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Sufficiency and duality of set-valued semi-infinite programming problems under generalized cone convexity

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Abstract

In this paper, we establish the sufficient KKT conditions of a set-valued semi-infinite programming problem (3.1) via contingent epiderivative. By using ρ-cone convexity assumptions, we also study the duality results of Mond-Weir type (5.1), Wolfe type (6.1), and mixed type (7.1) of the problem (3.1).

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1 Introduction

In the last few decades, authors like Hanson [18], Craven [6], Ben-Israel and Mond [3], Corley [5], Zalmai [27], Sheng and Liu [26] etc. are mainly interested in studying the analysis and applications of various types of optimization problems. One type of optimization problem is the semi-infinite programming problem. In recent times, many authors like Goberna and Lopez [17], Hettich and Kortanek [19], Lopez and Still [22], and Shapiro [25] have studied the semi-infinite programming problem in optimization theory. In 2005, Shapiro [24] studied the Lagrangian duality of convex semi-infinite programming problems. In 2010, Kostyukova and Tchemisova [21] established sufficient optimality conditions of convex semi-infinite programming problems. In 2012, Mishra and Jaiswal [23] established the sufficient optimality conditions of nondifferentiable multiobjective semi-infinite programming problems under generalized convexity assumptions and studied Mond-Weir type duality theorems.

In this paper, we consider a semi-infinite programming problem, where the objective function and functions attached to constraints are set-valued maps. We call it a set-valued semi-infinite programming problem. We are mainly interested to establish

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the sufficient Karush-Kuhn-Tucker (KKT) conditions of a set-valued semi-infinite programming problem (3.1) under generalized cone convexity assumptions. We also study the duality results of Mond-Weir (5.1), Wolfe (6.1), and mixed (7.1) types for the weak solutions of the problem (3.1).

This paper is organized as follows. In Section 2, we recall some definitions and preliminary concepts of set-valued mappings. In Section 3, we introduce a set-valued semi-infinite programming problem (3.1) and establish the sufficient (KKT) conditions. We also prove the duality results of various types under generalized cone convexity assumptions.

2 Definitions and Preliminaries

Let K be a nonempty subset of $\mathbb{R}^m$. Then K is called a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Also, K is called pointed if $K \cap (-K) = \{\theta_{\mathbb{R}^m}\}$, solid if $\text{int}(K) \neq \emptyset$, closed if $\overline{K} = K$ and convex if $\lambda K + (1-\lambda)K \subseteq K$, for all $\lambda \in [0,1]$, where $\text{int}(K)$ and $\overline{K}$ denote the interior and closure of K, respectively and $\theta_{\mathbb{R}^m}$ is the zero element of $\mathbb{R}^m$. Let us define the positive orthant $\mathbb{R}^m_+$ of $\mathbb{R}^m$ by

$$\mathbb{R}^m_+ = \{y = (y_1, ..., y_m) \in \mathbb{R}^m : y_i \geq 0, \forall i = 1, ..., m\}.$$

It is clear that $\mathbb{R}^m_+$ is a solid pointed closed convex cone and $\text{int}(\mathbb{R}^m_+) \cup \{\theta_{\mathbb{R}^m}\}$ is a solid pointed convex cone in $\mathbb{R}^m$.

Definition 1. Let $\emptyset \neq B \subseteq \mathbb{R}^m$ and $y' \in \overline{B}$. The contingent cone $T(B, y')$ to B at y' is defined as

An element $y \in T(B, y')$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$, with $\lambda_n \to 0^+$ and $\{y_n\}$ in $\mathbb{R}^m$, with $y_n \to y$, such that

$$y' + \lambda_n y_n \in B, \forall n \in \mathbb{N},$$

or, there exist sequences $\{t_n\}$, with $t_n > 0$ and $\{y'_n\}$ in B, with $y'_n \to y'$, such that

$$t_n(y'_n - y') \to y.$$

Let $2^{\mathbb{R}^m}$ be the set of all subsets of $\mathbb{R}^m$ and K be a solid pointed convex cone in $\mathbb{R}^m$. Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$. The effective domain, range, graph, and epigraph of the set-valued map F are defined as

$$\text{dom}(F) = \{x \in \mathbb{R}^n : F(x) \neq \emptyset\},$$

$$F(A) = \bigcup_{x \in A} F(x), \text{ for any } \emptyset \neq A \subseteq \mathbb{R}^n,$$

$$\text{gr}(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\},$$

and

$$\text{epi}(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x) + K\}.$$

Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map, with $\text{dom}(F) = \mathbb{R}^n$ and $y' \in F(x')$. In 1997, Jahn and Rauh [20] introduced the notion of contingent epiderivative of set-valued maps.

Definition 2 ([20])**.** A single-valued map $D_\uparrow F(x', y') : \mathbb{R}^n \to \mathbb{R}^m$ whose epigraph coincides with the contingent cone to the epigraph of F at (x', y'), i.e.

$$\text{epi}(D_\uparrow F(x', y')) = T(\text{epi}(F), (x', y')),$$

is said to be the contingent epiderivative of F at (x', y').

When $f : \mathbb{R}^n \to \mathbb{R}$ is a real-valued map being continuous at x' and convex,

$$D_{\uparrow} f(x', f(x'))(u) = f'(x')(u), \forall u \in X,$$

where $f'(x')(u)$ is the directional derivative of f at x' in the direction u.

Definition 3 ([4])**.** Let A be a non-empty convex subset of $\mathbb{R}^n$. A set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$, with $A \subseteq \text{dom}(F)$, is called K-convex on A if $\forall x_1, x_2 \in A$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + K.$$

It is clear that if the set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is K-convex on A, then epi(F) is a convex subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Lemma 4. *If $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is a K-convex set-valued map on a convex subset A of $\mathbb{R}^n$, then for all $x, x' \in A$ and $y' \in F(x')$,*

$$F(x) - y' \subseteq D_{\uparrow} F(x', y')(x - x') + K.$$

3 Formulation of the Main Problem

We consider a semi-infinite programming problem in the setting of set-valued maps. Let U be a countably infinite subset of $\mathbb{R}^p$, $\emptyset \neq A \subseteq \mathbb{R}^n$, and $F = (F_1, F_2, ..., F_m) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$, $G : \mathbb{R}^n \times U \to 2^{\mathbb{R}}$ be two set-valued maps with

$$A \subseteq \text{dom}(F) \text{ and } A \times U \subseteq \text{dom}(G).$$

Let $B_1, B_2, ..., B_m$ be $n \times n$ positive semi-definite (symmetric) real matrices. Consider a set-valued semi-infinite programming problem (3.1)

$$\begin{aligned} &\underset{x \in A}{\text{minimize}} \quad (F_1(x) + (x^T B_1 x)^{\frac{1}{2}}, F_2(x) + (x^T B_2 x)^{\frac{1}{2}}, ..., F_m(x) + (x^T B_m x)^{\frac{1}{2}}) \\ &\text{subject to} \quad G(x, u) \cap (-\mathbb{R}_+) \neq \emptyset, \forall u \in U. \end{aligned} \tag{3.1}$$

The feasible set of the problem (3.1) is defined by

$$S = \{x \in A : G(x, u) \cap (-\mathbb{R}_+) \neq \emptyset, \forall u \in U\}.$$

Definition 5. A point $(x', y') \in \mathbb{R}^n \times \mathbb{R}^m$, with $x' \in S$ and $y' = (y'_1, y'_2, ..., y'_m) \in F(x')$, is called a minimizer of the problem (3.1) if for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, with $x \in S$ and $y = (y_1, y_2, ..., y_m) \in F(x)$,

$$\begin{aligned} &(y_1 + (x^T B_1 x)^{\frac{1}{2}}, y_2 + (x^T B_2 x)^{\frac{1}{2}}, ..., y_m + (x^T B_m x)^{\frac{1}{2}}) \\ &- (y'_1 + (x'^T B_1 x')^{\frac{1}{2}}, y'_2 + (x'^T B_2 x')^{\frac{1}{2}}, ..., y'_m + (x'^T B_m x')^{\frac{1}{2}}) \notin (-\mathbb{R}^m_+) \setminus \{\theta_{\mathbb{R}^m}\}. \end{aligned}$$

Definition 6. A point $(x', y') \in \mathbb{R}^n \times \mathbb{R}^m$, with $x' \in S$ and $y' = (y'_1, y'_2 ..., y'_m) \in F(x')$, is called a weak minimizer of the problem (3.1) if for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, with $x \in S$ and $y = (y_1, y_2, ..., y_m) \in F(x)$,

$$\begin{aligned} &(y_1 + (x^T B_1 x)^{\frac{1}{2}}, y_2 + (x^T B_2 x)^{\frac{1}{2}}, ..., y_m + (x^T B_m x)^{\frac{1}{2}}) \\ &- (y'_1 + (x'^T B_1 x')^{\frac{1}{2}}, y'_2 + (x'^T B_2 x')^{\frac{1}{2}}, ..., y'_m + (x'^T B_m x')^{\frac{1}{2}}) \notin (-\text{int}(\mathbb{R}^m_+)). \end{aligned}$$

Let J be the index set, such that $U = \{u_j : j \in J\}$. Let $x' \in A$. Denote a set $J(x')$ by

$$J(x') = \{j \in J : 0 \in G(x', u_j)\}.$$

Throughout this chapter, we assume that $J(x') \neq \emptyset$.

For special case, when $f = (f_1, f_2, ..., f_m) : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \times U \to \mathbb{R}$ are single-valued maps, we have a multiobjective semi-infinite programming problem ([23]) as

$$\begin{aligned} \underset{x \in A}{\text{minimize}} \quad & (f_1(x) + (x^T B_1 x)^{\frac{1}{2}}, f_2(x) + (x^T B_2 x)^{\frac{1}{2}}, ..., f_m(x) + (x^T B_m x)^{\frac{1}{2}}) \\ \text{subject to} \quad & g(x, u) \in (-\mathbb{R}_+), \forall u \in U, \end{aligned}$$

by considering $F_i(x) = \{f_i(x)\}$, $i = 1, 2, ..., m$, and $G(x, u) = \{g(x, u)\}$ in the problem (3.1).

Das and Nahak [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] introduced the notion of ρ-cone convexity of set-valued maps. They establish the sufficient KKT conditions and develop the duality results for various types of set-valued optimization problems under contingent epiderivative and ρ-cone convexity assumptions. For $\rho = 0$, we have the usual notion of cone convex set-valued maps introduced by Borwein [4].

Definition 7 ([7, 10])**.** Let A be a convex subset of $\mathbb{R}^n$, $e \in \text{int}(K)$ and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map, with $A \subseteq \text{dom}(F)$. Then F is said to be ρ-K-convex with respect to e on A, if there exists $\rho \in \mathbb{R}$ such that

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \rho\lambda(1 - \lambda)\|x_1 - x_2\|^2 e + K, \quad \forall x_1, x_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

In [10], we give an example of ρ-cone convex set-valued map, which is not cone convex. We also characterize ρ-cone convexity of set-valued maps in terms of contingent epiderivative.

Theorem 8 ([10])**.** *Let A be a convex subset of $\mathbb{R}^n$, $e \in \text{int}(K)$ and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be ρ-K-convex with respect to e on A. Let $x' \in A$, and $y' \in F(x')$.*
Then,

$$F(x) - y' \subseteq D_{\uparrow}F(x', y')(x - x') + \rho\|x - x'\|^2 e + K, \forall x \in A.$$

4 Optimality Conditions

Let $\overline{x}_i \in \mathbb{R}^n$, $i = 1, 2, ..., m$. Define maps $.^T B_i \overline{x}_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, ..., m$, by

$$(.^T B_i \overline{x}_i)(x) = x^T B_i \overline{x}_i, \forall x \in \mathbb{R}^n.$$

The gradient vector of $.^T B_i \overline{x}_i$, denoted by $\nabla(.^T B_i \overline{x}_i)$, is given by

$$\nabla(.^T B_i \overline{x}_i) = B_i \overline{x}_i.$$

Let $x' \in A$ and $j \in J(x')$. Define a set-valued map $G(., u_j) : \mathbb{R}^n \to 2^{\mathbb{R}}$ by

$$G(., u_j)(x) = G(x, u_j), \forall x \in \text{dom}(G).$$

We establish the sufficient KKT conditions of the set-valued semi-infinite programming problem (3.1) by using ρ-cone convexity assumptions.

Theorem 9 (Sufficient optimality conditions)**.** *Let A be a nonempty convex subset of $\mathbb{R}^n$, $x' \in S$, and $y' = (y'_1, y'_2, ..., y'_m) \in F(x')$. Let $\overline{x}_i \in \mathbb{R}^n$, $i = 1, 2, ..., m$ and $z' = (z'_j)_{j\in J}$, with $z'_j \in G(x', u_j) \cap (-\mathbb{R}_+)$. Let $\rho_i, \rho'_i, \rho''_j \in \mathbb{R}$, for $i = 1, 2, ..., m$ and $j \in J(x')$. Suppose that F_i, $.^T B_i\overline{x}_i$, $i = 1, 2, ..., m$, and $G(., u_j)$, $j \in J(x')$, are ρ_i-$\mathbb{R}_+$-convex, ρ'_i-$\mathbb{R}_+$-convex, and ρ''_j-$\mathbb{R}_+$-convex set-valued maps, respectively, with respect to 1, on A. Assume that the contingent epiderivatives $D_\uparrow F_i(x', y'_i)$ and $D_\uparrow G(., u_j)(x', z'_j)$ exist. If there exist $y^*_i > 0$, $i = 1, 2, ..., m$, and $z^*_j \geq 0$, $j \in J(x')$, with $z^*_j \neq 0$, for finitely many j, and*

$$\sum_{i=1}^{m} y^*_i(\rho_i + \rho'_i) + \sum_{j\in J(x')} z^*_j \rho''_j \geq 0, \tag{4.1}$$

satisfying the following conditions

$$\Big(\sum_{i=1}^{m} y^*_i(D_\uparrow F_i(x', y'_i) + (B_i\overline{x}_i)^T) + \sum_{j\in J(x')} z^*_j D_\uparrow G(., u_j)(x', z'_j)\Big)(x - x') \geq 0, \forall x \in A, \tag{4.2}$$

$$z^*_j z'_j = 0, \forall j \in J(x'), \tag{4.3}$$

$$\overline{x}_i^T B_i\overline{x}_i \leq 1, i = 1, 2, ..., m, \tag{4.4}$$

and

$$(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i\overline{x}_i, i = 1, 2, ..., m. \tag{4.5}$$

Then (x', y') is a weak minimizer of the problem (3.1).

Proof. Suppose that (x', y') is not a weak minimizer of the problem (3.1). Then there exist $x \in S$ and $y = (y_1, ..., y_m) \in F(x)$, such that

$$\begin{aligned}&(y_1 + (x^T B_1 x)^{\frac{1}{2}}, y_2 + (x^T B_2 x)^{\frac{1}{2}}, ..., y_m + (x^T B_m x)^{\frac{1}{2}})\\ <&(y'_1 + (x'^T B_1 x')^{\frac{1}{2}}, y'_2 + (x'^T B_2 x')^{\frac{1}{2}}, ..., y'_m + (x'^T B_m x')^{\frac{1}{2}}).\end{aligned}$$

As $y^* \in \mathbb{R}^m_+ \setminus \{\theta_{\mathbb{R}^m}\}$, we have

$$\sum_{i=1}^{m} y^*_i(y_i + (x^T B_i x)^{\frac{1}{2}}) < \sum_{i=1}^{m} y^*_i(y'_i + (x'^T B_i x')^{\frac{1}{2}}). \tag{4.6}$$

Since F_i, $i = 1, 2, ..., m$, is ρ_i-$\mathbb{R}_+$-convex with respect to 1, on A and $(x', y'_i) \in \mathrm{gr}(F_i)$, we have

$$F_i(x) - y'_i - \rho_i\|x - x'\|^2 \subseteq D_\uparrow F_i(x', y'_i)(x - x') + \mathbb{R}_+.$$

Hence,

$$y_i - y'_i - \rho_i\|x - x'\|^2 \in D_\uparrow F_i(x', y'_i)(x - x') + \mathbb{R}_+.$$

Therefore,

$$y^*_i(y_i - y'_i) - \rho_i\|x - x'\|^2 y^*_i \geq y^*_i D_\uparrow F_i(x', y'_i)(x - x'). \tag{4.7}$$

Again, as $.^T B_i\overline{x}_i$, $i = 1, 2, ..., m$ and $G(., u_j)$, $j \in J(x')$, are ρ'_i-$\mathbb{R}_+$-convex and ρ''_j-$\mathbb{R}_+$-convex, respectively, with respect to 1, on A, we have

$$x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i - \rho'_i\|x - x'\|^2 \geq (B_i\overline{x}_i)^T (x - x')$$

and

$$G(x, u_j) - z'_j - \rho''_j\|x - x'\|^2 \subseteq D_\uparrow G(., u_j)(x', z'_j)(x - x') + \mathbb{R}_+.$$

As $x \in S$, there exists $z_j \in G(x, u_j) \cap (-\mathbb{R}_+), j \in J(x')$. So, we have

$$z_j - z'_j - \rho''_j\|x - x'\|^2 \in D_\uparrow G(., u_j)(x', z'_j)(x - x') + \mathbb{R}_+.$$

Therefore,

$$y_i^*(x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i) - \rho'_i\|x - x'\|^2 y_i^* \geq y_i^*(B_i\overline{x}_i)^T (x - x') \tag{4.8}$$

and

$$z_j^*(z_j - z'_j) - \rho''_j\|x - x'\|^2 z_j^* \geq z_j^* D_\uparrow G(., u_j)(x', z'_j)(x - x'). \tag{4.9}$$

From (4.7), (4.8), and (4.9), we have

$$\begin{aligned}
&\sum_{i=1}^m y_i^*(y_i - y'_i + x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i) + \sum_{j\in J(x')} z_j^*(z_j - z'_j)\\
&- \|x - x'\|^2 \sum_{i=1}^m y_i^*(\rho_i + \rho'_i) - \|x - x'\|^2 \sum_{j\in J(x')} z_j^*\rho''_j\\
&\geq \Big(\sum_{i=1}^m y_i^*(D_\uparrow F_i(x', y'_i) + (B_i\overline{x}_i)^T) + \sum_{j\in J(x')} z_j^* D_\uparrow G(., u_j)(x', z'_j)\Big)(x - x') \geq 0.
\end{aligned}$$

From (4.1), we have

$$\begin{aligned}
&\sum_{i=1}^m y_i^*(y_i - y'_i + x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i) + \sum_{j\in J(x')} z_j^*(z_j - z'_j)\\
&\geq \|x - x'\|^2\Big(\sum_{i=1}^m y_i^*(\rho_i + \rho'_i) + \sum_{j\in J(x')} z_j^*\rho''_j\Big)\\
&\geq 0.
\end{aligned}$$

As $z_j^* z'_j = 0$ and $z_j \in (-\mathbb{R}_+)$, $\forall j \in J(x')$, we have

$$\sum_{j\in J(x')} z_j^*(z_j - z'_j) \leq 0.$$

Hence,

$$\sum_{i=1}^m y_i^*(y_i - y'_i + x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i) \geq 0.$$

By the generalized Schwarz inequality, we have

$$(x^T B_i x)^{\frac{1}{2}} (\overline{x}_i^T B_i\overline{x}_i)^{\frac{1}{2}} \geq x^T B_i\overline{x}_i.$$

Again, from (4.5), we have

$$(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \overline{x}_i, i = 1, 2, ..., m.$$

Therefore,

$$\sum_{i=1}^{m} y_i^*(y_i - y_i' + (x^T B_i x)^{\frac{1}{2}} (\overline{x}_i^T B_i \overline{x}_i)^{\frac{1}{2}} - (x'^T B_i x')^{\frac{1}{2}}) \geq 0.$$

From (4.4), we have $\overline{x}_i^T B_i \overline{x}_i \leq 1, i = 1, 2, ..., m$. So, we have

$$(x^T B_i x)^{\frac{1}{2}} \geq (x^T B_i x)^{\frac{1}{2}} (\overline{x}_i^T B_i \overline{x}_i)^{\frac{1}{2}}.$$

Hence,

$$\sum_{i=1}^{m} y_i^*(y_i + (x^T B_i x)^{\frac{1}{2}}) \geq \sum_{i=1}^{m} y_i^*(y_i' + (x'^T B_i x')^{\frac{1}{2}}),$$

which contradicts (4.6). Therefore, (x', y') is a weak minimizer of the problem (3.1). □

Example 10. We study a set-valued semi-infinite programming problem for the case, where the objective function and functions attached to constraints are set-valued maps. When the set-valued maps reduce to single-valued maps, our problem coincides with the classical semi-infinite programming problem. So our problem is more generalized in the context of set-valued optimization problems.

We illustrate the problem by the following example.
Let $A = \{(t, 0) : t \in [-1, 1]\} \subset \mathbb{R}^2$.
Let $U = \{(n, n) : n \in \mathbb{N}\}$ be a countably infinite subset of $\mathbb{R}^2$.
Consider two set-valued maps $F : A \subset \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by

$$F(t, 0) = \begin{cases} \{(x - 2t^2, x^2 - 2t^2) : x \geq 0\}, & \text{if } 0 \leq t \leq 1, \\ \{(x - 2t^2, x - 2t^2) : x < 0\}, & \text{if } -1 \leq t < 0, \end{cases}$$

and $G : A \times U \subset \mathbb{R}^2 \times U \to 2^{\mathbb{R}}$ defined as

$$G((t, 0), u) = \begin{cases} \{x^2 + 3t^2 : x \geq 0\}, & \text{if } 0 \leq t \leq 1, u = \{(1, 1)\} \\ \{x + 3t^2 : x < 0\}, & \text{if } -1 \leq t < 0, u \in U \setminus \{(1, 1)\}. \end{cases}$$

Then F is (-2)-$\mathbb{R}^2_+$-convex with respect to $(1, 1)$ and $G(., u)$, $u \in U$, is 3-$\mathbb{R}_+$-convex with respect to 1 on A.
Let $\overline{x}_1 = (1, -1), \overline{x}_2 = (1, 1) \in \mathbb{R}^2$. Consider two positive semi-definite (symmetric) real matrices

$$B_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} 2 & 6 \\ 6 & 18 \end{pmatrix}$$

Then $.^T B_i \overline{x}_i$, $i = 1, 2$, are 2-$\mathbb{R}_+$-convex and 3-$\mathbb{R}_+$-convex, respectively with respect to 1 on A.
Let $x' = (0, 0)$ and $y' = (0, 0) \in F(x')$.
Here the index set $J = \mathbb{N}$ such that $U = \{u_j : j \in J\}$, where $u_j = (j, j)$. Hence

$$J(x') = \{j \in \mathbb{N} : 0 \in G(x', u_j)\} = \{1\}.$$

It is clear that for $y^* = (1, 1)$ and $z^* = 1$, the sufficient optimality conditions are satisfied. Therefore, $((0, 0), (0, 0))$ is a weak minimizer of the problem (3.1).

5 Mond-Weir Type Dual

We consider a Mond-Weir type dual (5.1) of (3.1), where F_i and $G(., u_j)$ are contingent epiderivable set-valued maps.

$$\begin{aligned}
&\text{maximize} \quad (y'_1 + (x'^T B_1 \overline{x}_1), y'_2 + (x'^T B_2 \overline{x}_2), ..., y'_m + (x'^T B_m \overline{x}_m)) \qquad (5.1)\\
&\text{subject to}\\
&\Big(\sum_{i=1}^{m} y_i^*(D_\uparrow F_i(x', y'_i) + (B_i \overline{x}_i)^T) + \sum_{j \in J(x')} z_j^* D_\uparrow G(., u_j)(x', z'_j)\Big)(x - x')\\
&\geq 0, \forall x \in A,\\
&\sum_{j \in J(x')} z_j^* z'_j \geq 0,\\
&\overline{x}_i^T B_i \overline{x}_i \leq 1, \overline{x}_i \in \mathbb{R}^n, i = 1, 2, ..., m,\\
&x' \in A, y' = (y'_1, y'_2, ..., y'_m) \in F(x'), z' = (z'_j)_{j \in J}, z'_j \in G(x', u_j), j \in J,\\
&y_i^* > 0, i = 1, 2, ..., m, \sum_{i=1}^{m} y_i^* = 1, z^* = (z_j^*)_{j \in J}, z_j^* \geq 0, j \in J,\\
&\text{and } z_j^* \neq 0, \text{ for finitely many } j \in J.
\end{aligned}$$

Definition 11. A feasible point $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ of (5.1) is called a weak maximizer of (5.1) if for all feasible points $(x, y, \widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_m, z, y_1^*, z_1^*)$ of (5.1),

$$\begin{aligned}
&(y'_1 + (x'^T B_1 \overline{x}_1), y'_2 + (x'^T B_2 \overline{x}_2), ..., y'_m + (x'^T B_m \overline{x}_m))\\
&\not< (y_1 + (x^T B_1 \widehat{x}_1), y_2 + (x^T B_2 \widehat{x}_2), ..., y_m + (x^T B_m \widehat{x}_m)),
\end{aligned}$$

where $y = (y_1, y_2, ..., y_m), y' = (y'_1, y'_2, ..., y'_m) \in \mathbb{R}^m$.

Theorem 12 (Weak duality). *Let A be a nonempty convex subset of $\mathbb{R}^n$ and x_0 be an element of the feasible set S of (3.1). Let $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ be a feasible point of the problem (5.1). Let $\rho_i, \rho'_i, \rho''_j \in \mathbb{R}$, for $i = 1, 2, ..., m$ and $j \in J(x')$. Suppose that F_i, $.^T B_i \overline{x}_i$, $i = 1, 2, ..., m$, and $G(., u_j)$, $j \in J(x')$, are ρ_i-$\mathbb{R}_+$-convex, ρ'_i-$\mathbb{R}_+$-convex, and ρ''_j-$\mathbb{R}_+$-convex set-valued maps, respectively, with respect to 1, on A, satisfying (4.1). Assume that the contingent epiderivatives $D_\uparrow F_i(x', y'_i)$ and $D_\uparrow G(., u_j)(x', z'_j)$ exist. Then,*

$$\begin{aligned}
&(F_1(x_0) + (x_0^T B_1 x_0)^{\frac{1}{2}}, F_2(x_0) + (x_0^T B_2 x_0)^{\frac{1}{2}}, ..., F_m(x_0) + (x_0^T B_m x_0)^{\frac{1}{2}})\\
&- (y'_1 + (x'^T B_1 \overline{x}_1), y'_2 + (x'^T B_2 \overline{x}_2), ..., y'_m + (x'^T B_m \overline{x}_m)) \subseteq \mathbb{R}^m \setminus (-\text{int}(\mathbb{R}^m_+)).
\end{aligned}$$

Proof. We prove the theorem by the method of contradiction. Suppose that there exists $y_i \in F_i(x_0)$, $i = 1, 2, ..., m$, such that

$$\begin{aligned}
&(y_1 + (x_0^T B_1 x_0)^{\frac{1}{2}}, y_2 + (x_0^T B_2 x_0)^{\frac{1}{2}}, ..., y_m + (x_0^T B_m x_0)^{\frac{1}{2}})\\
&- (y'_1 + (x'^T B_1 \overline{x}_1), y'_2 + (x'^T B_2 \overline{x}_2), ..., y'_m + (x'^T B_m \overline{x}_m)) \in (-\text{int}(\mathbb{R}^m_+)).
\end{aligned}$$

As $y^* \in \mathbb{R}^m_+ \setminus \{\theta_{\mathbb{R}^m}\}$, we have

$$\sum_{i=1}^{m} y_i^*(y_i + (x_0^T B_i x_0)^{\frac{1}{2}}) < \sum_{i=1}^{m} y_i^*(y'_i + (x'^T B_i \overline{x}_i)). \qquad (5.2)$$

As F_i, $i = 1, 2, ..., m$, is ρ_i-$\mathbb{R}_+$-convex with respect to 1 on A and $(x', y'_i) \in \mathrm{gr}(F_i)$, we have

$$F_i(x_0) - y'_i - \rho_i \|x_0 - x'\|^2 \subseteq D_\uparrow F_i(x', y'_i)(x_0 - x') + \mathbb{R}_+.$$

Hence,

$$y_i - y'_i - \rho_i \|x_0 - x'\|^2 \in D_\uparrow F_i(x', y'_i)(x_0 - x') + \mathbb{R}_+.$$

Therefore,

$$y^*_i(y_i - y'_i) - \rho_i \|x_0 - x'\|^2 y^*_i \geq y^*_i D_\uparrow F_i(x', y'_i)(x_0 - x'). \tag{5.3}$$

As $.^T B_i \overline{x}_i$, $i = 1, 2, ..., m$, is ρ'_i-$\mathbb{R}_+$-convex with respect to 1 on A, we have

$$x_0^T B_i \overline{x}_i - x'^T B_i \overline{x}_i - \rho'_i \|x_0 - x'\|^2 \geq (B_i \overline{x}_i)^T (x_0 - x').$$

Therefore,

$$y^*_i(x_0^T B_i \overline{x}_i - x'^T B_i \overline{x}_i) - \rho'_i \|x_0 - x'\|^2 y^*_i \geq y^*_i (B_i \overline{x}_i)^T (x_0 - x'). \tag{5.4}$$

Again, as $G(., u_j)$, $j \in J(x')$, is ρ''_j-$\mathbb{R}_+$-convex with respect to 1 on A and $z'_j \in G(x', u_j)$, $j \in J(x')$, we have

$$G(x_0, u_j) - z'_j - \rho''_j \|x_0 - x'\|^2 \subseteq D_\uparrow G(., u_j)(x', z'_j)(x_0 - x') + \mathbb{R}_+.$$

As $x_0 \in S$, there exists $z_j \in G(x_0, u_j) \cap (-\mathbb{R}_+)$, $j \in J(x')$, we have

$$z_j - z'_j - \rho''_j \|x_0 - x'\|^2 \in D_\uparrow G(., u_j)(x', z'_j)(x_0 - x') + \mathbb{R}_+.$$

Hence,

$$z^*_j(z_j - z'_j) - \rho''_j \|x_0 - x'\|^2 z^*_j \geq z^*_j D_\uparrow G(., u_j)(x', z'_j)(x_0 - x'). \tag{5.5}$$

From (5.3), (5.4), and (5.5), we have

$$\begin{aligned}
&\sum_{i=1}^m y^*_i(y_i - y'_i + x_0^T B_i \overline{x}_i - x'^T B_i \overline{x}_i) + \sum_{j \in J(x')} z^*_j(z_j - z'_j) \\
&- \|x_0 - x'\|^2 \sum_{i=1}^m y^*_i(\rho_i + \rho'_i) - \|x_0 - x'\|^2 \sum_{j \in J(x')} z^*_j \rho''_j \\
&\geq \Big(\sum_{i=1}^m y^*_i (D_\uparrow F_i(x', y'_i) + (B_i \overline{x}_i)^T) + \sum_{j \in J(x')} z^*_j D_\uparrow G(., u_j)(x', z'_j) \Big)(x_0 - x') \\
&\geq 0.
\end{aligned}$$

From (4.1), we have

$$\begin{aligned}
&\sum_{i=1}^m y^*_i(y_i - y'_i + x_0^T B_i \overline{x}_i - x'^T B_i \overline{x}_i) + \sum_{j \in J(x')} z^*_j(z_j - z'_j) \\
&\geq \|x_0 - x'\|^2 \Big(\sum_{i=1}^m y^*_i(\rho_i + \rho'_i) + \sum_{j \in J(x')} z^*_j \rho''_j \Big) \\
&\geq 0.
\end{aligned}$$

As $\sum_{j \in J(x')} z_j^* z_j' \geq 0$, $z_j^* \geq 0$, and $z_j \in (-\mathbb{R}_+)$, $j \in J(x')$, we have

$$\sum_{j \in J(x')} z_j^*(z_j - z_j') \leq 0.$$

Hence,

$$\sum_{i=1}^{m} y_i^*(y_i - y_i' + x_0^T B_i \overline{x}_i - x'^T B_i \overline{x}_i) \geq 0.$$

By the generalized Schwarz inequality, we have

$$(x_0^T B_i x_0)^{\frac{1}{2}} (\overline{x}_i^T B_i \overline{x}_i)^{\frac{1}{2}} \geq x_0^T B_i \overline{x}_i.$$

Again, from the constraints of (5.1), we have

$$\overline{x}_i^T B_i \overline{x}_i \leq 1, i = 1, 2, ..., m.$$

Hence, $(x_0^T B_i x_0)^{\frac{1}{2}} \geq x_0^T B_i \overline{x}_i$. It shows that

$$\sum_{i=1}^{m} y_i^*(y_i + (x_0^T B_i x_0)^{\frac{1}{2}}) \geq \sum_{i=1}^{m} y_i^*(y_i' + (x'^T B_i \overline{x}_i)),$$

which contradicts (5.2). Therefore,

$$\begin{aligned}&(y_1 + (x_0^T B_1 x_0)^{\frac{1}{2}}, y_2 + (x_0^T B_2 x_0)^{\frac{1}{2}}, ..., y_m + (x_0^T B_m x_0)^{\frac{1}{2}}) \\ &- (y_1' + (x'^T B_1 \overline{x}_1), y_2' + (x'^T B_2 \overline{x}_2), ..., y_m' + (x'^T B_m \overline{x}_m)) \notin (-\text{int}(\mathbb{R}_+^m)).\end{aligned}$$

Hence,

$$\begin{aligned}&(F_1(x_0) + (x_0^T B_1 x_0)^{\frac{1}{2}}, F_2(x_0) + (x_0^T B_2 x_0)^{\frac{1}{2}}, ..., F_m(x_0) + (x_0^T B_m x_0)^{\frac{1}{2}}) \\ &- (y_1' + (x'^T B_1 \overline{x}_1), y_2' + (x'^T B_2 \overline{x}_2), ..., y_m' + (x'^T B_m \overline{x}_m)) \subseteq \mathbb{R}^m \setminus (-\text{int}(\mathbb{R}_+^m)),\end{aligned}$$

which completes the proof of the theorem. □

Theorem 13 (Strong duality). *Let (x', y') be a weak minimizer of the problem (3.1), $z' = (z_j')_{j \in J}$, $z_j' \in G(x', u_j) \cap (-\mathbb{R}_+), j \in J$, and $\overline{x}_i \in \mathbb{R}^n, i = 1, 2, ..., m$. Assume that for some $y_i^* > 0$, $i = 1, 2, ..., m$, with $\sum_{i=1}^{m} y_i^* = 1$ and $z_j^* \geq 0$, with $z_j^* \neq 0$ for finitely many $j \in J$ and $z_j^* = 0$, $\forall j \in J \setminus J(x')$, Eqs. (4.2) - (4.5) are satisfied at $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$. Then $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ is a feasible solution of (5.1). Furthermore, if the weak duality Theorem 12 holds between the problems (3.1) and (5.1), then $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ is a weak maximizer of (5.1).*

Proof. As (4.2) - (4.5) are satisfied at $(x', y', \overline{x}_1, ..., \overline{x}_m, z', y^*, z^*)$, we have

$$\Big(\sum_{i=1}^{m} y_i^*(D_\uparrow F_i(x', y_i') + (B_i \overline{x}_i)^T) + \sum_{j \in J(x')} z_j^* D_\uparrow G(., u_j)(x', z_j')\Big)(x - x') \geq 0, \forall x \in A,$$

$$z_j^* z_j' = 0, \forall j \in J(x'),$$

$$\overline{x}_i^T B_i \overline{x}_i \leq 1, i = 1, 2, ..., m,$$

and

$$(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \overline{x}_i, i = 1, 2, ..., m.$$

Hence, $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ is a feasible solution of (5.1).
Suppose that the weak duality Theorem 12 holds between the problems (3.1) and (5.1). If $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ is not a weak maximizer of (5.1), then there exists a feasible point $(x, y, \widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_m, z, y_1^*, z_1^*)$ of (5.1), such that

$$\begin{aligned}&(y'_1 + (x'^T B_1 \overline{x}_1), y'_2 + (x'^T B_2 \overline{x}_2), ..., y'_m + (x'^T B_m \overline{x}_m))\\ &- (y_1 + (x^T B_1 \widehat{x}_1), y_2 + (x^T B_2 \widehat{x}_2), ..., y_m + (x^T B_m \widehat{x}_m)) \in (-\text{int}(\mathbb{R}^m_+)).\end{aligned}$$

where $y = (y_1, ..., y_m) \in \mathbb{R}^m$. As $(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \overline{x}_i$, $i = 1, 2, ..., m$, we have

$$\begin{aligned}&(y'_1 + (x'^T B_1 x')^{\frac{1}{2}}, y'_2 + (x'^T B_2 x')^{\frac{1}{2}}, ..., y'_m + (x'^T B_m x')^{\frac{1}{2}})\\ &- (y_1 + (x^T B_1 \widehat{x}_1), y_2 + (x^T B_2 \widehat{x}_2), ..., y_m + (x^T B_m \widehat{x}_m)) \in (-\text{int}(\mathbb{R}^m_+)).\end{aligned}$$

It contradicts the weak duality Theorem 12 between (3.1) and (5.1). Consequently, $(x', y', \overline{x}_1, ..., \overline{x}_m, z', y^*, z^*)$ is a weak maximizer of (5.1). □

Theorem 14 (Converse duality)**.** *Let A be a nonempty convex subset of $\mathbb{R}^n$. Suppose that $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ is a feasible point of the problem (5.1), with*

$$(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \overline{x}_i, i = 1, 2, ..., m.$$

Let $\rho_i, \rho'_i, \rho''_j \in \mathbb{R}$, for $i = 1, 2, ..., m$ and $j \in J(x')$. Suppose that F_i, $.^T B_i \overline{x}_i$, $i = 1, 2, ..., m$, and $G(., u_j)$, $j \in J(x')$, are ρ_i-$\mathbb{R}_+$-convex, ρ'_i-$\mathbb{R}_+$-convex, and ρ''_j-$\mathbb{R}_+$-convex set-valued maps, respectively, with respect to 1, on A, satisfying the condition (4.1). Assume that the contingent epiderivatives $D_\uparrow F_i(x', y'_i)$ and $D_\uparrow G(., u_j)(x', z'_j)$ exist. If $x' \in S$, then (x', y') is a weak minimizer of (3.1).

Proof. Suppose that (x', y') is not a weak minimizer of the problem (3.1). Then there exist $x \in S$ and $y = (y_1, y_2, ..., y_m) \in F(x)$, such that

$$\begin{aligned}&(y_1 + (x^T B_1 x)^{\frac{1}{2}}, y_2 + (x^T B_2 x)^{\frac{1}{2}}, ..., y_m + (x^T B_m x)^{\frac{1}{2}})\\ &- (y'_1 + (x'^T B_1 x')^{\frac{1}{2}}, y'_2 + (x'^T B_2 x')^{\frac{1}{2}}, ..., y'_m + (x'^T B_m x')^{\frac{1}{2}}) \in (-\text{int}(\mathbb{R}^m_+)).\end{aligned}$$

As $y^* \in \mathbb{R}^m_+ \setminus \{\theta_{\mathbb{R}^m}\}$, we have

$$\sum_{i=1}^m y_i^* (y_i + (x^T B_i x)^{\frac{1}{2}}) < \sum_{i=1}^m y_i^* (y'_i + (x'^T B_i x')^{\frac{1}{2}}). \tag{5.6}$$

Since F_i, $i = 1, 2, ..., m$, is ρ_i-$\mathbb{R}_+$-convex with respect to 1, on A and $(x', y'_i) \in \text{gr}(F_i)$, we have

$$F_i(x) - y'_i - \rho_i \|x - x'\|^2 \subseteq D_\uparrow F_i(x', y'_i)(x - x') + \mathbb{R}_+.$$

Hence,

$$y_i - y'_i - \rho_i \|x - x'\|^2 \in D_\uparrow F_i(x', y'_i)(x - x') + \mathbb{R}_+.$$

Therefore,

$$y_i^*(y_i - y_i') - \rho_i\|x - x'\|^2 y_i^* \geq y_i^* D_\uparrow F_i(x', y_i')(x - x'). \tag{5.7}$$

Again, as $.^T B_i\overline{x}_i$, $i = 1, 2, ..., m$, and $G(., u_j)$, $j \in J(x')$, are ρ_i'-$\mathbb{R}_+$-convex and ρ_j''-$\mathbb{R}_+$-convex, respectively, with respect to 1, on A, we have

$$x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i - \rho_i'\|x - x'\|^2 \geq (B_i\overline{x}_i)^T(x - x')$$

and

$$G(x, u_j) - z_j' - \rho_j''\|x - x'\|^2 \subseteq D_\uparrow G(., u_j)(x', z_j')(x - x') + \mathbb{R}_+.$$

As $x \in S$, there exists $z_j \in G(x, u_j) \cap (-\mathbb{R}_+)$, $\forall j \in J(x')$. So,

$$z_j - z_j' - \rho_j''\|x - x'\|^2 \in D_\uparrow G(., u_j)(x', z_j')(x - x') + \mathbb{R}_+.$$

Therefore,

$$y_i^*(x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i) - \rho_i'\|x - x'\|^2 y_i^* \geq y_i^*(B_i\overline{x}_i)^T(x - x') \tag{5.8}$$

and

$$z_j^*(z_j - z_j') - \rho_j''\|x - x'\|^2 z_j^* \geq z_j^* D_\uparrow G(., u_j)(x', z_j')(x - x'). \tag{5.9}$$

From (5.7), (5.8), and (5.9), we have

$$\begin{aligned}
&\sum_{i=1}^m y_i^*(y_i - y_i' + x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i) + \sum_{j\in J(x')} z_j^*(z_j - z_j') \\
&- \|x - x'\|^2 \sum_{i=1}^m y_i^*(\rho_i + \rho_i') - \|x - x'\|^2 \sum_{j\in J(x')} z_j^*\rho_j'' \\
&\geq \Big(\sum_{i=1}^m y_i^*(D_\uparrow F_i(x', y_i') + (B_i\overline{x}_i)^T) + \sum_{j\in J(x')} z_j^* D_\uparrow G(., u_j)(x', z_j')\Big)(x - x') \geq 0.
\end{aligned}$$

By (4.1), we have

$$\begin{aligned}
&\sum_{i=1}^m y_i^*(y_i - y_i' + x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i) + \sum_{j\in J(x')} z_j^*(z_j - z_j') \\
&\geq \|x - x'\|^2\Big(\sum_{i=1}^m y_i^*(\rho_i + \rho_i') + \sum_{j\in J(x')} z_j^*\rho_j''\Big) \\
&\geq 0.
\end{aligned}$$

As $\sum_{j\in J(x')} z_j^* z_j' \geq 0$, $z_j^* \geq 0$, and $z_j \in (-\mathbb{R}_+)$, $j \in J(x')$, we have

$$\sum_{j\in J(x')} z_j^*(z_j - z_j') \leq 0.$$

Hence,

$$\sum_{i=1}^m y_i^*(y_i - y_i' + x^T B_i\overline{x}_i - x'^T B_i\overline{x}_i) \geq 0.$$

By the generalized Schwarz inequality, we have

$$(x^T B_i x)^{\frac{1}{2}} (\overline{x}_i^T B_i \overline{x}_i)^{\frac{1}{2}} \geq x^T B_i \overline{x}_i.$$

Again, by assumption, we have

$$(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \overline{x}_i, i = 1, 2, ..., m.$$

Therefore,

$$\sum_{i=1}^{m} y_i^* (y_i - y_i' + (x^T B_i x)^{\frac{1}{2}} (\overline{x}_i^T B_i \overline{x}_i)^{\frac{1}{2}} - (x'^T B_i x')^{\frac{1}{2}}) \geq 0.$$

As $\overline{x}_i^T B_i \overline{x}_i \leq 1, i = 1, 2, ..., m$, (from the constraints of (5.1)), we have

$$(x^T B_i x)^{\frac{1}{2}} \geq (x^T B_i x)^{\frac{1}{2}} (\overline{x}_i^T B_i \overline{x}_i)^{\frac{1}{2}}.$$

Hence,

$$\sum_{i=1}^{m} y_i^* (y_i + (x^T B_i x)^{\frac{1}{2}}) \geq \sum_{i=1}^{m} y_i^* (y_i' + (x'^T B_i x')^{\frac{1}{2}}),$$

which contradicts (5.6). Therefore, (x', y') is a weak minimizer of the problem (3.1). □

6 Wolfe Type Dual

We consider a Wolfe type dual (6.1) of (3.1), where F_i and $G(., u_j)$ are contingent epiderivable set-valued maps.

$$\text{maximize} \quad (y_1' + (x'^T B_1 \overline{x}_1), y_2' + (x'^T B_2 \overline{x}_2), ..., y_m' + (x'^T B_m \overline{x}_m)) + \Big(\sum_{j \in J(x')} z_j^* z_j' \Big) \mathbf{1}_{\mathbb{R}^m} \tag{6.1}$$

subject to

$$\Big(\sum_{i=1}^{m} y_i^* (D_\uparrow F_i(x', y_i') + (B_i \overline{x}_i)^T) + \sum_{j \in J(x')} z_j^* D_\uparrow G(., u_j)(x', z_j') \Big)(x - x')$$
$$\geq 0, \forall x \in A,$$
$$\overline{x}_i^T B_i \overline{x}_i \leq 1, \overline{x}_i \in \mathbb{R}^n, i = 1, 2, ..., m,$$
$$x' \in A, y' = (y_1', y_2', ..., y_m') \in F(x'), z' = (z_j')_{j \in J(x')}, z_j' \in G(x', u_j), j \in J,$$
$$y_i^* > 0, i = 1, 2, ..., m, \sum_{i=1}^{m} y_i^* = 1, \text{ and } z^* = (z_j^*)_{j \in J}, z_j^* \geq 0, j \in J,$$
$$\text{and } z_j^* \neq 0, \text{ for finitely many } j \in J.$$

Definition 15. A feasible point $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ of (6.1) is called a weak maximizer of (6.1) if for all feasible points $(x, y, \widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_m, z, y_1^*, z_1^*)$ of (6.1),

$$(y_1' + (x'^T B_1 \overline{x}_1), y_2' + (x'^T B_2 \overline{x}_2), ..., y_m' + (x'^T B_m \overline{x}_m)) + \Big(\sum_{j \in J(x')} z_j^* z_j' \Big) \mathbf{1}_{\mathbb{R}^m}$$
$$\not< (y_1 + (x^T B_1 \widehat{x}_1), y_2 + (x^T B_2 \widehat{x}_2), ..., y_m + (x^T B_m \widehat{x}_m)) + \Big(\sum_{j \in J(x')} z_j^* z_j \Big) \mathbf{1}_{\mathbb{R}^m},$$

where $y = (y_1, y_2, ..., y_m), y' = (y_1', y_2', ..., y_m') \in \mathbb{R}^m$.

We prove the duality results of Wolfe type of the problem (3.1). The proofs are very similar to Theorems 12 - 14, and hence omitted.

Theorem 16 (Weak duality). *Let A be a nonempty convex subset of $\mathbb{R}^n$ and $x_0 \in S$. Let $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ be a feasible point of the problem (6.1). Let $\rho_i, \rho'_i, \rho''_j \in \mathbb{R}$, for $i = 1, 2, ..., m$ and $j \in J(x')$. Suppose that F_i, $.^T B_i \overline{x}_i$, $i = 1, 2, ..., m$, and $G(., u_j)$, $j \in J(x')$, are ρ_i-$\mathbb{R}_+$-convex, ρ'_i-$\mathbb{R}_+$-convex, and ρ''_j-$\mathbb{R}_+$-convex set-valued maps, respectively, with respect to 1, on A, satisfying (4.1). Let the contingent epiderivatives $D_\uparrow F_i(x', y'_i)$ and $D_\uparrow G(., u_j)(x', z'_j)$ exist. Then,*

$$\begin{aligned}&(F_1(x_0) + (x_0^T B_1 x_0)^{\frac{1}{2}}, F_2(x_0) + (x_0^T B_2 x_0)^{\frac{1}{2}}, ..., F_m(x_0) + (x_0^T B_m x_0)^{\frac{1}{2}})\\ \nless\ &(y'_1 + (x'^T B_1 \overline{x}_1), y'_2 + (x'^T B_2 \overline{x}_2), ..., y'_m + (x'^T B_m \overline{x}_m)) + \Big(\sum_{j \in J(x')} z_j^* z'_j\Big) \boldsymbol{1}_{\mathbb{R}^m}.\end{aligned}$$

Theorem 17 (Strong duality). *Let (x', y') be a weak minimizer of the problem (3.1), $z' = (z'_j)_{j \in J}$, $z'_j \in G(x', u_j) \cap (-\mathbb{R}_+), j \in J$, and $\overline{x}_i \in \mathbb{R}^n, i = 1, 2, ..., m$. Assume that for some $y_i^* > 0$, $i = 1, 2, ..., m$, with $\sum_{i=1}^{m} y_i^* = 1$ and $z_j^* \geq 0$, with $z_j^* \neq 0$ for finitely many $j \in J$ and $z_j^* = 0$, $\forall j \in J \setminus J(x')$, Eqs. (4.2) - (4.5) are satisfied at $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$. Then $(x', y', \overline{x}_1, ..., \overline{x}_m, z', y^*, z^*)$ is a feasible solution of the problem (6.1). Furthermore, if the weak duality Theorem 16 holds between the problems (3.1) and (6.1), then $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ is a weak maximizer of (6.1).*

Theorem 18 (Converse duality). *Let A be a nonempty convex subset of $\mathbb{R}^n$. Suppose that $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ is a feasible point of the problem (6.1), with*

$$(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \overline{x}_i, i = 1, 2, ..., m$$

and

$$\sum_{j \in J(x')} z_j^* z'_j \geq 0.$$

Let $\rho_i, \rho'_i, \rho''_j \in \mathbb{R}$, for $i = 1, 2, ..., m$ and $j \in J(x')$. Suppose that F_i, $.^T B_i \overline{x}_i$, $i = 1, 2, ..., m$, and $G(., u_j)$, $j \in J(x')$, are ρ_i-$\mathbb{R}_+$-convex, ρ'_i-$\mathbb{R}_+$-convex, and ρ''_j-$\mathbb{R}_+$-convex set-valued maps, respectively, with respect to 1, on A, satisfying the condition (4.1). Assume that the contingent epiderivatives $D_\uparrow F_i(x', y'_i)$ and $D_\uparrow G(., u_j)(x', z'_j)$ exist. If $x' \in S$, then (x', y') is a weak minimizer of (3.1).

7 Mixed Type Dual

We consider a mixed type dual (7.1) of (3.1), where F_i and $G(.,u_j)$ are contingent epiderivable set-valued maps.

$$\text{maximize} \quad (y'_1+(x'^T B_1\overline{x}_1), y'_2+(x'^T B_2\overline{x}_2),...,y'_m+(x'^T B_m\overline{x}_m))+\Big(\sum_{j\in J(x')} z_j^* z'_j\Big)\mathbf{1}_{\mathbb{R}^m} \tag{7.1}$$

subject to

$$\Big(\sum_{i=1}^m y_i^*(D_\uparrow F_i(x',y'_i)+(B_i\overline{x}_i)^T)+\sum_{j\in J(x')} z_j^* D_\uparrow G(.,u_j)(x',z'_j)\Big)(x-x')$$
$$\geq 0, \forall x\in A,$$
$$z_j^* z'_j \geq 0, \forall j\in J(x'),$$
$$\overline{x}_i^T B_i \overline{x}_i \leq 1, \overline{x}_i\in\mathbb{R}^n, i=1,2,...,m,$$
$$x'\in A, y'=(y'_1,y'_2,...,y'_m)\in F(x'), z'=(z'_j)_{j\in J}, z'_j\in G(x',u_j), j\in J,$$
$$y_i^*>0, i=1,2,...,m, \sum_{i=1}^m y_i^*=1, \text{ and } z^*=(z_j^*)_{j\in J}, z_j^*\geq 0, j\in J,$$
$$\text{and } z_j^*\neq 0, \text{ for finitely many } j\in J.$$

Definition 19. A feasible point $(x',y',\overline{x}_1,\overline{x}_2,...,\overline{x}_m,z',y^*,z^*)$ of (7.1) is called a weak maximizer of (7.1) if for all feasible points $(x,y,\widehat{x}_1,\widehat{x}_2,...,\widehat{x}_m,z,y_1^*,z_1^*)$ of (7.1),

$$(y'_1+(x'^T B_1\overline{x}_1), y'_2+(x'^T B_2\overline{x}_2),...,y'_m+(x'^T B_m\overline{x}_m))+\Big(\sum_{j\in J(x')} z_j^* z'_j\Big)\mathbf{1}_{\mathbb{R}^m}$$
$$\not< (y_1+(x^T B_1\widehat{x}_1), y_2+(x^T B_2\widehat{x}_2),...,y_m+(x^T B_m\widehat{x}_m))+\Big(\sum_{j\in J(x')} z_j^* z_j\Big)\mathbf{1}_{\mathbb{R}^m},$$

where $y=(y_1,y_2,...,y_m), y'=(y'_1,y'_2,...,y'_m)\in\mathbb{R}^m$.

We prove the duality results of mixed type of the problem (3.1). The proofs are very similar to Theorems 12 - 14, and hence omitted.

Theorem 20 (Weak duality)**.** *Let A be a nonempty convex subset of $\mathbb{R}^n$ and $x_0\in S$. Let $(x',y',\overline{x}_1,\overline{x}_2,...,\overline{x}_m,z',y^*,z^*)$ be a feasible point of the problem (7.1). Let $\rho_i,\rho'_i,\rho''_j\in\mathbb{R}$, for $i=1,2,...,m$ and $j\in J(x')$. Suppose that F_i, $.^T B_i\overline{x}_i$, $i=1,2,...,m$, and $G(.,u_j)$, $j\in J(x')$, are ρ_i-$\mathbb{R}_+$-convex, ρ'_i-$\mathbb{R}_+$-convex, and ρ''_j-$\mathbb{R}_+$-convex set-valued maps, respectively, with respect to 1, on A, satisfying the condition (4.1). Let the contingent epiderivatives $D_\uparrow F_i(x',y'_i)$ and $D_\uparrow G(.,u_j)(x',z'_j)$ exist. Then,*

$$(F_1(x_0)+(x_0^T B_1 x_0)^{\frac{1}{2}}, F_2(x_0)+(x_0^T B_2 x_0)^{\frac{1}{2}},...,F_m(x_0)+(x_0^T B_m x_0)^{\frac{1}{2}})$$
$$\not< (y'_1+(x'^T B_1\overline{x}_1), y'_2+(x'^T B_2\overline{x}_2),...,y'_m+(x'^T B_m\overline{x}_m))+\Big(\sum_{j\in J(x')} z_j^* z'_j\Big)\mathbf{1}_{\mathbb{R}^m}.$$

Theorem 21 (Strong duality)**.** *Let (x',y') be a weak minimizer of the problem (3.1), $z'=(z'_j)_{j\in J}$, $z'_j\in G(x',u_j)\cap(-\mathbb{R}_+), j\in J$, and $\overline{x}_i\in\mathbb{R}^n, i=1,2,...,m$. Assume that for some $y_i^*>0$, $i=1,2,...,m$, with $\sum_{i=1}^m y_i^*=1$ and $z_j^*\geq 0$, with $z_j^*\neq 0$ for* *finitely many* $j \in J$ *and* $z_j^* = 0$, $\forall j \in J \setminus J(x')$, *Eqs.* (4.2) - (4.5) *are satisfied at* $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$. *Then* $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ *is a feasible solution of the problem* (7.1)*. Furthermore, if the weak duality Theorem 20 holds between the problems* (3.1) *and* (7.1)*, then* $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ *is a weak maximizer of* (7.1)*.*

Theorem 22 (Converse duality)**.** *Let* A *be a nonempty convex subset of* $\mathbb{R}^n$*. Suppose that* $(x', y', \overline{x}_1, \overline{x}_2, ..., \overline{x}_m, z', y^*, z^*)$ *is a feasible point of the problem* (7.1)*, with* $(x'^T B_i x')^{\frac{1}{2}} = x'^T B_i \overline{x}_i, i = 1, 2, ..., m$*. Let* $\rho_i, \rho_i', \rho_j'' \in \mathbb{R}$*, for* $i = 1, 2, ..., m$ *and* $j \in J(x')$*. Suppose that* F_i*,* $.^T B_i \overline{x}_i$*,* $i = 1, 2, ..., m$*, and* $G(., u_j)$*,* $j \in J(x')$*, are* ρ_i*-*$\mathbb{R}_+$*-convex,* ρ_i'*-*$\mathbb{R}_+$*-convex, and* ρ_j''*-*$\mathbb{R}_+$*-convex set-valued maps, respectively, with respect to 1, on* A*, satisfying the condition* (4.1)*. Assume that the contingent epiderivatives* $D_\uparrow F_i(x', y_i')$ *and* $D_\uparrow G(., u_j)(x', z_j')$ *exist. If* $x' \in S$*, then* (x', y') *is a weak minimizer of* (3.1)*.*

8 Conclusions

In this chapter, we establish the sufficient KKT conditions of a set-valued semi-infinite programming problem (3.1) by using ρ-cone convexity assumptions. The duality results of Mond-Weir, Wolfe, and mixed types of the problem (3.1) are also studied.

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