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# Extension ideals and closure operator in a 0-distributive lattice

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**Abstract**
In a bounded 0-distributive lattice, extension ideals are defined and studied. The set of all extension ideals forms a sublattice of the lattice of all $\alpha$-ideals. A necessary and sufficient condition for a 0-distributive lattice to be quasi-complemented using extension ideals is established. A closure operator on the ideal lattice is introduced using the extension ideals and it is shown that the set of all ideals of $L$ closed with respect to this closure operator are the $\alpha$-ideals. Finally, $\alpha$-ideals are characterized in terms of extension ideals.

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## 1 Introduction

As a generalization of the concept of distributive lattices on one hand and pseudo-complemented lattices on the other, 0-distributive lattices are introduced by Varlet [6]. C. Jayaram [2] defined and studied $\alpha$-ideals in a 0-distributive lattice. In [2], the author established the existence of the smallest $\alpha$-ideal containing the given ideal and noted that the set of all $\alpha$-ideals forms a complete distributive lattice with specially defined binary operations on it. Additional properties of $\alpha$-ideals in a 0-distributive lattice are obtained by Pawar et. al. in [3], [4] and [5].

In this paper we introduce and study some properties of extension ideals in a 0-distributive lattice. It is proved that extension ideal is the smallest $\alpha$-ideal containing the given element (or equivalently, the given principal ideal). With the help of extension ideals a closure operator on the lattice of all ideals of a 0-distributive lattice is introduced and some characterizations of $\alpha$-ideals in terms of extension ideals are established. It is observed that the set of all extension ideals of a 0-distributive lattice forms a sublattice of the lattice of all $\alpha$-ideals. The lattice of all ideals of the lattice of extension ideals is proved to be isomorphic with the lattice of all $\alpha$-ideals of a 0-distributive lattice. A necessary and sufficient condition for a 0-distributive lattice to be quasi-complemented using extension ideals is given.

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## 2 Preliminaries

In this section we collect some basic concepts which we need in the sequel. For lattice theoretical concepts which have now become a common place we refer to Grätzer [1]. A lattice $L$ with 0 is said to be 0-distributive if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$ for any $a, b, c$ in $L$. Throughout this paper $L$ will denote a bounded 0-distributive lattice with bounds 0 and 1 unless otherwise specified. For any non-empty subset $A$ of $L$, define $A^* = \{x \in L : x \wedge a = 0, \text{ for each a} \in \text{A}\}$. Note that $A^*$ is an ideal for any non-empty subset $A$ of $L$. By $A^{**}$we mean $(A^*)^*$. If $A = \{a\}$ $(a \in L)$, then $A^* = \{a\}^* = (a]^*$, where $(a] = \{x \in L : x \leq a\}$ is the principal ideal generated by $a$.

An element $a \in L$ is said to be dense in $L$ if $\{a\}^* = \{0\}$. For all $a, b, c \in L$ we have $\{a\}^{**} \cap \{b\}^{**} = \{a \wedge b\}^{**}$ and $\{a\}^* \cap \{b\}^* = \{a \vee b\}^*$. An ideal $I$ of $L$ is an $\alpha$-ideal if $\{x\}^{**} \subseteq I$ for each $x \in I$. Let $I(L)$ denote the set of all ideals of $L$. Then $(I(L), \wedge, \vee)$ is a lattice where $I \wedge J = I \cap J$ and $I \vee J = (I \cup J]$, for any two ideals $I$ and $J$ of $L$.

## 3 Extension ideals

In this section we define and study *extension ideals* in $L$. For any $a \in L$, define $Ex(a) = \{x \in L : \{a\}^* \subseteq \{x\}^*\}$. We call this set extension of $a$. First we prove some properties of $Ex(a)$ for $a \in L$.

**Theorem 1.** *For $a, b \in L$ the following properties hold.*

*(i)* $(a] \subseteq Ex(a)$.

*(ii)* $a = 0$ if and only if $Ex(a) = \{0\}$.

*(iii)* $a \in Ex(b) \Rightarrow Ex(a) \subseteq Ex(b)$.

*(iv)* $a \leq b \Rightarrow Ex(a) \subseteq Ex(b)$.

*(v)* $a \in D$ if and only if $Ex(a) = L$, *where $D$ denotes the set of all dense elements in $L$.*

*(vi)* $Ex(a) \cap Ex(b) = Ex(a \wedge b)$.

*Proof.* (i) Let $x \in (a]$. Then $x \leq a$. Therefore $\{a\}^* \subseteq \{x\}^*$ proving that $x \in Ex(a)$. Thus $(a] \subseteq Ex(a)$.

(ii) If $a = 0$, then $Ex(a) = Ex(0) = \{x \in L : \{0\}^* \subseteq \{x\}^*\} = \{x \in L : L \subseteq \{x\}^*\} = \{x \in L : \{x\}^* = L\} = \{0\}$. Conversely if $Ex(a) = \{0\}$, then $a \in Ex(a)$ implies $a = 0$.

(iii) Since $a \in Ex(b)$ we have $\{b\}^* \subseteq \{a\}^*$. Let $x \in Ex(a)$. Then $\{a\}^* \subseteq \{x\}^*$. Therefore $\{b\}^* \subseteq \{x\}^*$ and consequently $x \in Ex(b)$. This proves $Ex(a) \subseteq Ex(b)$.

(iv) Since $a \leq b$ we have $\{b\}^* \subseteq \{a\}^*$. Let $x \in Ex(a)$. Then $\{a\}^* \subseteq \{x\}^*$. Therefore $\{b\}^* \subseteq \{x\}^*$ and hence $x \in Ex(b)$. Thus $Ex(a) \subseteq Ex(b)$.

(v) First suppose $a \in D$ i.e. $a$ is a dense element. Therefore $\{a\}^* = \{0\}$. Then $Ex(a) = \{x \in L : \{a\}^* \subseteq \{x\}^*\} = \{x \in L : \{0\} \subseteq \{x\}^*\}$. But $\{0\} \subseteq \{x\}^*$ hold for all $x \in L$. Therefore $Ex(a) = L$.
Conversely suppose $Ex(a) = L$. Therefore $\{a\}^* \subseteq \{y\}^*$ for all $y \in L$. Let $x \in \{a\}^*$ . Then $x \in \{y\}^*$ for all $y \in L$. In particular $x \in \{x\}^*$ implies $x \wedge x = 0$. Therefore $x = 0$.

Thus $\{a\}^* = \{0\}$ proving that $a \in D$.

(vi) By (iv), $Ex(a \wedge b) \subseteq Ex(a) \cap Ex(a)$. Let $x \in Ex(a) \cap Ex(b)$. Then $\{a\}^* \subseteq \{x\}^*$ and $\{b\}^* \subseteq \{x\}^*$. Let $z \in \{a \wedge b\}^*$ then $z \wedge (a \wedge b) = 0$. By associativity we get $z \wedge a \in \{b\}^*$ that is $z \wedge a \in \{x\}^*$. This implies $(z \wedge a) \wedge x = 0$ or $(z \wedge x) \wedge a = 0$. This yields $z \wedge x \in \{a\}^*$, that is $z \wedge x \in \{x\}^*$. Therefore $(z \wedge x) \wedge x = 0$ i.e. $z \wedge x = 0$. This shows $z \in \{x\}^*$ proving that $\{a \wedge b\}^* \subseteq \{x\}^*$ and consequently $x \in Ex(a \wedge b)$. Thus $Ex(a) \cap Ex(b) \subseteq Ex(a \wedge b)$. From both the inclusions we get $Ex(a \wedge b) = Ex(a) \cap Ex(a)$. □

The following property of $Ex(a)$ $(a \in L)$ is crucial for our further study.

**Theorem 2.** *For any $a \in L$, $Ex(a)$ is the smallest $\alpha$-ideal in $L$ containing $a$.*

*Proof.* As $a \in Ex(a)$, $Ex(a)$ is non-empty. Let $x, y \in L$ such that $x \leq y$ and $y \in Ex(a)$. Then $\{a\}^* \subseteq \{y\}^* \subseteq \{x\}^*$ implies $x \in Ex(a)$. Let $x, y \in Ex(a)$. Then $\{a\}^* \subseteq \{x\}^* \cap \{y\}^*$ implies $\{a\}^* \subseteq \{x \vee y\}^*$ (since $L$ is 0-distributive). Hence $x \vee y \in Ex(a)$. Therefore $Ex(a)$ is an ideal in L. Let $x \in Ex(a)$ and $t \in \{x\}^{**}$. Then $\{a\}^* \subseteq \{x\}^* \subseteq \{t\}^*$ implies $t \in Ex(a)$. This shows that $\{x\}^{**} \subseteq Ex(a)$ and hence $Ex(a)$ is an $\alpha$-ideal in $L$ containing $a$. Let $I$ be an $\alpha$-ideal in $L$ containing $a$ and $x \in Ex(a)$. Then $\{a\}^* \subseteq \{x\}^*$ implies $\{x\}^{**} \subseteq \{a\}^{**}$. Since $x \in \{x\}^{**}$ we get $x \in \{a\}^{**}$. As $I$ is an $\alpha$-ideal in $L$ with $a \in I$ we have $\{a\}^{**} \subseteq I$. Therefore $x \in I$. Hence $Ex(a) \subseteq I$. This proves $Ex(a)$ is the smallest $\alpha$-ideal in $L$ containing $a$. □

**Definition 3.** For any $a \in L$ we call the ideal $Ex(a)$ an extension ideal.

Thus by Theorem 2, the extension ideal $Ex(a)$ is the smallest $\alpha$-ideal in $L$ containing $a$. Following Jayaram [2], for an ideal $I$ in $L$, the set $I^e = \{x \in L : \{a\}^* \subseteq \{x\}^* \text{ for some } a \in I\}$ is the smallest $\alpha$-ideal containing the ideal $I$. Thus an ideal $I$ in $L$ is an $\alpha$-ideal if and only if $I = I^e$. We note the following facts.

**Remark 4.** (i) For any $a \in L$, if $I = (a]$ then $I^e = (a]^e = Ex(a)$.

(ii) The principal ideal $(a]$ $(a \in L)$ is an $\alpha$ - ideal if and only if $(a] = Ex(a)$.

(iii) $Ex(0) = (0]$ and $Ex(1) = (1] = L$.

(iv) For an ideal $I$ in $L$, $I^e = \bigcup\{Ex(x) \ : \ x \in I\}$.

(v) Let $\mathcal{A}^\alpha(L)$ be the set of all extension ideals in $L$ i.e. $\mathcal{A}^\alpha(L) = \{Ex(a) \ : \ a \in L\}$. Then $\mathcal{A}^\alpha(L) \subseteq I(L)$ (by Theorem 2).

By Theorem 1 (vi), $Ex(a \wedge b) = Ex(a) \cap Ex(b)$. As $Ex(a) \cap Ex(b) = Ex(a) \wedge Ex(b)$ we get $Ex(a \wedge b) = Ex(a) \wedge Ex(b)$. Thus $\mathcal{A}^\alpha(L)$ is a $\wedge$ - closed subset of $I(L)$.

(vi) $\mathcal{A}^\alpha(L)$ is not necessarily a sublattice of $I(L)$. For this consider the bounded 0-distributive lattice $L = \{0, a, b, c, d, 1\}$ as shown by the Hasse diagram in Figure 1. Here $(b)^e = \{0, c\}$ and $(c)^e = \{0, a, b\}$. But $(b)^e \vee (c)^e = \{0, a, b, c, d\} \notin \mathcal{A}^\alpha(L)$. Hence the set $\mathcal{A}^\alpha(L)$ is a poset under set inclusion but need not be a sublattice of the lattice $I(L)$.

Let $I^\alpha(L)$ denote the set of all $\alpha$-ideals in $L$. As $\mathcal{A}^\alpha(L) \subseteq I^\alpha(L) \subseteq I(L)$ (by Theorem 2), the example shows that $I^\alpha(L)$ is not necessarily a sublattice of $I(L)$. According to [2], $(I^\alpha(L), \sqcap, \sqcup)$ is complete distributive lattice where $I \sqcap J = I \cap J$ and $I \sqcup J = (I \vee J)^e$ for $I,\ J \in I^\alpha(L)$.

Further we have

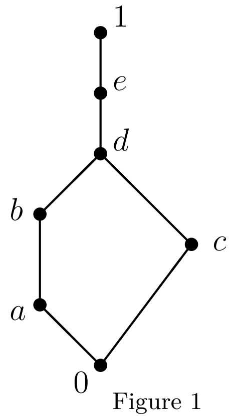

Figure 1

**Theorem 5.** *$(\mathcal{A}^\alpha(L), \sqcap, \sqcup)$ is a bounded distributive lattice.*

*Proof.* Let $Ex(a), Ex(b) \in \mathcal{A}^\alpha(L)$ for $a, b \in L$. Then, by using Theorem 2 and the facts noted in Remark 4, we have $Ex(a) \sqcap Ex(b) = Ex(a) \cap Ex(b) = Ex(a \wedge b)$ and $Ex(a) \sqcup Ex(b) = (a]^e \sqcup (b]^e = ((a] \vee (b])^e = (a \vee b]^e = Ex(a \vee b)$. Thus we have $Ex(a) \sqcap Ex(b) = Ex(a \wedge b)$ and $Ex(a) \sqcup Ex(b) = Ex(a \vee b)$. Further $Ex(0) = \{0\} \in \mathcal{A}^\alpha(L)$ and $Ex(1) = L \in \mathcal{A}^\alpha(L)$. This shows that $(\mathcal{A}^\alpha(L), \sqcap, \sqcup)$ is a bounded sublattice of the lattice $(I^\alpha(L), \sqcap, \sqcup)$. Hence $(\mathcal{A}^\alpha(L), \sqcap, \sqcup)$ is a bounded distributive lattice. □

**Corollary 6.** *The lattice $\mathcal{A}^\alpha(L)$ is a homomorphic image of the lattice $L$.*

*Proof.* Define $\theta : L \to \mathcal{A}^\alpha(L)$ by $\theta(a) = Ex(a)$ for each $a \in L$. Then for all $a, b \in L$, $\theta(a \wedge b) = Ex(a \wedge b) = Ex(a) \cap Ex(b) = Ex(a) \sqcap Ex(b) = \theta(a) \sqcap \theta(b)$ and $\theta(a \vee b) = Ex(a \vee b) = Ex(a) \sqcup Ex(b) = \theta(a) \sqcup \theta(b)$ hold. Hence $\theta$ is a homomorphism. As $\theta$ is onto, the result follows. □

**Remark 7.** Note that $\theta$ is not necessarily one-one. For this consider the 0-distributive lattice of the example given in Remark 4 (vi). Here for $a \neq b$ in $L$ we have $(a)^e = (b)^e$.

According to Varlet [6], a lattice $L$ with 0 is quasi-complemented if given $a \in L$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is dense in $L$.

**Theorem 8.** *$L$ is quasi-complemented if and only if the lattice $\mathcal{A}^\alpha(L)$ is a Boolean lattice.*

*Proof.* In view of Theorem 5, it is enough to prove that $L$ is quasi-complemented if and only if the lattice $\mathcal{A}^\alpha(L)$ is complemented. Assume that $L$ is quasi-complemented. Let $Ex(a) \in \mathcal{A}^\alpha(L)$. Then as $a \in L$, there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is dense in $L$. Since $a \vee b$ is dense, $Ex(a \vee b) = L$ (see Theorem 1 (v)). Now $Ex(a) \sqcap Ex(b) = Ex(a \wedge b) = Ex(0)$ and $Ex(a) \sqcup Ex(b) = Ex(a \vee b) = L$. This shows $Ex(b)$ is a complement of $Ex(a)$ in $\mathcal{A}^\alpha(L)$. Hence the lattice $\mathcal{A}^\alpha(L)$ is complemented.
Conversely, assume that the lattice $\mathcal{A}^\alpha(L)$ is complemented. Let $a \in L$. Then for $Ex(a) \in \mathcal{A}^\alpha(L)$, there exists $Ex(b) \in \mathcal{A}^\alpha(L)$ such that $Ex(a) \sqcap Ex(b) = Ex(0)$ and $Ex(a) \sqcup Ex(b) = L$. Thus $Ex(a \wedge b) = Ex(0)$ and $Ex(a \vee b) = L$ imply $a \wedge b = 0$ and $a \vee b$ is dense in $L$ (see Theorem 1 (v)). Hence $L$ is quasi-complemented. □

## 4 Closure operator

In this section we introduce a closure operator on the ideal lattice $I(L)$ of $L$ using the extension ideals in $L$ and show that the set of all closed ideals of $L$ with respect to this closure operator are the $\alpha$-ideals in $L$. For any ideal $I$ in $L$, define $\rho(I) = \{Ex(a) : a \in I\}$. Then we have:

**Theorem 9.**

*(i) For any ideal $I$ of $L$, $\rho(I)$ is an ideal of $\mathcal{A}^{\alpha}(L)$.*

*(ii) For any two ideals $I$ and $J$ of $L$, $I \subseteq J \Rightarrow \rho(I) \subseteq \rho(J)$.*

*Proof.* (i). Let $I$ be any ideal of $L$. As $0 \in I$, $Ex(0) = \{0\} \in \rho(I)$. Hence $\rho(I)$ is non-empty subset of $\mathcal{A}^{\alpha}(L)$. Let $Ex(a), Ex(b) \in \mathcal{A}^{\alpha}(L)$ be such that $Ex(a) \subseteq Ex(b)$ and $Ex(b) \in \rho(I)$. $Ex(b) \in \rho(I)$ implies $Ex(b) = Ex(x)$ for some $x \in I$. Thus $Ex(a) = Ex(a) \sqcap Ex(b) = Ex(a) \sqcap Ex(x) = Ex(a \wedge x)$. As $a \wedge x \in I$, we get $Ex(a) \in \rho(I)$. Let $Ex(a), Ex(b) \in \rho(I)$. Then $Ex(a) = Ex(x)$ for some $x \in I$ and $Ex(b) = Ex(y)$ for some $y \in I$. Hence $Ex(a) \sqcup Ex(b) = Ex(x) \sqcup Ex(y) = Ex(x \vee y)$. As $x \vee y \in I$, we get $Ex(x \vee y) \in \rho(I)$. Hence $Ex(a) \sqcup Ex(b) \in \rho(I)$. Therefore $\rho(I)$ is an ideal in $\mathcal{A}^{\alpha}(L)$.
(ii). Let $I$ and $J$ be two ideals of $L$ such that $I \subseteq J$. Let $Ex(a) \in \rho(I)$. Then $Ex(a) = Ex(x)$ for some $x \in I$. But then $x \in J$ implies $Ex(a) = Ex(x) \in \rho(J)$. Hence $\rho(I) \subseteq \rho(J)$. □

Let $I(\mathcal{A}^{\alpha}(L))$ denote the lattice of all ideals of the lattice $\mathcal{A}^{\alpha}(L)$. Then $(I(\mathcal{A}^{\alpha}(L), \wedge, \vee)$ is a lattice where $I \wedge J = I \cap J$ and $I \vee J = (I \cup J]$, for any two ideals $I$ and $J$ in $I(\mathcal{A}^{\alpha}(L))$. As $\rho(I)$ is an ideal of $\mathcal{A}^{\alpha}(L)$, for any ideal $I$ of $L$, the mapping $\rho : I(L) \to I(\mathcal{A}^{\alpha}(L))$ is well defined. Further we have:

**Theorem 10.** *The mapping $\rho : I(L) \to I(\mathcal{A}^{\alpha}(L))$ is a $\{0, 1\}$ homomorphism.*

*Proof.* Let $I$ and $J$ be any ideals in $I(L)$. Then by Theorem 9 (ii) we have, $\rho(I \wedge J) = \rho(I \cap J) \subseteq \rho(I) \cap \rho(J)$. Let $Ex(x) \in \rho(I) \cap \rho(J)$. Then $Ex(x) \in \rho(I)$ and $Ex(x) \in \rho(J)$ imply $Ex(x) = Ex(i)$ for some $i \in I$ and $Ex(x) = Ex(j)$ for some $j \in J$. Thus $Ex(x) = Ex(i) \sqcap Ex(j) = Ex(i \wedge j)$. As $i \wedge j \in I \cap J$, $Ex(x) \in \rho(I \wedge J)$. This shows that $\rho(I) \cap \rho(J) \subseteq \rho(I \wedge J)$. Combining both the inclusions we get $\rho(I \wedge J) = \rho(I) \cap \rho(J) = \rho(I) \wedge \rho(J)$. Now, again applying Theorem 9 (ii), we have $\rho(I) \vee \rho(J) \subseteq \rho(I \vee J)$. Let $Ex(x) \in \rho(I \vee J)$. Hence $Ex(x) = Ex(y)$ for some $y \in I \vee J$. Therefore $y \leq i \vee j$ for some $i \in I$ and $j \in J$. Using Theorem 1 (iv), we get $Ex(y) \subseteq Ex(i \vee j)$. Thus $Ex(x) = Ex(y) \subseteq Ex(i) \sqcup Ex(j)$. As $Ex(i) \sqcup Ex(j) \in \rho(I) \vee \rho(J)$ we get $Ex(x) \in \rho(I) \vee \rho(J)$. Hence $\rho(I \vee J) \subseteq \rho(I) \vee \rho(J)$. Combining both the inclusions we get $\rho(I \vee J) = \rho(I) \vee \rho(J)$.
This proves that $\rho : I(L) \to I(\mathcal{A}^{\alpha}(L))$ is a homomorphism. Also $\rho((0]) = \{Ex(0)\} =$ zero of $I(\mathcal{A}^{\alpha}(L))$ and $\rho((1]) = Ex(1) = L$, show $\rho$ is a $\{0, 1\}$ homomorphism. □

For any ideal $\overline{I}$ of $\mathcal{A}^{\alpha}(L)$, define $\overleftarrow{\rho}(\overline{I}) = \{a \in L : Ex(a) \in \overline{I}\}$. Then we have:

**Theorem 11.**

*(i) For any ideal $\overline{I}$ in $\mathcal{A}^{\alpha}(L)$, $\overleftarrow{\rho}(\overline{I})$ is an ideal in $L$.*

*(ii) For any ideals $\overline{I}$ and $\overline{J}$ of $\mathcal{A}^{\alpha}(L)$, $\overline{I} \subseteq \overline{J} \Rightarrow \overleftarrow{\rho}(\overline{I}) \subseteq \overleftarrow{\rho}(\overline{J})$.*

*Proof.* (i). Let $\overline{I}$ be any ideal in $\mathcal{A}^{\alpha}(L)$. Since $Ex(0) = \{0\} \in \overline{I}$, we have $0 \in \overleftarrow{\rho}(\overline{I})$. Thus $\overleftarrow{\rho}(\overline{I})$ is non-empty subset of $L$. Let $a, b \in L$ such that $a \leq b$ and $b \in \overleftarrow{\rho}(\overline{I})$. Then by Theorem 1 (iv) we get $Ex(a) \subseteq Ex(b)$ and $Ex(b) \in \overline{I}$. Since $\overline{I}$ is an ideal it follows that $Ex(a) \in \overline{I}$ and consequently $a \in \overleftarrow{\rho}(\overline{I})$. Let $a, b \in \overleftarrow{\rho}(\overline{I})$. Then $Ex(a), Ex(b) \in \overline{I}$ and $\overline{I}$ being an ideal we have $Ex(a) \sqcup Ex(b) = Ex(a \vee b) \in \overline{I}$. But then $a \vee b \in \overleftarrow{\rho}(\overline{I})$. Therefore $\overleftarrow{\rho}(\overline{I})$ is an ideal in $L$.
ii). Let $\overline{I}$ and $\overline{J}$ be any two ideals of $\mathcal{A}^{\alpha}(L)$ such that $\overline{I} \subseteq \overline{J}$. Let $x \in \overleftarrow{\rho}(\overline{I})$. Then $Ex(x) \in \overline{I}$ implies $Ex(x) \in \overline{J}$. Hence $x \in \overleftarrow{\rho}(\overline{J})$. This shows that $\overleftarrow{\rho}(\overline{I}) \subseteq \overleftarrow{\rho}(\overline{J})$. □

By Theorem 9 (i) and Theorem 11 (i), we get the mappings $\rho : I(L) \to I(\mathcal{A}^{\alpha}(L))$ and $\overleftarrow{\rho} : I(\mathcal{A}^{\alpha}(L)) \to I(L)$. Hence $\rho \circ \overleftarrow{\rho} : I(\mathcal{A}^{\alpha}(L)) \to I(\mathcal{A}^{\alpha}(L))$ and $\overleftarrow{\rho} \circ \rho : I(L) \to I(L)$. We have the following properties about the images under these mappings.

**Theorem 12.** *The following properties hold in $L$.*

*(i) For an ideal $I$ in $L$, $\overleftarrow{\rho} \circ \rho(I) = I^e$.*

*(ii) For an ideal $\overline{I}$ in $\mathcal{A}^{\alpha}(L)$, $\rho \circ \overleftarrow{\rho}(\overline{I}) = \overline{I}$ i.e. $\rho \circ \overleftarrow{\rho}$ is an identity mapping on $I(\mathcal{A}^{\alpha}(L))$.*

*Proof.* (i). Let $I \in I(L)$ and $x \in \bigcup\{Ex(a) \ : \ a \in I\}$. Hence $x \in Ex(a)$ for some $a \in I$. Then $Ex(x) \subseteq Ex(a)$ for some $a \in I$. Now $a \in I$ implies $Ex(a) \in \rho(I)$. Since $\rho(I)$ is an ideal, $Ex(x) \in \rho(I)$ and hence $x \in \overleftarrow{\rho} \circ \rho\,(I)$. This shows that $\bigcup\{Ex(a) \ : \ a \in I\} \subseteq \overleftarrow{\rho} \circ \rho(I)$. Conversely, assume that $x \in \overleftarrow{\rho} \circ \rho(I)$, for some $I \in I(L)$. Then $Ex(x) \in \rho(I)$ implies $Ex(x) = Ex(a)$ for some $a \in I$. Thus $Ex(x) = Ex(a) \subseteq \bigcup\{Ex(a) \ : \ a \in I\}$. Since $x \in Ex(x)$, it follows that $x \in \bigcup\{Ex(a) \ : \ a \in I\}$. Thus we get $\overleftarrow{\rho} \circ \rho(I) \subseteq \bigcup\{Ex(a) \ : \ a \in I\}$. From both the inclusions we have $\overleftarrow{\rho} \circ \rho(I) = \bigcup\{Ex(a) \ : \ a \in I\}$. As $I^e = \bigcup\{Ex(a) \ : \ a \in I\}$ (see Remark 4 (iv)), we get $\overleftarrow{\rho} \circ \rho(I) = I^e$.
(ii). Let $\overline{I}$ be any ideal of $\mathcal{A}^{\alpha}(L)$. Let $Ex(x) \in \rho \circ \overleftarrow{\rho}(\overline{I}) = \rho(\overleftarrow{\rho}(\overline{I}))$. Hence $Ex(x) = Ex(y)$ for some $y \in \overleftarrow{\rho}(\overline{I})$. But then $Ex(y) \in \overline{I}$ implies $Ex(x) \in \overline{I}$. This gives $\rho \circ \overleftarrow{\rho}(\overline{I}) \subseteq \overline{I}$. Conversely, let $Ex(x) \in \overline{I}$. Then $x \in \overleftarrow{\rho}(\overline{I})$ implies $Ex(x) \in \rho\left(\overleftarrow{\rho}(\overline{I})\right)$ (since $\overleftarrow{\rho}(\overline{I})$ is an ideal of $L$). Therefore $Ex(x) \in \rho \circ \overleftarrow{\rho}(\overline{I})$. Hence $\overline{I} \subseteq \rho \circ \overleftarrow{\rho}(\overline{I})$. From both the inclusions we get $\rho \circ \overleftarrow{\rho}(\overline{I}) = \overline{I}$. Hence $\rho \circ \overleftarrow{\rho}$ is an identity mapping on $I(\mathcal{A}^{\alpha}(L))$. □

Recall that closure operator on a lattice $L$ is a mapping $f : L \to L$ satisfying the following conditions: (i) $x \leq f(x)$ (ii) $x \leq y \Rightarrow f(x) \leq f(y)$ (iii) $f(f(x)) = f(x)$. Now we prove our main result.

**Theorem 13.** *The mapping $\overleftarrow{\rho} \circ \rho : I(L) \to I(L)$ is a closure operator on the lattice $I(L)$.*

*Proof.* (i) Let $I \in I(L)$ and $x \in I$. Then $Ex(x) \in \rho(I)$ and $\rho(I)$ is an ideal of $\mathcal{A}^{\alpha}(L)$ (by Theorem 9 (i)) imply $x \in \overleftarrow{\rho} \circ \rho(I)$. Hence $I \subseteq \overleftarrow{\rho} \circ \rho(I)$.
(ii) Let $I, J \in I(L)$ and $I \subseteq J$. By Theorem 9 and 11, $\rho$ and $\overleftarrow{\rho}$ are isotone mappings, therefore we get $\overleftarrow{\rho} \circ \rho(I) \subseteq \overleftarrow{\rho} \circ \rho(J)$.
(iii) Let $I \in I(L)$. As $\overleftarrow{\rho} \circ \rho(I)$ is an ideal of $L$, by using (i) and (ii) we get $\overleftarrow{\rho} \circ \rho(I) \subseteq \overleftarrow{\rho} \circ \rho(\overleftarrow{\rho} \circ \rho(I))$. Conversely, let $x \in \overleftarrow{\rho} \circ \rho(\overleftarrow{\rho} \circ \rho(I))$. Then $Ex(x) \in \rho(\overleftarrow{\rho} \circ \rho(I))$ implies $Ex(x) = Ex(y)$ for some $y \in \overleftarrow{\rho} \circ \rho(I)$. But then $Ex(x) = Ex(y) \in \rho(I)$. Therefore $x \in \overleftarrow{\rho} \circ \rho(I)$. Thus we have $\overleftarrow{\rho} \circ \rho(\overleftarrow{\rho} \circ \rho(I)) \subseteq \overleftarrow{\rho} \circ \rho(I)$. Combining both the inclusions we get $\overleftarrow{\rho} \circ \rho(\overleftarrow{\rho} \circ \rho(I)) = \overleftarrow{\rho} \circ \rho(I)$.
From(i), (ii) and (iii) we get $\overleftarrow{\rho} \circ \rho$ is a closure operator on $I(L)$. □

Let $\mathcal{C}(L)$ denote the set of all ideals in $L$ which are closed with respect to the closure operator $\overleftarrow{\rho} \circ \rho$ defined on $I(L)$. Thus $\mathcal{C}(L) = \{I \in I(L) \ : \ \overleftarrow{\rho} \circ \rho(I) = I\}$.

**Remark 14.** $\mathcal{C}(L) = \{I \in I(L) \ : \ \overleftarrow{\rho} \circ \rho(I) = I\} = \{I \in I(L) \ : \ I^e = I\}$ (see Theorem 12 (i) ).

We know that $(I^\alpha (L), \sqcap, \sqcup)$ is a complete distributive lattice where $I \sqcap J = I \cap J$ and $I \sqcup J = (I \vee J)^e$ for $I, J \in I^\alpha(L)$. The following theorem is of intrinsic interest.

**Theorem 15.** *$\mathcal{C}(L) = I^\alpha (L)$ and $I \sqcup J = (I \vee J)^e = \overleftarrow{\rho} \circ \rho(I \vee J)$ for $I, J \in I^\alpha(L)$.*

*Proof.* Using Theorem 12 (i), we have

$$\begin{aligned} I \in \mathcal{C}(L) \quad &\Leftrightarrow \quad \overleftarrow{\rho} \circ \rho(I) = I \\ &\Leftrightarrow \quad I^e = I \\ &\Leftrightarrow \quad I \in I^\alpha(L). \end{aligned}$$

Hence $\mathcal{C}(L) = I^\alpha (L)$. For $I, J \in I^\alpha(L)$, $I \sqcup J = (I \vee J)^e = \overleftarrow{\rho} \circ \rho(I \vee J)$ (see Theorem 12 (i)). ∎

Interestingly we have:

**Theorem 16.** *The lattice $I^\alpha (L)$ is isomorphic with the lattice $I(\mathcal{A}^\alpha (L))$.*

*Proof.* Define the mapping $\psi : I^\alpha (L) \to I(\mathcal{A}^\alpha (L))$ by $\psi(I) = \rho(I)$ for each $I \in I^\alpha (L)$.
(i) Clearly $\psi$ is a well defined mapping.
(ii) Let $\psi(I) = \psi(J)$ for $I, J \in I^\alpha (L)$. Then $\rho(I) = \rho(J)$ implies $\overleftarrow{\rho} \circ \rho(I) = \overleftarrow{\rho} \circ \rho(J)$. Since $I, J \in I^\alpha (L) = \mathcal{C}(L)$, we get $I = J$. This shows that the mapping $\psi$ is one-one.
(iii) Let $\overline{I}$ be any ideal of $\mathcal{A}^\alpha (L)$. Then $\overleftarrow{\rho}(\overline{I})$ is an ideal in $L$ ( by Theorem 11 (i)) and $\rho \circ \overleftarrow{\rho}(\overline{I}) = \overline{I}$ (by Theorem 12 (ii)). Then $\overleftarrow{\rho} \circ \rho\left(\overleftarrow{\rho}(\overline{I})\right) = \overleftarrow{\rho}(\rho(\overleftarrow{\rho}(\overline{I}))) = \overleftarrow{\rho}(\overline{I})$. This shows that $\overleftarrow{\rho}(\overline{I}) \in \mathcal{C}(L) = I^\alpha (L)$. As $\psi(\overleftarrow{\rho}(\overline{I})) = \overline{I}$, $\psi$ is onto.
(iv) Let $I, J \in I^\alpha (L)$. Then by using Theorem 10

$$\begin{aligned} \psi(I \ \sqcap \ J) &= \psi(I \cap J) \\ &= \rho(I \cap J) \\ &= \rho(I) \cap \rho(J) \\ &= \psi(I) \cap \psi(J) \end{aligned}$$

Also, using Theorem 9

$$\begin{aligned} \psi(I \ \sqcup \ J) &= \rho(I \vee J) \\ &= \rho(\overleftarrow{\rho} \circ \rho(I \vee J)) \end{aligned}$$

Since $\rho \circ \overleftarrow{\rho}$ is an identity map

$$\begin{aligned} \psi(I \ \sqcup \ J) &= \rho(I \vee J) \\ &= \rho(I) \vee \rho(J) \\ &= \psi(I) \vee \psi(J). \end{aligned}$$

This shows that the mapping $\psi$ is a homomorphism.
From (i) - (iv) we get $\psi$ is an isomorphism. ∎

Many characterizations of an $\alpha$-ideal in a 0-distributive lattice are established in [3] and [4]. In the next theorem we give some more characterizations of an $\alpha$-ideal in a 0-distributive lattice in terms of principal $\alpha$-ideals.

**Theorem 17.** *For any ideal $I$ in $L$, following statements are equivalent.*

*(i)* $I \in I^{\alpha}(L)$

*(ii)* *For* $x, y \in L$, $Ex(x) = Ex(y)$, $x \in I \Rightarrow y \in I$.

*(iii)* $I = \bigcup\{Ex(x) \ : \ x \in I\}$.

*Proof.* $(i) \Rightarrow (ii)$ Let $I \in I^{\alpha}(L)$. Then by Theorem 15, $I \in \mathcal{C}(L)$ i.e. $I = \overleftarrow{\rho} \circ \rho(I)$. Let $x, y \in L$ such that $Ex(x) = Ex(y)$ and $x \in I$. Then $x \in I = \overleftarrow{\rho} \circ \rho(I)$ implies $Ex(x) \in \rho(I)$. But then $Ex(y) \in \rho(I)$ and consequently $y \in \overleftarrow{\rho} \circ \rho(I) = I$ and the implication follows.
$(ii) \Rightarrow (iii)$ Let $I$ be an ideal in $L$. We know that $(x] \subseteq Ex(x)$ for all $x \in L$ (see Theorem 1 (i)). As $I = \bigcup\{(x] \ : \ x \in I\}$ we get $I \subseteq \bigcup\{Ex(x) \ : \ x \in I\}$. For the converse inclusion, let $y \in \bigcup\{Ex(x) \ : \ x \in I\}$ . Then $y \in Ex(x)$ for some $x \in I$. Therefore $Ex(y) \subseteq Ex(x)$. Thus $Ex(y) = Ex(x) \cap Ex(y) = Ex(x \wedge y)$. As $x \wedge y \in I$, by assumption (ii) we get $y \in I$. Thus $\bigcup\{Ex(x) \ : \ x \in I\} \subseteq I$. Combining both the inclusions we get $I = \bigcup\{Ex(x) \ : \ x \in I\}$.
$(iii) \Rightarrow (i)$ Let $I$ be an ideal of $L$ and $I = \bigcup\{Ex(x) \ : \ x \in I\}$. Let $y \in I$. Then $y \in \bigcup\{Ex(x) \ : \ x \in I\}$ implies $y \in Ex(a)$ for some $a \in I$. Let $t \in \{y\}^{**}$. Then $\{a\}^{*} \subseteq \{y\}^{*} \subseteq \{t\}^{*}$. Therefore $t \in Ex(a)$. Thus $t \in \bigcup\{Ex(x) \ : \ x \in I\} = I$. This shows that $\{y\}^{**} \subseteq I$ and hence $I$ is an $\alpha$ - ideal in $L$ i.e. $I \in I^{\alpha}(L)$. □

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