conditions of Theorem (7) hold and so this system of $\psi-$ Caputo fractional functional differential equation has a solution.

5 Conclusion

The main reason behind the unpopularity of fractional calculus is that there are many nonequivalent definitions for integral and differential operators in it. Hence nowadays many researchers concentrate on defining generalized operators, from which the classical definitions can be obtained. Different phenomena can be interpreted with the help of systems of equations more effectively than with single equation. In this paper we concentrated on generalized fractional differential operators in $k-$systems and proved the existence and uniqueness of solutions of a $k-$systems of $\psi-$caputo fractional neutral functional differential equations under the specified conditions using Krasnoselskii's Fixed Point theorem and Banach's Fixed Point theorem respectively. Finally, we give an example to illustrate our results.

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