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An accurate approximation method for solving fractional order boundary value problems

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Abstract

In the present work, the generalized Chebyshev polynomials are used as basis functions in a collocation scheme to solve a class of fractional differential equations along with boundary conditions arising in science, engineering, and mathematical physics. By means of the collocation points and the matrix operations, the proposed scheme transforms the fractional boundary value problems (FBVPs) into a matrix equation, and this matrix equation corresponds to a set of linear algebraic equations consist of polynomial coefficients. An error estimation based on the residual function is performed to show the accuracy of the results. Hence, an improvement of the approximate solutions are obtained based upon this error estimation. Illustrative examples are given to demonstrate the validity and applicability of the method. Comparisons between the numerical results of the proposed method with existing results are done in order to show that the new method is efficient.

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1 Introduction

In the present work, a new simple but accurate collocation algorithm based on Chebyshev polynomials is developed to obtain an approximate solution of the following fractional differential equation [1]

$$Y''(x) + \mathcal{D}_\star^{(\gamma)} Y(x) = f(x), \quad 0 \leqslant x \leqslant 1, \tag{1.1}$$

supplemented with the boundary conditions

$$Y(0) = \beta_0, \quad Y(1) = \beta_1, \tag{1.2}$$

with given β_0, β_1 as real constants. Here, the function $f(x)$ is assumed to be continuous on $[0, 1]$ and $\mathcal{D}_\star^{(\gamma)}$ the standard Caputo fractional derivative operator and $n-1 < \gamma < n$, $n \in \mathbb{N}$. The discussion about the existence and uniqueness of the solution of fractional two-point boundary value problems can be found in [2], [36], [23], [32], [35]. The fractional order boundary value problems appear in the description of many physical stochastic-transport processes and in the inspection of liquid filtration which arises in a strongly porous's medium [33], see also the monographs of Kilbas et al. [17] and the references therein.

Although, the appearance of the fractional calculus as well as the fractional differential equations (FDEs) are as old as the classical calculus, but they have recently proved to be powerful and valuable in the modeling of many phenomena in various fields of science and engineering [21, 26, 17]. To model many real world problems, it has turned out the use of the fractional-order derivatives are more adequate than the integer-order ones. That is due to the fact that the fractional derivatives and integrals enable the description of the memory properties of various materials and processes [26]. Therefore, one needs to extend the concept of the ordinary differentiation as well as the integration to an arbitrary non-integer order. However, most of the resulting FDEs do not have an exact analytical solution, so the approximative and numerical techniques are preffered in identifying the solutions behaviour of such fractional equations. Numerous analytical and numerical methods have been developed to solve the FDEs. Among others, we mention some schemes such as the fractional linear multistep method [18], the adomian decomposition method [29, 16], the variational iteration method [19], the generalized Taylor method [20], the spline techniqes [35, 1], the Adams-type predictor-corrector method [6], the spectral collocation approach [8, 9, 11, 14, 12], the local discontinuous Galerkin method [10, 13, 15], and the sinc-Galerkin method [28], to name but a few.

The following approximative and numerical schemes have been proposed for the model problem (1.1)-(1.2) and closely related problems, to the best of our knowledge. These include the quadratic spline method [35], exponential spline technique [1], and sinc-Galerkin scheme [28]. Recently, considerable attention has been given to the establishment of techniques to solve the fractional differential equations using the orthogonal functions. The main characteristic of this technique is that it reduces the solution of the differential equations to the solution of a system of algebraic equations. Historically this approach was originated from the use of Fourier [25], Walsh [5] and block-pulse functions [27] and was later extended to other classical orthogonal polynomials such as Chebyshev, Legendre, Hermite, and Laguerre polynomials [31]. In most of the presented works, the use of the numerical techniques in conjunction with the operational matrices in differentiation and integration operators of some orthogonal polynomials, to solve the fractional differential equations on finite and infinite intervals, produced highly accurate solutions for such equations, see [3] for a recent review.

In this work, we are aiming to propose an approximation algorithm as an extension of the above mentioned papers. Our approach is based on the generalized fractional order of the Chebyshev orthogonal functions of the first kind to get an approximative solution of (1.1) accurately on the interval $[0, 1]$. The main idea of the proposed technique based on using these (orthogonal) functions along with the collocation points is that, it converts the differential or integral operator involved in (1.1) (1.2) to an algebraic form, thus greatly reduces the computational effort.

The content of this work is organized as follows. In the next section 2, some preliminary definitions of the fractional calculus and the relevant properties are introduced. Hence, we define the Chebyshev polynomials of fractional order and their properties. Section 3 is devoted to the presentation of the proposed collocation scheme applied to the fractional boundary value problems. Section 4 is devoted to the error analysis technique based on the residual function for the present method. Improving the Chebyshev collocation method is then introduced with the aid of the residual error function. In Section 5, we perform some experiments to illustrate the high accuracy and efficiency of the scheme. Finally, Section 5 provides a conclusion.

2 Basic definitions

In this section, first, some definitions and fundamental facts of the fractional calculus are given. Hence, some basic definitions of (generalized) Chebyshev polynomials and theorems, which are useful for our subsequent sections have been introduced.

2.1 Fractional calculus

Definition 1. Assuming that $g(x)$ is n-times continuously differentiable, the fractional derivative $\mathcal{D}_\star^{(\gamma)}$ of $g(x)$ of order $\gamma > 0$ in the Caputo's sense is defined as

$$\mathcal{D}_\star^{(\gamma)} g(x) = \begin{cases} I^{n-\gamma} g^{(n)}(x) & \text{if } n-1 < \gamma < n, \\ g^{(n)}(x), & \text{if } \gamma = n, \quad n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where

$$I^\gamma g(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \frac{g(s)}{(x-s)^{1-\gamma}}\, ds, \quad x > 0.$$

The properties of the operator $\mathcal{D}_\star^{(\gamma)}$ can be found in [26]. We make use of the followings

$$\mathcal{D}_\star^{(\gamma)}(C) = 0 \quad (C \text{ is a constant}), \tag{2.2}$$

$$\mathcal{D}_\star^{(\gamma)}\, x^\beta = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\gamma)} x^{\beta-\gamma}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geqslant \lceil \gamma \rceil, \text{ or } \beta \notin \mathbb{N}_0 \text{ and } \beta > \lfloor \gamma \rfloor, \\ 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \gamma \rceil. \end{cases} \tag{2.3}$$

2.2 Chebyshev functions

The Chebyshev polynomials play an outstanding role in classical as well as modern numerical computation [7]. It is known that the classical Chebyshev polynomials (of the first kind) are defined on $[-1, 1]$. Starting with $\mathcal{T}_0(t) = 1$ and $\mathcal{T}_1(t) = t$, these polynomials satisfy the following recurrence relation

$$\mathcal{T}_{n+1}(t) = 2t\,\mathcal{T}_n(t) - \mathcal{T}_{n-1}(t), \quad n = 1, 2, \ldots.$$

By introducing the change of variable, $x = 1 - 2(\frac{t}{L})^\alpha$, $\alpha > 0$, $L > 0$, one obtains the shifted version of the polynomials defined on $[0, L]$ will be denoted as $\mathcal{T}_n^\alpha(x) = \mathcal{T}_n(t)$. This transformation was introduced in [24]. The explicit analytical form of $\mathcal{T}_n^\alpha(x)$ of degree $(n\alpha)$ is given for $n = 0, 1, \ldots$

$$\mathcal{T}_n^\alpha(x) = \sum_{k=0}^{n} t_{n,k}\, x^{\alpha k}, \quad t_{n,k} = (-1)^k \frac{n\, 2^{2k}\, (n+k-1)!}{(n-k)!\, L^{\alpha k}\, (2k)!}, \quad k = 0, 1, \ldots, n, \tag{2.4}$$

with $t_{0,k} = 1$ for all $k = 0, 1, \ldots, n$. It is proved in [24] that the set of the fractional polynomial functions $\{\mathcal{T}_0^\alpha, \mathcal{T}_1^\alpha, \ldots\}$ is orthogonal on $[0, L]$ with respect to the weight function, $w_{L,\alpha}(x) = \frac{x^{\alpha/2-1}}{\sqrt{L^\alpha - x^\alpha}}$; i.e.

$$\int_0^L \mathcal{T}_n^\alpha(x)\, \mathcal{T}_m^\alpha(x)\, w_{L,\alpha}(x) dx = \frac{\pi}{2\alpha} d_n \delta_{mn}, \quad n, m \geqslant 0.$$

Here, δ_{mn} is the Kronecker delta function, $d_0 = 2$ while $d_n = 1$ for $n \geqslant 1$. These polynomials also satisfy the following properties

$$\mathcal{T}_n^\alpha(0) = 1, \quad \mathcal{T}_n^\alpha(L) = (-1)^n.$$

2.2.1 Approximation of functions

Any square integrable function $u(x)$ in $(0, L)$, may be expanded in terms of shifted Chebyshev polynomials as

$$u(x) = \sum_{0}^{\infty} a_n \mathcal{T}_n^{\alpha}(x),$$

where the unknown coefficients a_n are obtained through the orthogonality properties of the shifted Chebyshev polynomials as follows

$$a_n = \frac{2\alpha}{\pi d_n} \int_0^L u(x)\, \mathcal{T}_n^{\alpha}(x)\, w_{L,\alpha}(x) dx, \quad n = 0, 1, \ldots.$$

However, in practice, one needs to deal with only the first $(N+1)$-terms shifted Chebyshev polynomials to find an approximate solution of model (1.1) expressed as

$$u_{N,\alpha}(x) = \sum_{n=0}^{N} a_n \mathcal{T}_n^{\alpha}(x), \quad 0 \leqslant x \leqslant L, \tag{2.5}$$

where the unknown coefficients a_n, $n = 0, 1, \ldots, N$ are sought. To proceed, we write $\mathcal{T}_n^{\alpha}(x), n = 0, 1, \ldots, N$ in the matrix form as follows

$$\mathbb{T}_{\alpha}(x) = \mathbb{B}_{\alpha}(x)\, \mathbb{D}, \tag{2.6}$$

where

$$\mathbb{T}_{\alpha}(x) = \begin{bmatrix} \mathcal{T}_0^{\alpha}(x) & \mathcal{T}_1^{\alpha}(x) & \ldots & \mathcal{T}_N^{\alpha}(x) \end{bmatrix},$$
$$\mathbb{B}_{\alpha}(x) = \begin{bmatrix} 1 & x^{\alpha} & x^{2\alpha} & \ldots & x^{N\alpha} \end{bmatrix},$$

and the upper triangular $(N+1) \times (N+1)$ matrix $\mathbb{D}$ takes the form

$$\mathbb{D} = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 0 & t_{1,1} & t_{2,1} & t_{3,1} & \ldots & t_{N-1,1} & t_{N,1} \\ 0 & 0 & t_{2,2} & t_{3,2} & \ldots & t_{N-1,2} & t_{N,2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & t_{N-1,N-1} & t_{N,N-1} \\ 0 & 0 & 0 & \ldots & 0 & 0 & t_{N,N} \end{bmatrix}.$$

By means of (2.6) one can write the relation (2.5) in the matrix form

$$u_{N,\alpha}(x) = \mathbb{T}_{\alpha}(x)\, \mathbb{A} = \mathbb{B}_{\alpha}(x)\, \mathbb{D}\, \mathbb{A}, \tag{2.7}$$

where the vector of unknown is defined as

$$\mathbb{A} = \begin{bmatrix} a_0 & a_1 & \ldots & a_N \end{bmatrix}^t.$$

We conclude the discussion about the shifted Chebyshev polynomials by considering their convergence result. The following theorem states that the approximation solution $u_{N,\alpha}(x)$ is convergent to $u(x)$ exponentially, if one increases the number of basis functions N [24].

Theorem 2. *Assuming that $\mathcal{D}_{\star}^{(k\alpha)} u(x) \in C[0, L]$ for $k = 0, 1, \ldots, N$ and let*

$$CT_N^{\alpha} = Span\langle \mathcal{T}_0^{\alpha}(x), \mathcal{T}_1^{\alpha}(x), \ldots, \mathcal{T}_{N-1}^{\alpha}(x) \rangle.$$

If $u_{N,\alpha} = \mathbb{T}_\alpha\,\mathbb{A}$ is the best approximation to u from CT_N^α, then the error bound is presented as follows:

$$\|u(x) - u_{N,\alpha}(x)\|_w \leqslant \frac{L^{N\alpha} M_\alpha}{2^N\,\Gamma(N\alpha+1)} \Big(\frac{\pi}{\alpha N!}\Big)^{1/2},$$

where $M_\alpha \geqslant |\mathcal{D}_\star^{(N\alpha)} u(x)|$, $x \in [0, L]$.

Ultimately, to obtain a solution in the form (2.5) for the problem (1.1) on the interval $0 < x \leqslant L$, we use the spectral collocation points as the roots of the generalized fractional order of the Chebyshev functions. According to [24], the following points are used

$$x_k = L\Big(\frac{1-t_k}{2}\Big)^{\frac{1}{\alpha}}, \quad k = 0, 1, \ldots, N, \tag{2.8}$$

where $t_k = \cos\big(\frac{2k+1}{N+1}\frac{\pi}{2}\big)$ are the zeros of the usual Chebyshev polynomials of degree $N+1$ on $(-1, 1)$.

3 Chebyshev-collocation method

Now, suppose the approximation of the solution $Y(x)$ of the linear BVPs (1.1) in terms of $(N+1)$-terms Chebyshev polynomials series denoted by $Y_{N,\alpha}(x)$ on the interval $[0, L]$. As previously stated, in the vector form one may write

$$Y(x) \approx Y_{N,\alpha}(x) = \mathbb{B}_\alpha(x)\,\mathbb{D}\,\mathbb{A}. \tag{3.1}$$

By inserting the collocation points (2.8) into (3.1), we get a system of matrix equations in the form

$$Y_{N,\alpha}(x_k) = \mathbb{B}_\alpha(x_k)\,\mathbb{D}\,\mathbb{A}, \quad k = 0, 1, \ldots, N.$$

These equations can be expressed in the following compact representation

$$\boldsymbol{Y} = \boldsymbol{B}\,\mathbb{D}\,\mathbb{A}, \quad \boldsymbol{Y} = \begin{bmatrix} Y_{N,\alpha}(x_0) \\ Y_{N,\alpha}(x_1) \\ \vdots \\ Y_{N,\alpha}(x_N) \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \mathbb{B}_\alpha(x_0) \\ \mathbb{B}_\alpha(x_1) \\ \vdots \\ \mathbb{B}_\alpha(x_N) \end{bmatrix}. \tag{3.2}$$

To proceed, we take the fractional derivative of order γ from both sides of (3.1) to get

$$\mathcal{D}_\star^{(\gamma)} Y_{N,\alpha}(x) = \mathcal{D}_\star^{(\gamma)}\,\mathbb{B}_\alpha(x)\,\mathbb{D}\,\mathbb{A}. \tag{3.3}$$

The computation of $\mathcal{D}_\star^{(\gamma)}\,\mathbb{B}_\alpha(x)$ can be easily obtained via the property (2.2) and (2.3) as follows

$$\mathbb{B}_\alpha^{(\gamma)}(x) = \mathcal{D}_\star^{(\gamma)}\,\mathbb{B}_\alpha(x) = [0 \quad \mathcal{D}_\star^{(\gamma)}\,x^\alpha \quad \ldots \quad \mathcal{D}_\star^{(\gamma)}\,x^{\alpha N}].$$

To obtain a system of matrix equations for the fractional derivative, we substitute the collocation points (2.8) into (3.3) to get

$$\mathcal{D}_\star^{(\gamma)} Y_{N,\alpha}(x_k) = \mathbb{B}_\alpha^{(\gamma)}(x_k)\,\mathbb{D}\,\mathbb{A}, \quad k = 0, 1 \ldots, N,$$

which can also be expressed in the matrix form

$$\boldsymbol{Y}^{(\gamma)} = \boldsymbol{B}^{(\gamma)}\,\mathbb{D}\,\mathbb{A}, \quad \boldsymbol{Y}^{(\gamma)} = \begin{bmatrix} \mathcal{D}_\star^{(\gamma)} Y_{N,\alpha}(x_0) \\ \mathcal{D}_\star^{(\gamma)} Y_{N,\alpha}(x_1) \\ \vdots \\ \mathcal{D}_\star^{(\gamma)} Y_{N,\alpha}(x_N) \end{bmatrix}, \quad \boldsymbol{B}^{(\gamma)} = \begin{bmatrix} \mathbb{B}_\alpha^{(\gamma)}(x_0) \\ \mathbb{B}_\alpha^{(\gamma)}(x_1) \\ \vdots \\ \mathbb{B}_\alpha^{(\gamma)}(x_N) \end{bmatrix}. \tag{3.4}$$

Our next goal is to find a relationship between $Y_{N,\alpha}(x)$ and its second derivative. To this end, it suffices to compute $\frac{d^2}{dx^2}\mathbb{B}_\alpha(x)$. Evidently, the calculation of integer-order derivatives of $\mathbb{B}_\alpha(x)$ strictly depends on the values of α and N. These tasks are also obtainable through the properties (2.2)-(2.3) using integer values of $\gamma = 1, 2$. For instance, by choosing $\alpha = 1/2$ and $N = 7$ we get

$$\mathbb{B}_{\frac{1}{2}}(x) = \begin{bmatrix} 1 & x^{1/2} & x & x^{3/2} & x^2 & x^{5/2} & x^3 & x^{7/2} \end{bmatrix}.$$

Differentiation two times with respect to x reveals that

$$\frac{d}{dx}\mathbb{B}_{\frac{1}{2}}(x) = \begin{bmatrix} 0 & 0 & 1 & \frac{3}{2}x^{1/2} & 2x & \frac{5}{2}x^{3/2} & 3x^2 & \frac{7}{2}x^{5/2} \end{bmatrix},$$
$$\frac{d^2}{dx^2}\mathbb{B}_{\frac{1}{2}}(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & \frac{15}{4}x^{1/2} & 6x & \frac{35}{4}x^{3/2} \end{bmatrix}.$$

Now, by defining

$$\ddot{\mathbb{B}}_\alpha(x) := \frac{d^2}{dx^2}\mathbb{B}_\alpha(x),$$

and using the relation (3.1) one obtains that

$$Y''_{N,\alpha}(x) = \ddot{\mathbb{B}}_\alpha(x)\,\mathbb{D}\,\mathbb{A}. \tag{3.5}$$

If we place the collocation points (2.8) into (3.5), we arrive at the following matrix expression

$$\ddot{\boldsymbol{Y}} = \ddot{\boldsymbol{B}}\,\mathbb{D}\,\mathbb{A}, \quad \ddot{\boldsymbol{Y}} = \begin{bmatrix} Y''_{N,\alpha}(x_0) \\ Y''_{N,\alpha}(x_1) \\ \vdots \\ Y''_{N,\alpha}(x_N) \end{bmatrix}, \quad \ddot{\boldsymbol{B}} = \begin{bmatrix} \ddot{\mathbb{B}}_\alpha(x_0) \\ \ddot{\mathbb{B}}_\alpha(x_1) \\ \vdots \\ \ddot{\mathbb{B}}_\alpha(x_N) \end{bmatrix}. \tag{3.6}$$

Now, we are in place to calculate the Chebyshev solutions of (1.1). The collocation procedure is based on computing these polynomial coefficients by the aid of collocation points defined in (2.8). This can be done by inserting the collocation points into the fractional BVPs to get the system

$$Y''(x_k) + \mathcal{D}_\star^{(\gamma)}\,Y(x_k) = f(x_k), \quad k = 0, 1, \dots, N.$$

In the matrix form we may write the above equations as

$$\ddot{\boldsymbol{Y}} + \boldsymbol{Y}^{(\gamma)} = \boldsymbol{F}, \tag{3.7}$$

where the right-hand side vector $\boldsymbol{F}$ of size $(N+1) \times 1$ takes the form

$$\boldsymbol{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}.$$

Substituting the relations (3.4) and (3.6) into (3.7), the fundamental matrix equation is obtained

$$\boldsymbol{W}\,\mathbb{A} = \boldsymbol{F}, \tag{3.8}$$

where

$$\boldsymbol{W} := \left(\ddot{\boldsymbol{B}} + \boldsymbol{B}^{(\gamma)}\right)\mathbb{D}.$$

Obviously, (3.8) is a linear matrix equation with a_n, $n = 0, 1, \ldots, N$, being the unknown Chebyshev coefficients to be sought.

We are left with the task of entering the boundary conditions (1.2) into the former matrix equation. To take into account the first condition $Y(0) = \beta_0$, we tend $x \to 0$ in (3.1) to get the following matrix representation

$$\widehat{\boldsymbol{W}}_0\,\mathbb{A} = \beta_0, \qquad \widehat{\boldsymbol{W}}_0 := \mathbb{B}_\alpha(0)\,\mathbb{D} = [1 \quad 1 \quad \ldots \quad 1].$$

Similarly, for the second condition $Y(1) = \beta_1$ we obtain the matrix expression

$$\widehat{\boldsymbol{W}}_1\,\mathbb{A} = \beta_1, \qquad \widehat{\boldsymbol{W}}_1 := \mathbb{B}_\alpha(1)\,\mathbb{D} = [\hat{w}_{1,0} \quad \hat{w}_{1,1} \quad \ldots \quad \hat{w}_{1,N}].$$

Consequently, by replacing the first and last rows of the augmented matrix $[\boldsymbol{W}; \boldsymbol{F}]$ by the row matrices $[\widehat{\boldsymbol{W}}_0; \beta_0]$ and $[\widehat{\boldsymbol{W}}_0; \beta_1]$ we arrive at the new augmented system

$$\left[\widehat{\boldsymbol{W}}; \widehat{\boldsymbol{F}}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & ; & \beta_0 \\ w_{1,0} & w_{1,1} & w_{1,2} & w_{1,3} & \ldots & w_{1,N} & ; & f(x_1) \\ w_{2,0} & w_{2,1} & w_{2,2} & w_{2,3} & \ldots & w_{2,N} & ; & f(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & ; & \vdots \\ w_{N-1,0} & w_{N-1,1} & w_{N-1,2} & w_{N-1,3} & \ldots & w_{N-1,N} & ; & f(x_{N-1}) \\ \hat{w}_{1,0} & \hat{w}_{1,1} & \hat{w}_{1,2} & \hat{w}_{1,3} & \ldots & \hat{w}_{1,N} & ; & \beta_1 \end{bmatrix}. \tag{3.9}$$

Thus, the unknown Chebyshev coefficients in (3.1) will be calculated via solving this linear system of equations. This task can be easily performed by means of the linear solvers.

4 Error estimation based on residual functions and improvement of solutions

In this section, the error estimation based on the residual function is introduced for the method and thus the approximate solution (1.1) is corrected by the residual correction technique. This technique were previously used in [22, 4, 30] and recently in [34]. This error estimation is useful, in particular, when the exact solution of the boundary value problems is not yet known and requires some tools to measure the accuracy of the proposed collocation scheme. Briefly speaking, our goal is to construct an approximate solution based on the already calculated Chebyshev solution $Y_{N,\alpha}(x)$ in the form

$$Y_{N,M,\alpha}(x) = Y_{N,\alpha}(x) + \widetilde{\mathcal{E}_{N,M,\alpha}}(x), \tag{4.1}$$

where $\widetilde{\mathcal{E}_{N,M,\alpha}}(x)$ is the Chebyshev solution of the error problem obtained using the residual error function as described below. Here, the positive constant M is selected such that $M > N$.

To continue, let us define the residual function for the present method as

$$\mathcal{R}_{N,\alpha}(x) := \mathcal{L}[Y_{N,\alpha}](x) - f(x) = Y''_{N,\alpha}(x) + \mathcal{D}_\star^{(\gamma)} Y_{N,\alpha}(x) - f(x). \tag{4.2}$$

Clearly, the approximate solution $Y_{N,\alpha}(x)$ is satisfied the following problem

$$\mathcal{L}[Y_{N,\alpha}](x) = f(x) + \mathcal{R}_{N,\alpha}(x), \quad Y_{N,\alpha}(0) = \beta_0, \quad Y_{N,\alpha}(1) = \beta_1. \tag{4.3}$$

Assuming that the function $Y(x)$ is the exact solution of (1.1), we define the error function $\mathcal{E}_{N,\alpha}(x)$ as

$$\mathcal{E}_{N,\alpha}(x) = Y(x) - Y_{N,\alpha}(x). \tag{4.4}$$

Putting (4.4) into (1.1) and (1.2) while exploiting (4.2)-(4.3), we arrive at the error differential equation with the homogeneous boundary conditions

$$\mathcal{E}''_{N,\alpha}(x) + \mathcal{D}_{\star}^{(\gamma)}\mathcal{E}_{N,\alpha}(x) = -\mathcal{R}_{N,\alpha}(x), \quad \mathcal{E}_{N,\alpha}(0) = 0, \quad \mathcal{E}_{N,\alpha}(1) = 0. \tag{4.5}$$

Now, we solve the error differential equation (4.5) by means of the Chebyshev-collocation scheme, already described in the last section, to get the approximation

$$\widetilde{\mathcal{E}_{N,M,\alpha}}(x) = \sum_{m=0}^{M} e_m\, \mathcal{T}_m^{\alpha}(x), \tag{4.6}$$

for the error function $\mathcal{E}_{N,\alpha}(x)$ for $M > N$. Once the approximate solution $\widetilde{\mathcal{E}_{N,M,\alpha}}(x)$ is obtained, the corrected solution $Y_{N,M,\alpha}(x)$ defined in (4.1) will be known. Moreover, we also compute the error function as well as the residual error function for the corrected approximate solution as

$$\mathcal{E}_{N,M,\alpha}(x) = Y(x) - Y_{N,M,\alpha}(x), \tag{4.7a}$$

$$\mathcal{R}_{N,M,\alpha}(x) = Y''_{N,M,\alpha}(x) + \mathcal{D}_{\star}^{(\gamma)}Y_{N,M,\alpha}(x) - f(x), \tag{4.7b}$$

for the purpose of comparison.

5 Illustrative Examples

In this section, to describe the efficiency and accuracy of the proposed Chebshev collocation method, some numerical examples are given and the comparisons are made with the results of the other methods. All numerical computations have been done using MATLAB R2017a.

Example 3. As the first example, we consider the linear fractional boundary value problem (1.1) with

$$f(x) = 2 + \frac{2}{\Gamma(3-\gamma)}x^{2-\gamma} - \frac{1}{\Gamma(2-\gamma)}x^{1-\gamma}, \quad 0 < \gamma \leqslant 1,$$

and $\beta_0 = \beta_1 = 0$. This problem with $\gamma = 1/2$ is considered in [28]. It can be easily verified that the exact solution of this FBVPs is $Y(x) = x^2 - x$ for any $0 < \gamma \leqslant 1$.

In all examples below, we take $L = 1$. By employing $N = 2$, we are looking for an approximate solution in the form $Y_{2,\alpha}(x) = \sum_{n=0}^{2} a_n \mathcal{T}_n^{\alpha}(x)$. To this end, we calculate the unknown coefficients a_0, a_1, and a_2. For this example we set $\alpha = 1$ and therefore the following spectral collocation points are used

$$\left\{x_0 = \frac{195}{2911}, x_1 = \frac{1}{2}, x_3 = \frac{2716}{2911}\right\}.$$

Using $\gamma = 1/2$, the corresponding matrices and vectors in the fundamental matrix equation (3.8) become

$$\mathbb{D}^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -8 & 8 \end{bmatrix}, \quad \boldsymbol{B}^{(1/2)} = \begin{bmatrix} 0 & 481/1647 & 92/3527 \\ 0 & 679/851 & 1383/2600 \\ 0 & 1418/1301 & 1162/857 \end{bmatrix}, \quad \ddot{\boldsymbol{B}} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} 4590/2647 \\ 4590/2647 \\ 5998/2647 \end{bmatrix}, \left[\widehat{\boldsymbol{W}}; \widehat{\boldsymbol{F}}\right] = \begin{bmatrix} 1 & 1 & 1 & ; & 0 \\ 0 & -1358/851 & 9017/650 & ; & 4590/2647 \\ 1 & -1 & 1 & ; & 0 \end{bmatrix}.$$

Solving the system corresponding to $\left[\widehat{\boldsymbol{W}}; \widehat{\boldsymbol{F}}\right]$, the coefficients matrix is found as

$$\mathbb{A} = [-1/8 \quad 0 \quad 1/8]^t.$$

Hence, inserting the determined coefficients into $Y_{2,1}(x)$ we get the approximate solution

$$Y_{2,1}(x) = \begin{bmatrix} 1 & 1-2x & 8x^2-8x+1 \end{bmatrix} \mathbb{A} = x^2 - x,$$

which is the desired exact solution. The numerical solutions, which are obtained using the Chebyshev-collocation scheme, at some points $x \in [0,1]$ for Example (3), are presented in Table 1. The corresponding exact solutions along with the absolute errors and the results obtained via residual error functions (4.2) are further reported in this table. In addition, in order to justify our results, the solutions are also compared with the numerical solutions computed using the sinc-Galerkin method (SGM). Obviously, our numerical solutions are in excellent agreement with the exact solutions compared to the SGM.

Table 1. The comparison of the numerical solutions and the absolute/residual errors in Example 3 for $\gamma = 1/2, \alpha = 1$, and $N = 2$ for various $x \in [0,1]$.

x	Exact	Chebyshev	$\lvert\mathcal{E}_{2,1}(x)\rvert$	$\lvert\mathcal{R}_{2,1}(x)\rvert$	SGM	Absolute Errors
0.0	0.0	0.0	0.0	1.28×10^{-16}	0.0	0
0.1	−0.09	−0.09	5.76×10^{-18}	1.08×10^{-16}	−0.0899988	1.15×10^{-6}
0.2	−0.16	−0.16	1.02×10^{-17}	1.04×10^{-16}	−0.159998	1.50×10^{-6}
0.3	−0.21	−0.21	1.34×10^{-17}	1.04×10^{-16}	−0.209998	1.85×10^{-6}
0.4	−0.24	−0.24	1.54×10^{-17}	1.07×10^{-16}	−0.239999	1.43×10^{-6}
0.5	−0.25	−0.25	1.60×10^{-17}	1.11×10^{-16}	−0.249999	1.04×10^{-6}
0.6	−0.24	−0.24	1.54×10^{-17}	1.17×10^{-16}	−0.239999	1.27×10^{-6}
0.7	−0.21	−0.21	1.34×10^{-17}	1.24×10^{-16}	−0.209999	5.20×10^{-7}
0.8	−0.16	−0.16	1.02×10^{-17}	1.32×10^{-16}	−0.16000015	1.59×10^{-7}
0.9	−0.09	−0.09	5.76×10^{-18}	1.42×10^{-16}	−0.0900004	3.50×10^{-7}
1.0	0.0	0.0	0.0	1.52×10^{-16}	0.0	0

Example 4. Consider the boundary value problem (1.1) with the right-hand side $f(x)$ taken as

$$f(x) = 20x^3 - 12x^2 + \frac{120}{\Gamma(6-\gamma)} x^{5-\gamma} - \frac{24}{\Gamma(5-\gamma)} x^{4-\gamma}, \quad 0 < \gamma \leqslant 1,$$

and also zero boundary conditions. In this case, the exact solution is $Y(x) = x^4(x-1)$ for any $0 < \gamma \leqslant 1$. This FBVPs is taken from [35] and [1].

We first take $\gamma = 3/10$ in the second example and set $N = 5$ as the number of basis functions. The parameter $\alpha = 1$ is also sufficient to get the desired approximation. The approximate solutions $Y_{5,1}(x)$ of this model problem using the Chebyshev basis functions in the interval $0 \leqslant x \leqslant 1$ are obtained as follows:

$$Y_{5,1}(x) = 0.999999999999999\, x^5 - 0.999999999999998\, x^4 - 1.2253730 \times 10^{-15}\, x^3 + 3.5162677643 \times 10^{-16}\, x^2 - 7.0610181123 \times 10^{-17}\, x + 3.5373746402 \times 10^{-74}.$$

The corresponding residual error function $\mathcal{R}_{5,1}(x)$ takes the form

$$\mathcal{R}_{5,1}(x) = 3.163714295 \times 10^{-15}\, x^2 - 1.280874739 \times 10^{-15}\, x - 2.371724879 \times 10^{-15}\, x^3 - 9.248187761 \times 10^{-18}\, x^{\frac{7}{10}} + 9.947106210 \times 10^{-17}\, x^{\frac{17}{10}} - 3.071162029 \times 10^{-16}\, x^{\frac{27}{10}} + 4.100356309 \times 10^{-16}\, x^{\frac{37}{10}} - 1.962058927 \times 10^{-16}\, x^{\frac{47}{10}} + 1.536515417 \times 10^{-16}$$

The above approximation solution and its related residual error function are visualized in Fig. 1. To validate our results, we also plot the exact solution, which represented by a solid line. Obviously, our approximated solution is matching up to the machine epsilon.

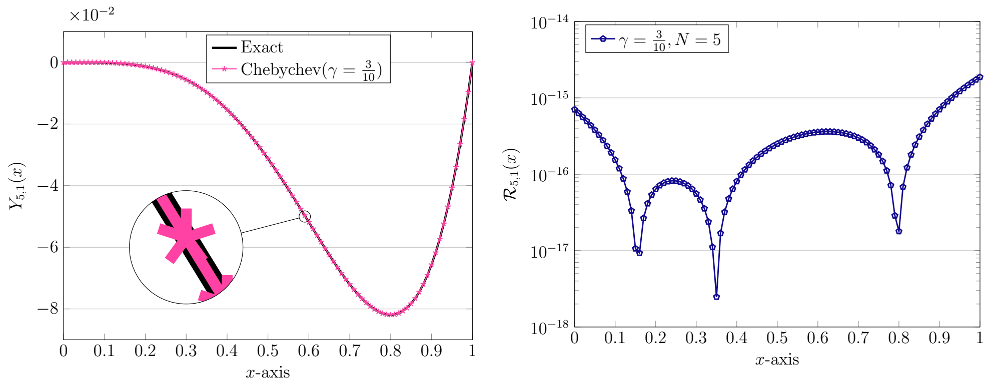

Figure 1. The comparison of the approximated and the exact solutions using the Chebyshev functions (left) and the corresponding residual error function (right) for Example 4 with $\gamma = 3/10$, $\alpha = 1$, and $N = 5$.

In Table 2, we report the numerical results as welll as the absolute errors correspond to $N = 5$ obtained by the Chebyshev-collocation procedure using $\gamma = 3/10$ and $\alpha = 1$ at some points $x \in [0, 1]$. A comparison in this table is also made with the exponential spline approach from [1]. As one can see in Table 2, the results obtained by our proposed scheme, are superior in terms of accuracy. The impact of utilizing the various values of the fractional orders $\gamma = 0, 0.25, 0.5, 0.75$, and $\gamma = 1$ are depicted in Fig. 2. From Fig. 2 it can be inferred that the same accuracies are achieved using different γ in the range $[0, 1]$.

In the last experiment, we show the benefits of using the fractional version of the Chebyshev polynomials. For this purpose, we construct an example that has a fractional solution.

Example 5. Let us consider (1.1) with the function $f(x)$ defined as

$$f(x) = \frac{15}{4}\sqrt{x} - \frac{35}{4}\sqrt[3]{x^2} + \frac{\Gamma(7/2)}{\Gamma(7/2-\gamma)} x^{5/2-\gamma} - \frac{\Gamma(9/2)}{\Gamma(9/2-\gamma)} x^{7/2-\gamma}, \quad 0 < \gamma \leqslant 1.$$

Using the zero boundary conditions, it is not difficult to show that the exact solution of this problem satisfies $Y(x) = \sqrt{x^5} - \sqrt{x^7}$.

Table 2. The comparison of the numerical solutions and the absolute errors in Example 4 with $\gamma = 3/10, \alpha = 1$, and $N = 5$ for various $x \in [0, 1]$.

x	Chebyshev	$\lvert\mathcal{E}_{5,1}(x)\rvert$	Exp-Spline	Max. Abs. Errors
0.000	0.000000000000000	7.03×10^{-16}	0	0
0.125	−0.000213623046875	7.29×10^{-17}	0.0026	2.8×10^{-3}
0.250	−0.002929687500000	8.17×10^{-17}	0.0028	5.7×10^{-3}
0.375	−0.012359619140625	3.99×10^{-17}	−0.0040	8.4×10^{-3}
0.500	−0.031250000000000	2.50×10^{-16}	−0.0211	1.0×10^{-2}
0.625	−0.057220458984375	3.61×10^{-16}	−0.0471	1.0×10^{-2}
0.750	−0.079101562500000	1.81×10^{-16}	−0.0710	8.1×10^{-3}
0.875	−0.073272705078125	4.93×10^{-16}	−0.0689	4.4×10^{-3}
1.000	0.000000000000000	1.87×10^{-15}	0	0

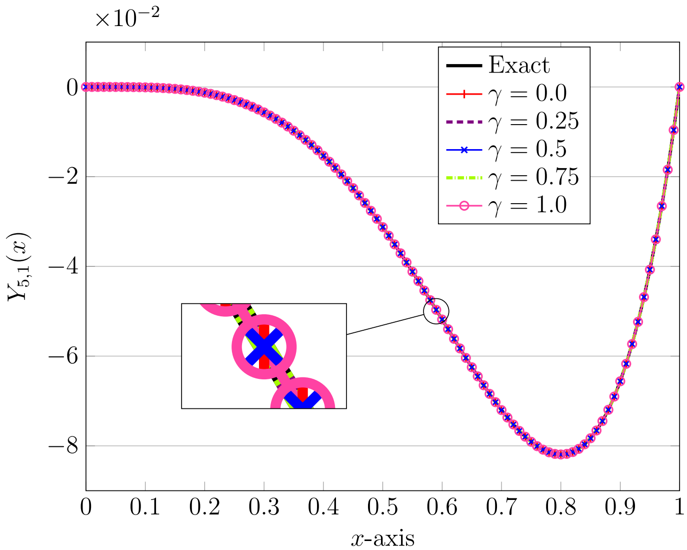

Figure 2. The approximated Chebyshev series solutions $Y_{N,\alpha}(x)$ for Example 4 using $N = 5,\ \alpha = 1$ for different $\gamma = 0, 0.25, 0.5, 0.75$, and $\gamma = 1.0$.

For this example, we fix $N = 7$ and use two values of $\alpha = 1$ and $\alpha = 1/2$, which indicate the difference between the fractional and non-fractional Chebyshev basis functions respectively. Setting $\gamma = 1/2$, the approximate solutions $Y_{7,1}(x)$ and $Y_{7,\frac{1}{2}}(x)$ obtained via (3.9) of the model (1.1) in the interval $[0, 1]$ are as follows

$$\begin{aligned} Y_{7,1}(x) = {} & 0.176260724015854\, x^7 - 0.792821466676062\, x^6 + 1.58262835557799\, x^5 \\ & - 2.23737888512802\, x^4 + 1.01020023333265\, x^3 + 0.268398536500592\, x^2 \\ & - 0.00728749762300631\, x, \end{aligned}$$

and

$$
\begin{aligned}
Y_{7,\frac{1}{2}}(x) = &-1.7687 \times 10^{-74} - 3.7183 \times 10^{-13}\, x - 7.5367 \times 10^{-14}\, x^2 \\
&- 3.6789 \times 10^{-14}\, x^3 + 1.7053 \times 10^{-13}\, x^{\frac{1}{2}} + 2.0876 \times 10^{-13}\, x^{\frac{3}{2}} \\
&+ 1.00000000000011\, x^{\frac{5}{2}} - 1.0\, x^{\frac{7}{2}}.
\end{aligned}
$$

Evidently, using the generalized Chebyshev basis functions leads to a considerably more accurate solution than usual ones. This fact can be further confirmed in the next two Figs. 3-4, in which we plot these approximations with the corresponding residual error functions $\mathcal{R}_{7,\alpha}(x)$ for $\alpha = 1, 1/2$.

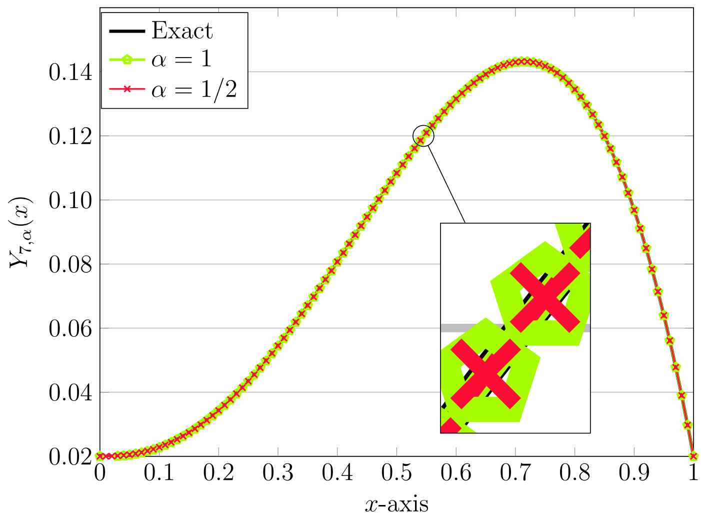

Figure 3. The approximated Chebyshev series solutions $Y_{7,\alpha}(x)$ for Example 5 using $\gamma = 1/2$ for two different $\alpha = 1$ and $\alpha = 1/2$.

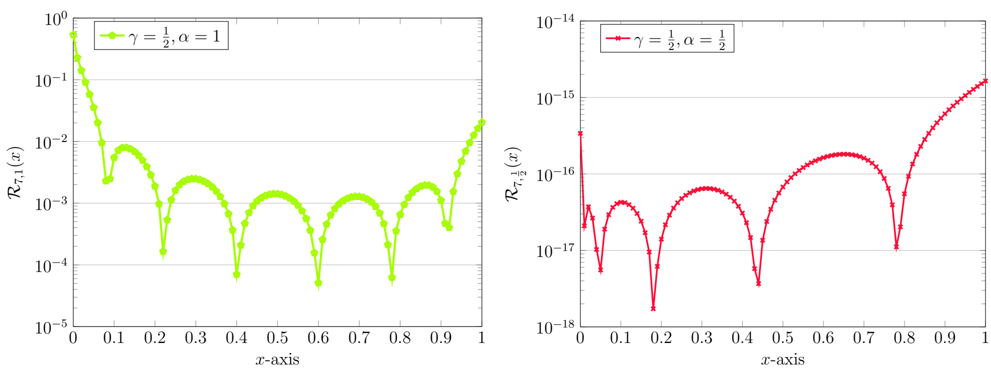

Figure 4. The comparison of the residual error functions using $\alpha = 1$ (left) and $\alpha = 1/2$ (right) for Example 5 with $\gamma = 1/2$ and $N = 7$.

Our next goal is to improve the current obtained solution $Y_{N,\alpha}(x)$ for Example 5 to get a solution of the form $Y_{N,M,\alpha}(x)$, where $N = 7$ and $\alpha = 1/2$. To this end, we solve the error problem (4.5) for $\mathcal{E}_{N,\alpha}(x)$. Choosing $M = 8$ and using the Chebeyshev-collocation procedure, the coefficient matrix of the error problem is calculated. These coefficients $e_0, e_1, \dots, e_8$ are approximately expressed as

$$e_0 = -8.1044 \times 10^{-15}, \quad e_1 = -6.8377 \times 10^{-15}, \quad e_2 = 8.226 \times 10^{-15},$$
$$e_3 = 6.8944 \times 10^{-15}, \quad e_4 = -1.4792 \times 10^{-16}, \quad e_5 = -5.6145 \times 10^{-17},$$

$$e_6 = 2.6377 \times 10^{-17}, \quad e_7 = -6.01 \times 10^{-19}, \quad e_8 = -4.0568 \times 10^{-33}.$$

By inserting the coefficient matrix into (4.6) and simplifying, we get the error function approximately as

$$\begin{aligned}\widetilde{\mathcal{E}_{7,8,\frac{1}{2}}}(x) &= -6.2836 \times 10^{-87} + 3.7183 \times 10^{-13}\, x + 7.5367 \times 10^{-14}\, x^2 \\ &+ 3.6789 \times 10^{-14}\, x^3 - 1.3293 \times 10^{-28}\, x^4 - 1.7053 \times 10^{-13}\, x^{1/2} \\ &- 2.0876 \times 10^{-13}\, x^{3/2} - 1.0962 \times 10^{-13}\, x^{5/2} + 4.9234 \times 10^{-15}\, x^{7/2}.\end{aligned}$$

Finally, we substitute the error function $\widetilde{\mathcal{E}_{7,8,\frac{1}{2}}}(x)$ into (4.1), the corrected approximate solution $Y_{7,8,\frac{1}{2}}(x)$ for $(N, M) = (7, 8)$ is calculated as

$$\begin{aligned}Y_{7,8,\frac{1}{2}}(x) &= -1.7687 \times 10^{-74} + 7.2131 \times 10^{-27}\, x - 2.3010 \times 10^{-28}\, x^2 \\ &+ 1.4667 \times 10^{-27}\, x^3 - 1.3293 \times 10^{-28}\, x^4 + 5.1996 \times 10^{-28}\, x^{1/2} \\ &- 7.0356 \times 10^{-27}\, x^{3/2} + 1.0\, x^{5/2} - 1.0\, x^{7/2}.\end{aligned}$$

Table 3 reports the comparison of the Chebyshev approximate solution $Y_{7,\frac{1}{2}}(x)$ and the corrected Chebyshev approximate solution $Y_{7,8,\frac{1}{2}}(x)$ with the exact solution $Y(x)$ at some points $x \in [0, 1]$ in Example 5. The excellent agreement between the exact and corrected approximate solutions is evident in this table. Note that we have only shown the numerical results up to 15 digits. In fact, using $M = N + 1$, the contribution of the correction term is a considerable achievement to the accuracy of the solutions compared to non-corrected results. To highlight the advantage of applying the correction technique, we also present the results obtained by the error functions defined in (4.4), (4.7a) and the residual error functions in (4.2), (4.7b). These results are shown in Table 4.

6 Conclusions

In this work, an accurate approximation algorithm based on the generalized Chebyshev functions is developed to solve the fractional-order differential equation under the boundary conditions. Utilizing the (fractional) Chebyshev functions together with the collocation points, the differential equations is transformed into an algebraic system of linear equations. Illustrative examples are given to demonstrate the efficiency and accuracy of the proposed method and a comparison between the method and other existing schemes is done. Moreover, the performance of fractional and non-fractional basis functions has been assessed and the reliability of the proposed technique is checked through defining the error as well as the residual error functions. The approximate solutions are further improved through these error functions.

It can be seen from Figures and Tables that the proposed scheme is not only a simple but also an accurate and powerful tool for obtaining the approximate solutions of FBVPs. From comparisons, it is observed that our results are more accurate than the numerical results of other existing well-known computational methods. The method can be easily extended to the solutions of higher-order FBVPs and systems appearing in the modelling of many problems in the science and engineering fields.

x	$Y(x)$ (Exact)	$Y_{7,\frac{1}{2}}(x)$	$Y_{7,8,\frac{1}{2}}(x)$
0.0	0.000000000000000	$-17.686873200833 \times 10^{-75}$	$-17.686873200840 \times 10^{-75}$
0.1	0.002846049894152	$2.84604989417444 \times 10^{-03}$	$2.84604989415154 \times 10^{-03}$
0.2	0.014310835055999	$14.3108350560179 \times 10^{-03}$	$14.3108350559987 \times 10^{-03}$
0.3	0.034506521122826	$34.5065211228392 \times 10^{-03}$	$34.5065211228255 \times 10^{-03}$
0.4	0.060715731075233	$60.7157310752413 \times 10^{-03}$	$60.7157310752329 \times 10^{-03}$
0.5	0.088388347648318	$88.3883476483224 \times 10^{-03}$	$88.3883476483184 \times 10^{-03}$
0.6	0.111541920370774	$111.541920370774 \times 10^{-03}$	$111.541920370774 \times 10^{-03}$
0.7	0.122989023900509	$122.989023900508 \times 10^{-03}$	$122.989023900509 \times 10^{-03}$
0.8	0.114486680447989	$114.486680447987 \times 10^{-03}$	$114.486680447989 \times 10^{-03}$
0.9	0.076843347142092	$76.8433471420900 \times 10^{-03}$	$76.8433471420916 \times 10^{-03}$
1.0	0.000000000000000	$24.7616224811668 \times 10^{-72}$	$24.3127412678843 \times 10^{-72}$

Table 3. The comparison of the numerical solutions in the Chebyshev and the corrected Chebyshev collocation methods in Example 5 for $(N, M) = (7, 8)$, $\gamma, \alpha = 1/2$.

	Chebyshev		Corrected Chebyshev	
x	$\lvert\mathcal{E}_{7,\frac{1}{2}}(x)\rvert$	$\lvert\mathcal{R}_{7,\frac{1}{2}}(x)\rvert$	$\lvert\mathcal{E}_{7,8,\frac{1}{2}}(x)\rvert$	$\lvert\mathcal{R}_{7,8,\frac{1}{2}}(x)\rvert$
0.0	$1.7686873 \times 10^{-74}$	$3.9527851 \times 10^{-16}$	$1.7686873 \times 10^{-74}$	$1.5352514 \times 10^{-31}$
0.1	$2.2899936 \times 10^{-14}$	$4.5204731 \times 10^{-17}$	$6.5569279 \times 10^{-28}$	$4.5813764 \times 10^{-30}$
0.2	$1.9204119 \times 10^{-14}$	$1.5524563 \times 10^{-17}$	$1.0108576 \times 10^{-27}$	$3.4952039 \times 10^{-30}$
0.3	$1.3711741 \times 10^{-14}$	$6.5574364 \times 10^{-17}$	$1.2093598 \times 10^{-27}$	$4.9166546 \times 10^{-30}$
0.4	$8.4135756 \times 10^{-15}$	$2.9473773 \times 10^{-17}$	$1.2839077 \times 10^{-27}$	$1.6561716 \times 10^{-29}$
0.5	$3.9783696 \times 10^{-15}$	$7.0097562 \times 10^{-17}$	$1.2542846 \times 10^{-27}$	$2.5916463 \times 10^{-29}$
0.6	$6.8238175 \times 10^{-16}$	$1.6365969 \times 10^{-16}$	$1.1354044 \times 10^{-27}$	$2.7219049 \times 10^{-29}$
0.7	$1.3641080 \times 10^{-15}$	$1.5692961 \times 10^{-16}$	$9.3978665 \times 10^{-28}$	$1.4805308 \times 10^{-29}$
0.8	$2.1366920 \times 10^{-15}$	$5.8523449 \times 10^{-17}$	$6.7858449 \times 10^{-28}$	$1.6783724 \times 10^{-29}$
0.9	$1.6619505 \times 10^{-15}$	$5.9956544 \times 10^{-16}$	$3.6208852 \times 10^{-28}$	$7.2771193 \times 10^{-29}$
1.0	$2.4743936 \times 10^{-71}$	$1.5881998 \times 10^{-15}$	$2.4312741 \times 10^{-71}$	$1.5814277 \times 10^{-28}$

Table 4. The comparison of the error and residual error functions in the Chebyshev and the corrected Chebyshev collocation methods in Example 5 for $(N, M) = (7, 8)$, $\gamma, \alpha = 1/2$.

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